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Small Prolate Spheroidal Antenna in a Dissipative Medium[1]

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A perfectly conducting prolate spheroidal antenna immersed in a conducting medium is analyzed. The dimensions of the spheroid are assumed to be small compared to a skin depth in the medium. A gain factor for the antenna as a receiver is computed as a function of the spheroid's axis ratio. The prolate spheroid is then assumed to be excited by a circular filament of magnetic current located in the plane of the spheroid's minor axes to approximate excitation by a magnetic toroid. Expressions for the antenna's input admittance and effective length are derived. Some experimental results verifying the derivation of the input conductance are included. A computation shows that at VLF, the antenna has a Q of 0.5 or less, and a relatively constant effective length.

1. Introduction

It is well known that the receiving properties of a small electric loop antenna can be enhanced by loading the loop with a permeable core. A theory developed by Wait [1953a, 1953b], valid for small loops in a uniform field, shows that the permeable core tends to concentrate magnetic flux within itself thereby increasing the effective area of the loop wound around the core. An analogous situation occurs with a spheroidal piece of metal in a medium where conduction currents dominate displacement currents. Lines of electric current tend to concentrate in the metal which is assumed to have a much greater conductivity than the dissipative medium surrounding it. W. L. Anderson [1961a, 1961b] seems to have been the first person to consider seriously the use of this phenomenon to improve the reception of antennas buried in the earth. In this paper, the basic ideas of Anderson are modified and extended to analyze the prolate spheroidal antenna immersed in a conducting medium.

The general problem of a small electric type antenna in a dissipative medium was investigated first by C. T. Tai, in which he analyzed the fundamental electric dipole in an insulating radome [Tai, 1947], and the thin biconical antenna in a spherical insulating radome [Tai, 1949]. Wheeler [1958] and Hansen [1963] summarize some properties of the electric type antenna and compare these with the properties of the magnetic loop antenna in an insulating radome. One purpose of the insulating radome in these analyses is to separate idealized circuit theory type sources from the dissipative medium. For, if these sources are in contact with a dissipative medium, then losses become unbounded. Equally valid methods of eliminating these unbounded losses can be made other ways. One example is given by King, Harrison, and Denton [1961], who consider a small uninsulated cylindrical antenna in a dissipative medium. Their method, however, requires the current distribution to vanish at the ends of the cylindrical antenna, which means that their antenna length cannot be too small. Another example is given in this paper in which the source is specified in terms of a practical toroidal winding.

Assume a perfectly conducting prolate spheroidal antenna that is small with respect to the skin depth in the dissipative medium surrounding it. A prolate spheroidal coordinate system is used [Magnus and Oberhettinger, 1954, p. 149, 157] (see fig. 1). Since the spheroid is assumed small, the quasi-static approximation to Maxwell's equations is valid in regions near the spheroid. Thus, Maxwell's equations are taken to be

$$\begin{aligned}\nabla\times\mathbf{H}&=\sigma\mathbf{E},\\ \nabla\times\mathbf{E}&=-\mathbf{M},\end{aligned}\tag{1}$$

where $\mathbf{M}$ is the magnetic current density vector.

In the following sections of this paper, advantage is taken of the azimuthal symmetry of the spheriod shown in figure 1. With this symmetry it is clear that

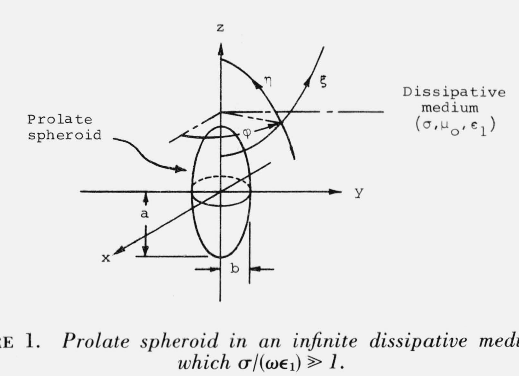

FIGURE 1. *Prolate spheroid in an infinite dissipative medium in which $\sigma/(\omega\epsilon_1)\gg 1$.*

[1] This work was sponsored by Contract Nonr 2798(01) (FBM).

