Directivity of Uniformly Spaced Optimum Endfire Arrays With Equal Sidelobes

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A mathematical formulation is devised for a uniformly spaced and uniformly progressively phased antenna array of a given number of isotropic elements such that the final radiation pattern is endfire with no large backlobe. The pattern also has the maximum number of sidelobes of equal level, modifying the result of an earlier paper, even when the element spacing is made arbitrarily small. The directive gain, the first null beamwidth, the half-power beamwidth, the current excitations, and the required phases for such an array are given in curves as a function of the element spacing. It is shown that the limiting radiation characteristics when the element spacing approaches zero can also be determined in a relatively simple manner.

1. Introduction

Since the publication of Dolph's paper [1946], many articles dealing with the optimization between the beamwidth and the sidelobe level of linear antenna arrays have appeared in the literature. In summary, different types of mathematical formulations have been used by different investigators for different arrays, depending on whether the array is broadside or endfire, whether the total number of elements in the array is odd or even, and whether the element spacing is greater or less than one-half wavelength [Dolph, 1946; Riblet, 1947; DuHamel, 1953; Pritchard, 1955]. A unified approach covering all these cases was described by Ma and Cheng [1961a].

The problem of maximizing the directivity of an array was investigated by Uzkov [1946]. He showed, by means of an orthogonal transformation in vector space, that the maximum directivity of an endfire array of n isotropic elements, as the element spacing approaches zero, is numerically equal to n^2. Other techniques for improving directivities of endfire arrays were also proposed [Hansen and Woodyard, 1938; Goward, 1947]. Bloch, Medhurst and Pool [1953] investigated similar problems by using the impedance relation between the array elements, but only a very limited calculation was given in their paper. Based on their method, a rather extensive calculation of the maximum directivity of an endfire array of half-wave dipoles was made later [Stearns, 1965]. Recently, Tai made some investigation of the optimum directivity of uniformly spaced broadside arrays [1964] and the ordinary endfire arrays [1963] of isotropic sources or dipoles. The sidelobe level of an array was not considered or limited in any of these papers.

As far as the directivity of an array with equal sidelobes is concerned, Reuss [1959] derived a formula for the Chebyshev linear broadside arrays consisting of infinitesimal dipoles, but his formula provides no information concerning the effect on the directivity when the element spacing changes. Stegen [1960] obtained an expression for the directivity in terms of Chebyshev polynomials, but his expression is valid only when the element spacing is an integral multiple of a half-wavelength. A general approach for dealing with this problem outlined by Ma and Cheng [1961a] was based on the idea of keeping the number of sidelobes as great as possible and with a (desired) equal level even when the element spacing was less than one-half wavelength. Pertinent numerical results on directive gains, however, were not reported there. More recently, while attempting to obtain some numerical result, it was discovered that, according to the procedures outlined by Ma and Cheng [1961a], the sidelobes in the final radiation pattern synthesized for an odd number of elements are not all equal in level, since there always is a higher sidelobe occurring near $\theta = \pi/2$, and that the final pattern for an even number of elements is not physically realizable (e.g., the "power" pattern function will become negative for some physical angles). It is the purpose of this paper to modify the procedures given before in such a way that the final pattern will have all sidelobes at an equal level. The modified procedures given in this paper will be valid for both an odd and even number of elements.

2. Mathematical Formulation

The field from a linear array of n equispaced elements as given in figure 1 can be mathematically represented by the polynomial:

$$E(z) = \sum_{k=0}^{n-1} a_k z^k, \tag{1}$$

where
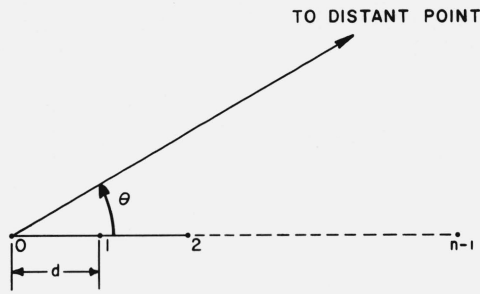
$$z = \exp{(j\psi)}, \tag{2}$$

FIGURE 1. *A linear array of* n *equi-spaced isotropic elements.*

$$\psi = \frac{2\pi}{\lambda} d \cos\theta + \alpha, \tag{3}$$

and a_k represents the excitation of the kth element. In this paper a_k are all real, implying that the array is uniform-progressively phased (α). It was previously shown [Cheng and Ma, 1960] that, by introducing a new variable, y,