$E_\varphi = H_\eta = H_\xi = 0$ and that H_φ, E_η, and E_ξ are independent of φ. It then follows that

$$\nabla \times \nabla \times \mathbf{H} = -\sigma \mathbf{M} \tag{2}$$

reduces to a scalar partial differential equation when $\mathbf{M}$ is oriented in the φ-direction, and is independent of φ. In regions where $\mathbf{M}=0$, (2) is separable and its solutions can be constructed as a product of linear combinations of associated Legendre functions[2] in both variables η and ξ:

$$H = \sum_{n=1}^{\infty} P_n^1(\eta)[A_n P_n^1(\xi) + B_n Q_n^1(\xi)] \,. \tag{3}$$

The second-kind solution in η, $Q_n^1(\eta)$, is not used because it does not remain bounded when $\eta = \pm 1$.

Using (1), the electric field components are expressed as

$$E_\eta = \frac{-1}{c\sigma\sqrt{\xi^2-\eta^2}} \frac{\partial}{\partial \xi}[\sqrt{\xi^2-1}\, H],$$

$$E_\xi = \frac{1}{c\sigma\sqrt{\xi^2-\eta^2}} \frac{\partial}{\partial \eta}[\sqrt{1-\eta^2}\, H], \tag{4}$$

where c is the semifocal length of the prolate spheroid. The metrical coefficients for prolate spheroidal coordinates are listed here for convenience since they are referred to often in the remainder of this paper:

$$h_\xi = c\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}},$$

$$h_\eta = c\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}},$$

$$h_\varphi = c\sqrt{(\xi^2-1)(1-\eta^2)}. \tag{5}$$

In terms of the assumptions already stated, the receiving properties of the prolate spheroid are derived in section 2, the driving point admittance is derived and compared with some experimental results in section 3, the effective length and equivalent circuit are discussed in section 4, and the conclusions are presented in section 5.

[2] Since there can be no confusion, the φ-subscript in H is deleted. Also, no special notation for P_n^1 and Q_n^1 is used to show that $|\eta| \leq 1$ and $\xi \geq 1$. In this paper the following definitions are used:

$$P_n^1(\eta) = -\sqrt{1-\eta^2}\frac{d}{d\eta}P_n(\eta) \qquad |\eta| \leq 1,$$

$$P_n^1(\xi) = \sqrt{\xi^2-1}\frac{d}{d\xi}P_n(\xi)$$

$$\xi > 1.$$

$$Q_n^1(\xi) = \sqrt{\xi^2-1}\frac{d}{d\xi}Q_n(\xi)$$

For other definitions and properties of associated Legendre functions of the first and second kind, see Magnus and Oberhettinger [1954, ch. 4].

2. Receiving Properties

Let an electromagnetic plane wave be incident on the prolate spheroid in figure 1 such that the electric vector is parallel to the z-axis. Let the spheroid be defined by the surface $\xi = \xi_0$. Since the spheroid is small, the incident electric field can be assumed uniform in the region about the spheroid. Thus,

$$\mathbf{E}^i = \mathbf{a}_z E^i$$

or, in terms of prolate spheroidal coordinates [Magnus and Oberhettinger, 1954, p. 149]:

$$E_\xi^i = (E^i)\eta\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}} = \frac{+E^i}{\sqrt{\xi^2-\eta^2}} P_1(\eta) P_1^1(\xi),$$

and

$$E_\eta^i = (E^i)\xi\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}} = \frac{-E^i}{\sqrt{\xi^2-\eta^2}} P_1(\xi) P_1^1(\eta). \tag{6}$$

If the metallic spheroid were not present, the current enclosed by a circle with radius b located in the x, y-plane at the origin would be

$$I_0 = \pi b^2 \sigma E^i. \tag{7}$$

With the spheroid present the current enclosed by the same circle would be

$$I = g I_0, \tag{8}$$

where g is a current collecting gain factor to be derived in this section. Anderson [1961b] previously derived a similar gain factor using a method different from that used here. He assumed a prolate spheroid of finite conductivity and then used the formal analogy between stationary currents, and static fields in ideal dielectrics [Panofsky and Phillips, 1962, p. 120]. For spheroids of finite conductivity this method, of course, led to electric fields inside the spheroid which are parallel to the z-axis. This result is valid only in the strictly stationary current case, and does not allow for the appreciable skin effect phenomenon in the metal which occurs in this quasi-static problem. Although Anderson's gain factor becomes the same as the one derived here when the conductivity of his spheroid becomes infinite, the method of derivation presented here seems preferable.

The incident field causes induced currents in the prolate spheroid and produces a scatter field which can be constructed as a weighted sum of Legendre functions as shown in (3):

$$H^s = \sum_{n=1}^{\infty} B_n P_n^1(\eta) Q_n^1(\xi)\,, \tag{9}$$

where the coefficients A_n in (3) are set equal to zero because $P_n^1(\xi) \to \infty$ as $\xi \to \infty$. The coefficients B_n are determined by requiring the total tangential electric field at the surface of the prolate spheroid to vanish. That is

$$E^i_\eta(\xi_0, \eta) + E^s_\eta(\xi_0, \eta) = 0. \tag{10}$$

Using (4) and (9), the only nonzero coefficient is B_1;

$$B_1 = -\frac{\sigma c E^i}{2} \frac{P_1(\xi_0)}{Q_1(\xi_0)}. \tag{11}$$

The total current flowing through the circle of radius b with the conducting prolate spheroid in place is

$$I = \left[\int_0^{2\pi} \int_0^1 h_\eta h_\varphi \sigma E_\xi d\eta d\varphi \right]_{\xi=\xi_0}, \tag{12}$$

where E_ξ is the total normal field from (4) and (6):

$$E_\xi = \frac{E^i}{\sqrt{\xi^2-\eta^2}} P_1(\eta) P_1^1(\xi) - \frac{E^i P_1(\xi_0) Q_1^1(\xi)}{2Q_1(\xi_0)\sqrt{\xi^2-\eta^2}} \frac{\partial}{\partial \eta} \left[\sqrt{1-\eta^2}\, P_1^1(\eta) \right]. \tag{13}$$

Performing the integrations, one obtains

$$I = \pi c^2 \sigma E^i \sqrt{\xi_0^2 - 1} \left[P_1^1(\xi_0) - \frac{P_1(\xi_0)}{Q_1(\xi_0)} Q_1^1(\xi_0) \right]. \tag{14}$$

which can be written as

$$I = -\frac{\pi c^2 \sigma E^i (\xi_0^2 - 1)}{Q_1(\xi_0)} W[P_1(\xi_0), Q_1(\xi_0)], \tag{15}$$

where the Wronskian of $P_1(\xi_0)$ and $Q_1(\xi_0)$ is [Jahnke and Emde, 1945, p. 114]

$$W[P_1(\xi_0), Q_1(\xi_0)] = \frac{-1}{\xi_0^2 - 1}. \tag{16}$$

The radius of the prolate spheroid is b as shown in figure 1, and is related to the prolate spheroidal surface ξ_0 by

$$b = c\sqrt{\xi_0^2 - 1}. \tag{17}$$

Using (7), (14) can now be written as

$$I = \frac{I_0}{(\xi_0^2 - 1) Q_1(\xi_0)}. \tag{18}$$

Therefore the gain defined by (8) is

$$g = \frac{1}{(\xi_0^2 - 1) Q_1(\xi_0)}. \tag{19}$$

Figure 2 shows the gain in dB as a function of the axes ratio a/b of the prolate spheroid.