$$y \equiv z + z^{-1} = 2\cos\psi = 2\cos\left(\frac{2\pi}{\lambda} d \cos\theta + \alpha\right), \tag{4}$$

one can express $|E(z)|^2$, the power pattern, as a polynominal of y,

$$|E(z)|^2 = E(z)E(z^{-1})$$
$$= \sum_{m=0}^{n-1} b_m(z^m + z^{-m})\Big|_{z+z^{-1}=y} \rightarrow \cdot P(y), \tag{5}$$

where

$$b_m = \sum_{i=0}^{n-1-m} a_i a_{m+i}, \quad 1 \leqslant m \leqslant (n-1), \tag{6}$$

$$b_0 = \frac{1}{2}\sum_{i=0}^{n-1} a_i^2. \tag{7}$$

Since $P(y)$ is a polynominal of $(n-1)$th degree with real coefficients, and is non-negative for all physical values of y in the interval $[-2, 2]$, it generally can only contain a combination of the following possible elementary factors:

(i) $(y + b_k)$ or $(b_k - y)$ with b_k real and $\geqslant 2$,
(ii) $(y + b_k)^2$ with b_k real and $|b_k| < 2$,
(iii) $(y^2 + 2b_{k1}y + b_{k1}^2 + b_{k2}^2)$ with b_{k1} and b_{k2} real.

In particular, for arrays producing the maximum possible number $(n-1)$ of physical nulls in the visible range $-2 \leqslant y \leqslant 2$, the only factors that can appear in $P(y)$ are $(y + 2)$ and $(y + b_k)^2$ with $|b_k| < 2$. Therefore, depending on whether n is odd or even, $P(y)$ for such an array must be:

$$P_o(y) = \prod_{k=1}^{\frac{n-1}{2}} (y + b_k)^2 \qquad \text{for odd } n, \tag{8}$$

$$P_e(y) = (y+2)\prod_{k=1}^{\frac{n-2}{2}} (y+b_k)^2 \text{ for even } n, \tag{9}$$

where all b_k are real, distinct, and $|b_k| < 2$. In (8) and (9), we have assumed that $y = 2$ is the location of the main beam. The corresponding array polynominals are, respectively,

$$E_o(z) = \prod_{k=1}^{\frac{n-1}{2}} (1 + b_k z + z^2) \qquad \text{for odd } n, \tag{10}$$

and

$$E_e(z) = (1+z)\prod_{k=1}^{\frac{n-2}{2}} (1 + b_k z + z^2) \text{ for even } n. \tag{11}$$

The excitation coefficients a_k in (1) can then be determined by expanding (10) or (11).

With this preliminary analysis given, the general procedure of synthesizing an array having the maximum number of nulls can now be summarized as follows: (i) to determine b_k for (8) or (9) according to a specification, and (ii) to determine the required excitation coefficients using (10) or (11).

3. Directivity of Endfire Arrays With Equal Sidelobes

For arrays having all the sidelobes equal in level, the locations of nulls $(y = -b_k)$ and sidelobes $(y = y_l)$ are related in a special manner, where

$$2 > (-b_1) > y_1 > (-b_2) > y_2 > \ldots$$
$$> y_{\frac{n-3}{2}} > (-b_{\frac{n-1}{2}}) > (-2) \text{ for odd } n, \tag{12}$$

and

$$2 > (-b_1) > y_1 > \ldots > y_{\frac{n-2}{2}} > (-2) \text{ for even } n. \tag{13}$$

The parameters b_k in (8) or (9), the corresponding excitation coefficients a_k in (1), and the location of sidelobes can be uniquely determined once a desired sidelobe level relative to the main beam or a desired first null location $(-b_1)$ is specified. The detailed derivations and procedures are omitted here, but can be found elsewhere [Ma and Cheng, 1961b].