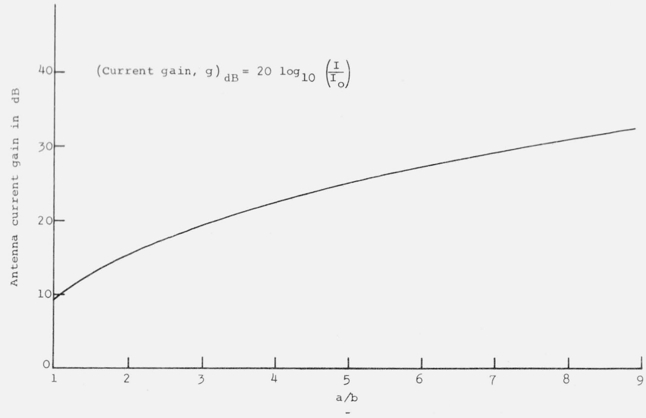

FIGURE 2. *Relative current gain of the prolate spheroidal antenna.*

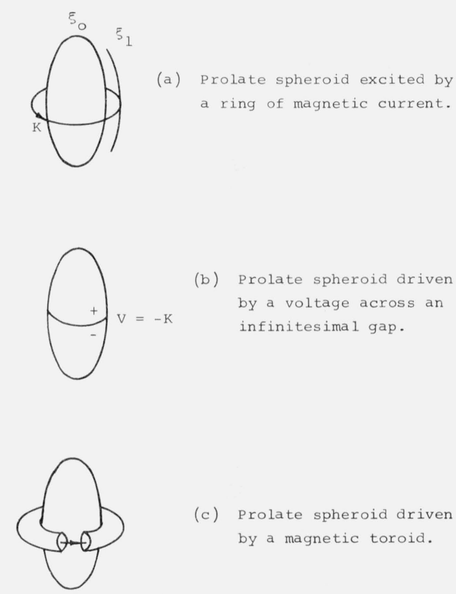

FIGURE 3. *Methods of prolate spheroid excitation.*

3. Input Admittance

To find the input admittance of a driven prolate spheroidal antenna, assume a ring of magnetic current K located at the center of the prolate spheroid to be the source, see figure 3a. If the spheroid is separated into halves and driven by a voltage source, then the location of the magnetic current ring is on the surface of the spheroid, ($\xi_1 = \xi_0$) as shown in figure 3b. In case the spheroid is driven by a torroidal winding then K is an approximation to the magnetic current of the torroid, and is located in the center of the toroid cross section, ($\xi_1 > \xi_0$), as shown in figure 3c. In any case

$$Y_{\text{in}} = -\frac{1}{V^2} \int_c \mathbf{H} \cdot K d\mathbf{l}, \tag{20}$$

where K is related to the voltage V by

$$V=-K,$$

and the integration is over the contour specified by the ring source.

In terms of the magnetic current density vector, the ring source is defined to be

$$\mathbf{M}=\mathbf{a}_\varphi K\frac{\delta(\eta-\eta_1)\delta(\xi-\xi_1)}{h_\eta h_\xi} \tag{21}$$

where $\delta(x)$ represents the Dirac delta-function. The primary quasi-static field from the ring source, that is, the field produced with no metallic spheroid present, can be obtained by substituting (21) into (2), assuming a solution of the form specified by (3), applying the orthogonality relationship to the functions of η, and applying the method of variation in parameters to the resultant differential equation involving ξ. The result is

$$H^p=\sum_{n=1}^{\infty}H_nP_n^1(\eta)\begin{cases}Q_n^1(\xi)P_n^1(\xi_1), & \xi>\xi_1\\ Q_n^1(\xi_1)P_n^1(\xi), & \xi<\xi_1\end{cases}, \tag{22}$$

where

$$H_n=\frac{\sigma K}{2}\frac{(2n+1)}{n^2(n+1)^2}P_n^1(0)\sqrt{\xi_1^2-1}\,. \tag{23}$$

The field scattered by the perfectly conducting prolate spheroid is constructed as follows:

$$H^s=\sum_{n=1}^{\infty}C_nH_nQ_n^1(\xi_1)P_n^1(\eta)Q_n^1(\xi), \qquad \xi>\xi_0, \tag{24}$$

where ξ_0 is the fixed ξ describing the surface of the prolate spheroid. The total magnetic field is the sum of the primary and scatter fields:

$$H_t=H^p+H^s. \tag{25}$$

The total tangential electric field must vanish at the surface of the prolate spheroid. Using this boundary condition, the unknown coefficients in (24) are determined. The results are

$$C_n=-\frac{P_n(\xi_0)}{Q_n(\xi_0)}, \qquad n=1,\,2,\,\ldots\,.$$

Substitution of H_t into (20) yields

$$Y_{\text{in}}=Y_0+\Delta G, \tag{26}$$

where

$$Y_0=-\frac{1}{K}\int_c H^p dl \tag{27}$$

and

$$\Delta G=-\frac{1}{K}\int_c H^s dl. \tag{28}$$

The expression for Y_0 in (27) is that part of the total admittance Y_{in} which is due to a magnetic toroid alone in a conducting medium. The quantity Y_0 has real and imaginary components and has been determined by Swain [1965]. The ΔG expression in (28) is the contribution to the total admittance Y_{in} due to the presence of the conducting prolate spheroid. Carrying out the integration in (28):

$$\Delta G=\pi c\sigma(\xi_1^2-1)\sum_{n=1}^{\infty}\frac{(2n+1)}{n^2(n+1)^2}[P_n^1(0)]^2\frac{P_n(\xi_0)}{Q_n(\xi_0)}[Q_n^1(\xi_1)]^2. \tag{29}$$