After a power polynomial $P(y)$ is determined, the directivity or the directive gain of an endfire array with equal sidelobes can be calculated according to the conventional definition [Silver, 1949]:

$$G = \frac{\left(\frac{4\pi}{\lambda} d\right) P(y_a)}{W_0}, \tag{14}$$

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where

$$W_0 = \int_{y_a}^{y_b} \frac{P(y)}{\sqrt{4-y^2}} \, dy, \tag{15}$$

with the limits given by

$$y_b = 2 \cos \psi_b = 2 \cos \left(-\frac{2\pi}{\lambda} d + \alpha \right) \quad \text{when } \theta = \pi$$

$$y_a = 2 \cos \psi_a = 2 \cos \left(\frac{2\pi}{\lambda} d + \alpha \right) \quad \text{when } \theta = 0. \tag{16}$$

Note that the polynominal $P(y)$ can always be rearranged in terms of $(4-y^2)$ such that the exact expression for W_0 (and hence for G) will be obtained by integrating (15) term by term.

4. A Transformation When $d < \lambda/2$

To keep the discussion more specific, let us consider the $P_0(y)$ given in (8). The whole pattern for $d = \frac{\lambda}{2}$ $\left(\text{or } \frac{2\pi}{\lambda} d = \pi \right)$ is plotted as in figure 2a when the ordinary endfire condition, $-\alpha = \frac{2\pi}{\lambda} d = \pi$, is used. Note that, in this case, there always is a large backlobe at $\theta = \pi$ having the same magnitude as the main lobe at
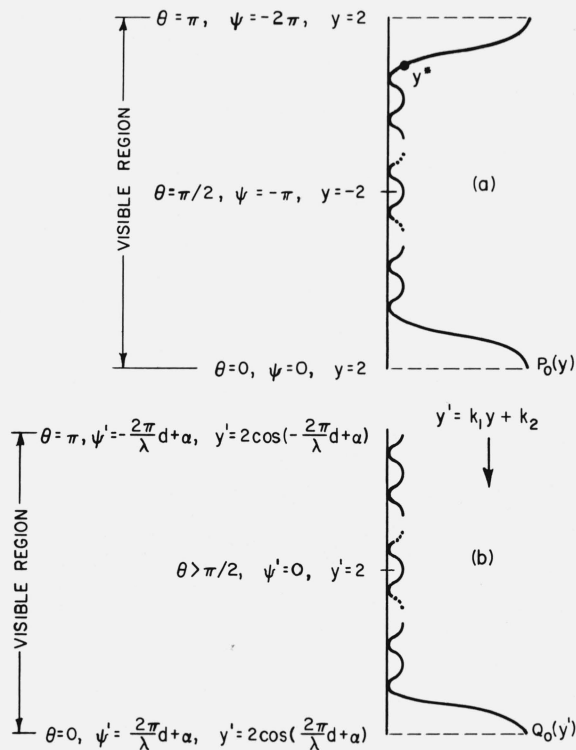

FIGURE 2. *Power pattern of an array with equal sidelobes (odd* n).
(a) Ordinary endfire, $-\alpha = \frac{2\pi}{\lambda} d = \pi$,
(b) Optimum endfire, $d < d^* < \frac{\lambda}{2}, \alpha + 0, -\alpha + \frac{2\pi}{\lambda} d$.

$\theta = 0$. If a smaller d is chosen, some portion of $P_0(y)$ in the left-hand side of figure 2a will be invisible. It is clear that there exists an element spacing $d^* < \frac{\lambda}{2}$ for the ordinary endfire array such that the whole visible region will be ended at $y^* = 2 \cos \left(-\frac{4\pi}{\lambda} d^* \right)$ and that the final pattern will have a sidelobe at $\theta = \pi$ having the same level as the other sidelobes. In this case, no modification for improving both the directivity and the beamwidth is possible if the sidelobe level remains unchanged. This actually is the case considered before by Rhodes [1953].

For $d < d^*$, some of the sidelobes will also be invisible, then only the visible portion of the pattern would be expanded into the region $0 \le \theta \le \pi$, making the beamwidth wider than the case when $d = d^*$. In order to make improvement under this circumstance, an optimum endfire condition $\alpha \neq -\frac{2\pi}{\lambda} d$, together with a linear transformation of variables such as

$$y' = k_1 y + k_2 \tag{17}$$

is devised such that the final pattern still remains much as it formerly did up to $y = y^*$. The parameters k_1, k_2 in (17), and α can be determined by the following conditions:

$$y' = 2 \cos \left(\frac{2\pi}{\lambda} d + \alpha \right) \quad \text{when } y = 2 \quad (\theta = 0),$$

$$y' = 2 \cos \left(\frac{-2\pi}{\lambda} d + \alpha \right) \quad \text{when } y = y^* \quad (\theta = \pi), \tag{18}$$