Since $r_1=c\sqrt{\xi_1^2-1}$, (29) can be written as

$$\frac{\Delta G}{\sigma r_1}=\pi\sqrt{\xi_1^2-1}\sum_{n=1}^{\infty}\frac{(2n+1)}{n^2(n+1)^2}\frac{P_n(\xi_0)}{Q_n(\xi_0)}[P_n^1(0)]^2[Q_n^1(\xi_1)]^2. \tag{30}$$

Figure 4 shows a plot of (30) as a function of the ratio (a/b) of the prolate spheroid for various (r_1/b) ratios.

The validity of the theoretical expression for ΔG has been corroborated by an experiment. For this experiment a model of the toroidal-prolate spheroid antenna about 2½ in. long was constructed. The model was immersed in a conducting medium of salt water (4.89 mhos/m) and measurements of the impedance were made at the input of the transmission line feeding the model. Two sets of data were taken, the first set was over a selected frequency range for the toroid alone; the second set was for the toroid-prolate spheroid combination for the same frequency range.

The electrical length effect of the transmission line was removed from both sets of data, then both sets of impedance data were converted to admittance form.

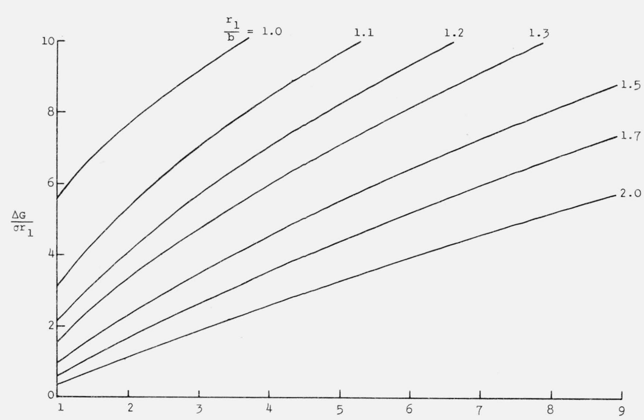

FIGURE 4. *Graph of normalized conductance.*

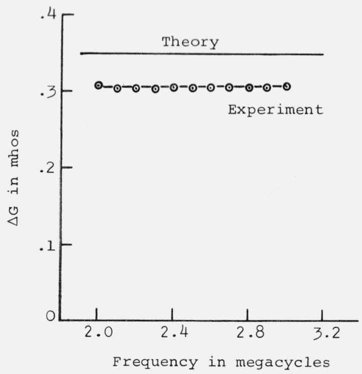

FIGURE 5. *Graph of* ΔG *determined theoretically and experimentally.*
($a/b=2.06$)

A change in conductance resulting from the addition of the prolate spheroid to the toroid antenna was computed and is shown plotted with the theoretical ΔG in figure 5. (The effect on ΔG in fig. 5 caused by a number of turns of wire wound on the toroid was taken into account using the method described in the following section.)

4. Effective Length and Equivalent Circuit

Combining the results of Swain [1965], who discusses a toroid in a dissipative medium, and the results of this paper, an equivalent circuit valid for the small toroid-driven prolate spheroidal antenna is shown in figure 6. The meaning of the various circuit parameters is described below:

R_1 is the self-resistance of the N-turn primary winding on the toroid.
L_1 is the self-inductance of the N-turn primary winding on the toroid.
L_2 is the self-inductance of the one turn secondary of the transformer.[3]
M is the mutual inductance between the primary and secondary.
$1/R_m$ is the conductance due to the toroid by itself (derived by Swain [1965]).
ΔG is the increase in conductance caused by the presence of the spheroid given by (29).
$1/(\omega L_p)$ is a susceptive component due to magnetic energy storage in the dissipative medium. The appendix contains a derivation of an approximate expression for this inductance, see (39). As the transformer in figure 6 approaches an ideal transformer, the effect of this susceptance becomes appreciable.