$$y' = 2 \quad \text{when } y = -2.$$

The first condition of (18) maps both the magnitude and location of the main lobe, the second condition assures that the maximum possible number of sidelobes at equal level be kept and that the large backlobe be avoided, and the third condition maps the sidelobe (an odd n) or the null (an even n) originally at $y = -2$ to $y' = 2$. The transformation (17) with the conditions given in (18) is shown in figure 2b. From this figure one can see that, for $d < d^*$, the transformation (17) always squeezes the whole pattern up to y^* into the visible region and, therefore, helps to shift the pattern toward the main lobe, making the beamwidth narrower and the directivity higher. For this reason the transformed array is called the optimum endfire array. Solving (18), one obtains

$$\tan \frac{\alpha}{2} = \frac{2 - \sqrt{2 + y^*}}{2 + \sqrt{2 + y^*}} \tan \frac{\pi d}{\lambda},$$

$$k_1 = -\sin^2 \left(\frac{\alpha}{2} + \frac{\pi d}{\lambda} \right) < 0, \tag{19}$$

$$k_2 = 2(1 + k_1) = 2 \cos^2 \left(\frac{\alpha}{2} + \frac{\pi d}{\lambda} \right) > 0,$$

where y^* can be easily determined by the following simple relation:

$$y^* = 2(1 - b_{(n+1)/4}), \text{ for } n \text{ odd and } \frac{n-1}{2} \text{ odd};$$

$$y^* = 2(1 + y_{(n-1)/4}), \text{ for } n \text{ odd and } \frac{n-1}{2} \text{ even}, \quad (20)$$

or

$$y^* = 2 + y_1 - b_{(n-2)/2} \text{ for even } n.$$

The phase α and the parameters k_1 and k_2 required to produce an optimum endfire array can be calculated from (19) once a $d < d^*$ is chosen. From the first equation of (19) it can be concluded that α is always located somewhere in the first quadrant in order to make a pattern of an endfire array optimum when $d < d^*$. Note that (19) also holds true when $d = d^*$ and yields $\alpha^* = \pi - \frac{2\pi}{\lambda} d^* > 0$. There seems to be a phase discontinuity of (π) as compared with the phase required by the ordinary endfire condition. This extra phase of π is actually taken care of by the fact that $k_1 = -1$ and $k_2 = 0$ for the case of $d = d^*$. Of course, this apparent discrepancy can be removed if one tries to map $y = -2$ onto $y' = -2$, replacing the third condition in (18). While this alternative transformation seems more natural, as far as the limiting case $d \to d^*$ is concerned, it will create a little mathematical difficulty in dealing with the limiting case to the other end $(d \to 0)$, as compared to what we have done [see (30) through (32)] by using the present approach. In any case, these two approaches yield the same array depending on whether one likes to have the final excitation coefficients a_k alternating signs (as is clearly shown in fig. 6) or to have the sign reversal absorbed in α if using the alternative transformation suggested above. As pointed out before, it is not necessary to make further modification in phases for the endfire arrays with equal sidelobes when $d = d^*$ (considered by Rhodes [1953]) because the ordinary endfire condition for phase still holds for that particular situation. Also, since this paper is to primarily deal with cases when $d < d^*$, the aforementioned two alternative approaches would not make any difference as far as the presentation is concerned.

The optimization discussed above specifically for $P_0(y)$ applies as well to $P_e(y)$ given in (9).

From (17) one has

$$y = k_1^{-1}(y' - k_2). \quad (21)$$

Substituting (21) into (8) and (9), the power polynomial becomes, respectively,

$$Q_0(y') = k_1^{-(n-1)} \prod_{k=1}^{\frac{n-1}{2}} (y' + b_k')^2 \text{ for odd } n, \quad (22)$$

$$Q_e(y') = k_1^{-(n-1)}(y' - 2) \prod_{k=1}^{\frac{n-2}{2}} (y' + b_k')^2$$

$$= |k_1^{-(n-1)}|(2 - y') \prod_{k=1}^{\frac{n-2}{2}} (y' + b_k')^2 \text{ for even } n, \quad (23)$$

where

$$b_k' = k_1 b_k - k_2. \quad (24)$$

Note that the second form of (23) is due to the fact that $k_1 < 0$, see (19), and $k_1^{-(n-1)} < 0$ for an even n. It is then clear from this second form that $Q_e(y') \geq 0$ for $-2 \leq y' \leq 2$, as it should be for a realizable power polynomial. Based on (22) and (23), the corresponding array polynomial becomes, respectively,

$$E_0'(z) = \sqrt{k_1^{-(n-1)}} \prod_{k=1}^{\frac{n-1}{2}} (1 + b_k'z + z^2) \qquad \text{for odd } n, \quad (25)$$

$$E'(z) = \sqrt{|k_1^{-(n-1)}|} (1 - z) \prod_{k=1}^{\frac{n-2}{2}} (1 + b_k'z + z^2) \quad \text{for even } n. \quad (26)$$

A new set of excitation coefficients required for synthesizing the optimum endfire array can be determined by expanding (25) or (26).

The directivity G', the first null beamwidth $(2\theta_1)$, and the half-power beamwidth $(2\theta_h)$ can be calculated respectively from

$$G' = \frac{4\pi \dfrac{d}{\lambda} Q(y_a')}{W_0'}, \quad (27)$$

$$-b_1' = 2 \cos\left(\frac{2\pi}{\lambda} d \cos\theta_1 + \alpha\right), \quad (28)$$

and

$$y_h' = 2 \cos\left(\frac{2\pi}{\lambda} d \cos\theta_h + \alpha\right), \quad (29)$$

where

$$W_0' = \int_{y_a'}^{y_b'} \frac{Q(y')}{\sqrt{4 - y'^2}} dy',$$

$$y_b' = 2 \cos\left(-\frac{2\pi}{\lambda} d + \alpha\right),$$

$$y_a' = 2 \cos\left(\frac{2\pi}{\lambda} d + \alpha\right),$$

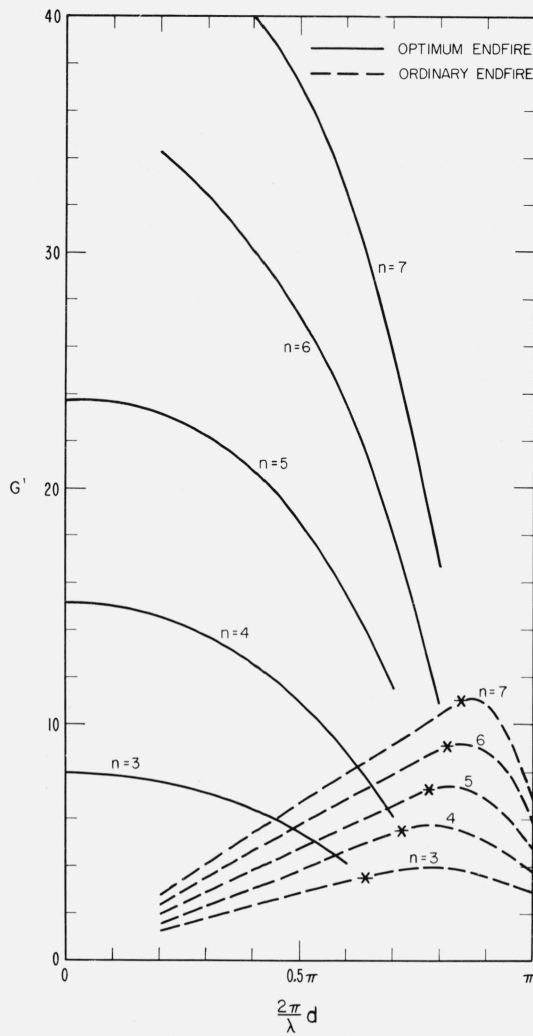and y_h' is determined by

$$Q(y_h') = \frac{1}{2} Q(y_a').$$

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FIGURE 3. *Directivities for optimum endfire arrays with equal sidelobes (− 20 dB).*

Numerical results, for $n = 3$ through $n = 7$ and for a sidelobe level 20 dB down from the main lobe, obtained by using these formulas are given in figures 3, 4, 5, 6, and 7. For comparison, the corresponding directivities for various d, if the ordinary endfire condition $-\alpha = \frac{2\pi}{\lambda} d$ is used throughout, are also included in figure 3 as the dashed curves. The points marked with * on these dashed curves are those when $d = d^*$.