[3] An investigation of the secondary resistance due to loss in the spheroid itself is not presented here but has been performed. It showed that this resistance is negligible for typical practical material, e.g., spheroids made of copper and a dissipative medium of sea water or earth.

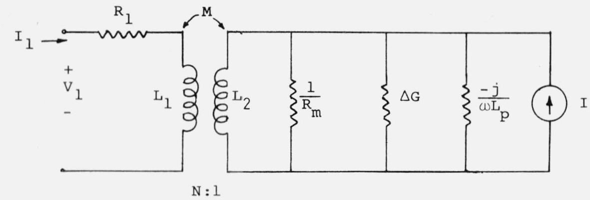

FIGURE 6. *Equivalent circuit for a toroid-driven prolate spheroidal antenna.*

The effective length and the driving point impedance of the toroid-driven prolate spheroidal antenna can be computed using elementary circuit theory once values have been assigned to the parameters in figure 6. The driving-point impedance is V_1/I_1 when $I=0$, and the effective length is V_1/E^i when $I_1=0$.

It is of interest to note that when the transformer in figure 6 becomes ideal, then the driving-point impedance is

$$Z_{\text{in}}=R_1+\frac{N^2}{Y_p}, \tag{31}$$

and the effective length is

$$l_{\text{eff}}=\frac{N\pi b^2\sigma g}{Y_p}, \tag{32}$$

where

$$Y_p=1/R_m+\Delta G-j/(\omega L_p).$$

The quasi-static approximations used in this paper do not contain some effects that could affect significantly the input impedance and effective length of the antenna. These effects can be introduced into the equivalent circuit in figure 6 by the addition of appropriate circuit parameters. Swain [1965] shows how to introduce a capacitor to correspond to parasitic capacities, and this paper shows how to introduce an inductance (L_p) to correspond to magnetic energy storage in the dissipative medium. Several other effects could be introduced depending upon the user's application and desire for refinement.

5. Conclusions

Equations (31) and (32) indicate that if the effects of L_p are small, and if the transformer in figure 6 approaches the ideal, then the prolate spheroidal antenna's driving-point impedance and effective length should be relatively insensitive with respect to frequency changes (provided the original quasi-static approximations are observed). A more detailed computation shows this to be so. For this computation, frequencies were assumed to be in the VLF band, the prolate spheroid was assumed to be 1 m long with an axes ratio (a/b) of 2.5, and the dissipative medium was assumed to be sea water ($\sigma=4$ mhos/m). The spheroid was assumed to be excited by a toroid with 26 turns evenly distributed about it. Figures 7 and 8 show the driving-point resistance and reactance resulting from the computations plotted versus frequency for several different assumed toroid permeabilities.