From figure 2b, it is seen that the smaller the d, the narrower will the beamwidth become, and the higher will the directivity be. In fact, the limiting radiation characteristics for $n = 3$, 4, and 5 when $d \to 0$, has also been obtained by taking the appropriate limiting process. Specifically, the equations in (19) have been replaced by (30) through (32), when $d \to 0$,

$$\frac{\alpha}{2} \sim \frac{2 - \sqrt{2 + y^*}}{2 + \sqrt{2 + y^*}} \frac{\pi d}{\lambda}, \tag{30}$$
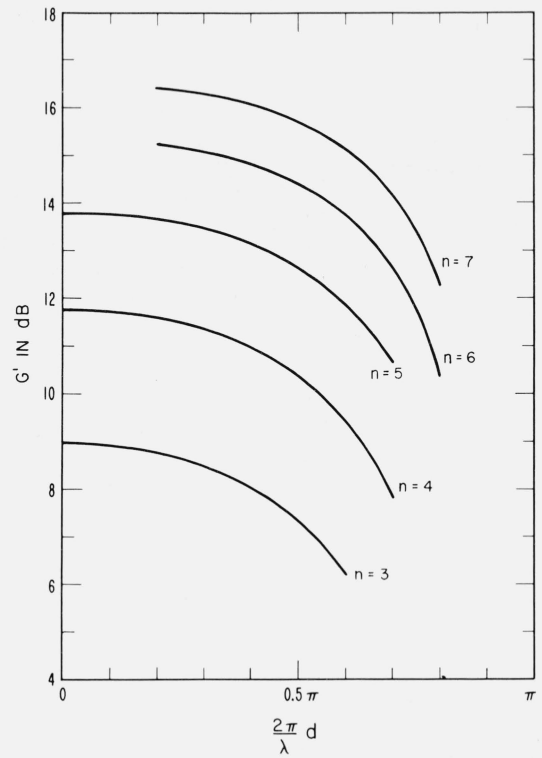

FIGURE 4. *Directivities in dB for optimum endfire arrays with equal sidelobes (− 20 dB).*

$$k_1 \sim -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \right)^2, \tag{31}$$

$$k_2 \sim 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \right)^2, \tag{32}$$

where

$$x = \frac{\alpha}{2} + \frac{\pi d}{\lambda} \sim \frac{4}{2 + \sqrt{2 + y^*}} \frac{\pi d}{\lambda}. \tag{33}$$

The number of terms to be carried for k_1 and k_2 in (31) and (32) is determined by n, the total number of elements in the array.

5. Conclusion

A new way of calculating the directivity of a uniformly spaced endfire array with equal sidelobes, when the element spacing is made arbitrarily small, has been formulated. Numerical results on directivities, beamwidths, current excitations, and phases for a typical endfire array have been presented (figs. 3 through 7). The limiting radiation characteristics for $n = 3$, 4, and 5 when the element spacing approaches zero have also been included. An extended table for $n = 3$ through $n = 30$ with the sidelobe level varying from 10 dB through 50 dB below the main lobe is being prepared as an NBS monograph to be published soon. The investigation has been carried
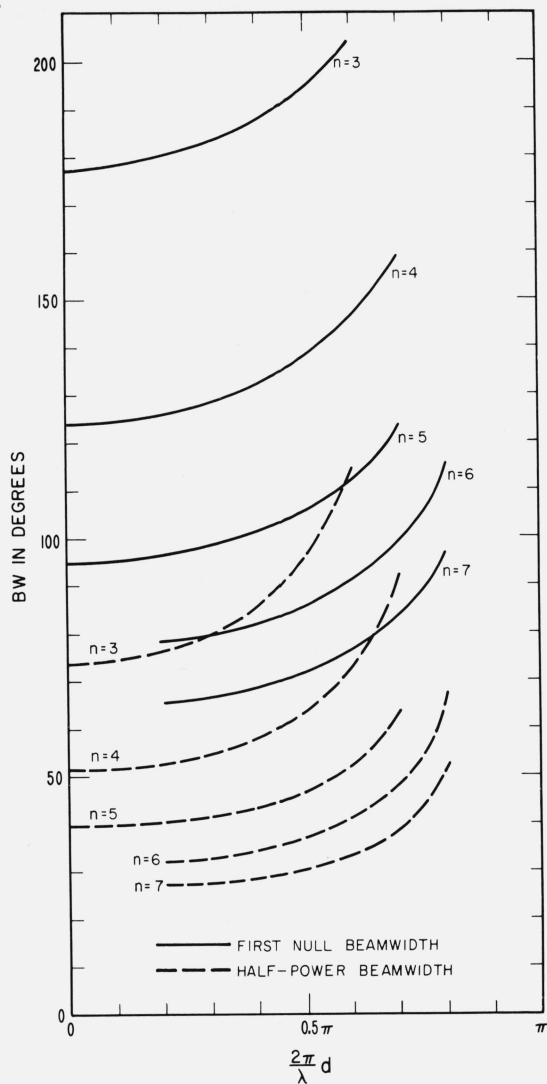
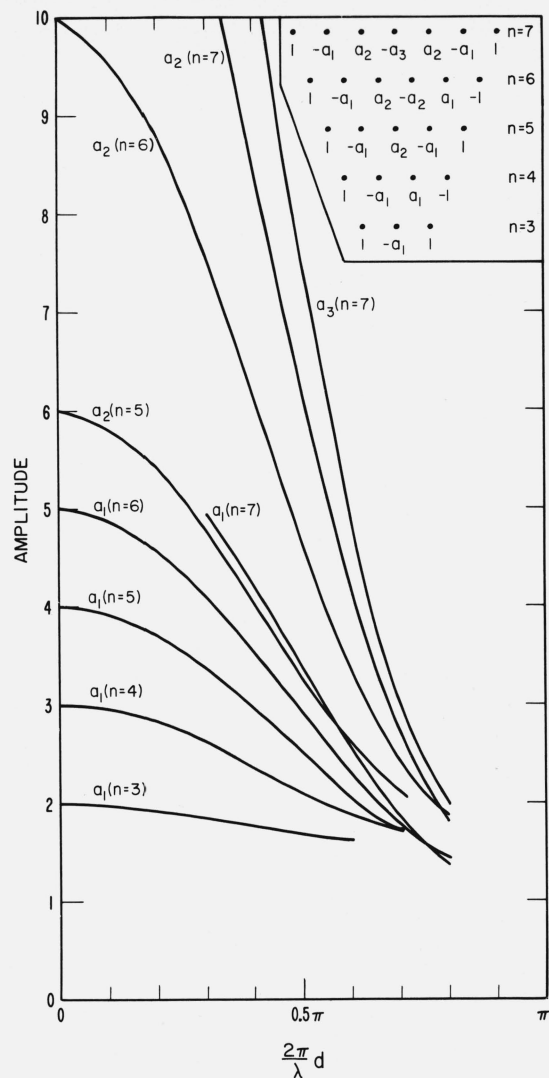FIGURE 5. *Beamwidths in degrees for optimum endfire arrays with equal sidelobes (− 20 dB).*

FIGURE 6. *Excitation coefficients for optimum endfire arrays with equal sidelobes (− 20 dB)*

out only on a theoretical basis. The actual designing of this type of so-called supergain array is perhaps far more difficult in view of increased current magnitudes [Yaru, 1951], mutual couplings between elements, high ohmic losses, low efficiency, low effective radiation resistances [Wilmotte, 1948], critical mechanical tolerances, narrow bandwidth, and increased stored energy [Taylor, 1948].

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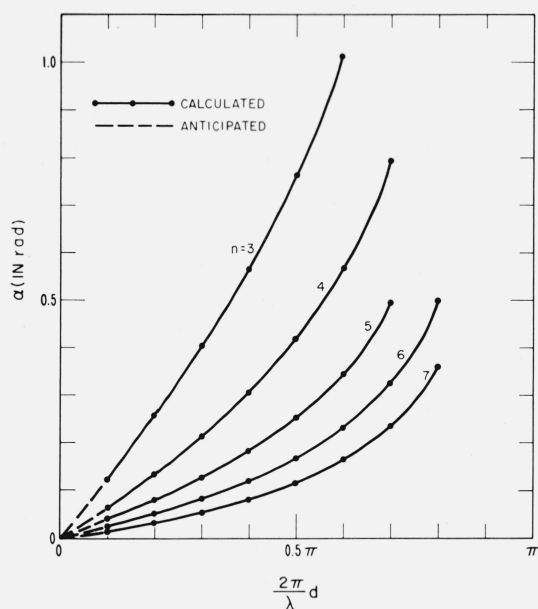
FIGURE 7. *Phases in radians required for optimum endfire arrays with equal sidelobes (−20 dB).*

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