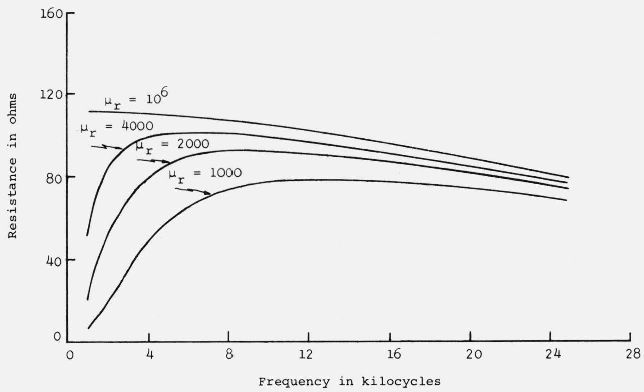

FIGURE 7. *Input resistance for computed example.*

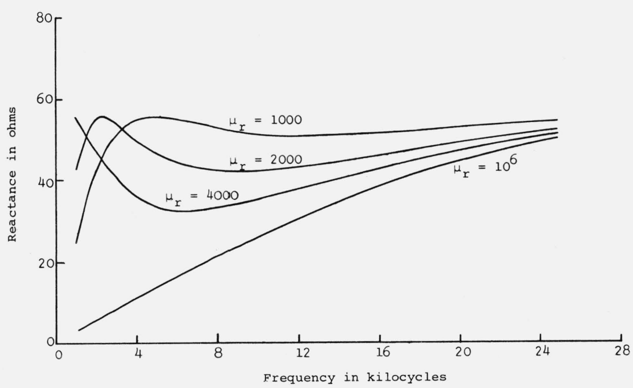

FIGURE 8. *Input reactance for computed example.*

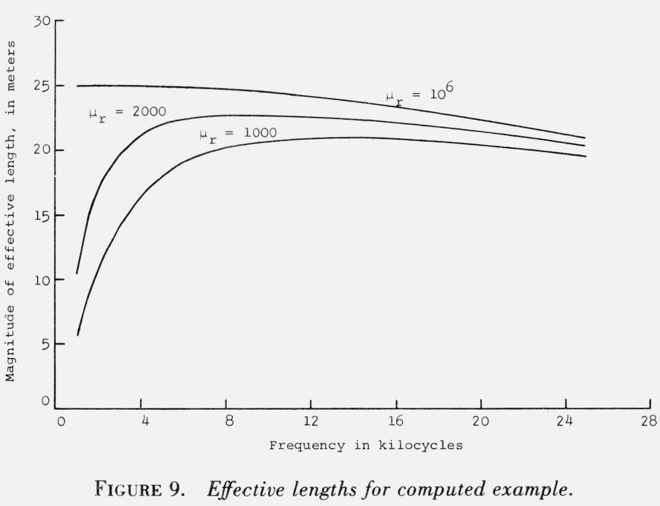

FIGURE 9. *Effective lengths for computed example.*

Figure 9 shows the magnitude of the effective length plotted versus frequency for the assumed toroid permeabilities. It is to be noted that a relative permeability of 10^6 corresponds very nearly to the ideal transformer case. However, practical permeabilities of the order of 2×10^3 or 4×10^3 exhibit desirable characteristics if broadband capabilities are desired at VLF. This example also shows that the Q of the antenna is 0.5 or less over the frequency band.

6. Appendix

The magnetic energy stored in the dissipative medium external to the prolate spheroid antenna is computed using

$$W=\frac{\mu_0}{2}\int_v H^2 dv, \tag{33}$$

where v indicates the region external to the spheroid. At a given frequency ω, this energy can be associated with an inductance:

$$L_p=\frac{V^2}{4\omega^2 W}, \tag{34}$$

where V is the peak voltage across the inductance.

In principle, it is possible to substitute (25) into (33) and (34), and compute an expression for L_p. However, the computation of (33) in prolate spheroidal coordinates presents some numerical difficulties because the orthogonality relations associated with Legendre functions cannot be used except for the degenerate case when the spheroid is a sphere.

It is of some value, however, to derive formulas for W and L_p for the sphere, because the result can be extended, by approximation, to spheroids that do not differ too much from spheres.

Thus, in the limit, as a prolate spheroid approaches a sphere,

$$c\rightarrow 0,$$

$$\eta\rightarrow \cos\theta,$$

$$\xi\rightarrow r,$$

where c is the semifocal length of the prolate spheroid, and where (r, θ, φ) represents a standard spherical coordinate system. The result of this limiting process yields

$$H_t=H^p+H^s, \tag{35}$$

where

$$H^p=-\frac{\sigma K}{2}\sum_{n=1}^{\infty}\frac{P_n^1(0)}{n(n+1)}P_n^1(\cos\theta)\begin{cases}\left(\dfrac{r}{r_1}\right)^n, & r<r_1\\ \left(\dfrac{r_1}{r}\right)^{n+1}, & r>r_1\end{cases}$$

$$H^s = -\frac{\sigma K}{2} \sum_{n=1}^{\infty} A_n \frac{P_n^1(0)}{n(n+1)} P_n^1(\cos\theta) \left(\frac{b}{r}\right)^{n+1}, \qquad r > b$$

$$A_n = \frac{n+1}{n} \left(\frac{b}{r_1}\right)^{2n+1}$$

The total magnetic energy stored external to the sphere is then

$$W = \frac{\mu_0}{2} \int_b^{\infty} dr \int_0^{\pi} r d\theta \int_0^{2\pi} r \sin\theta \, d\varphi H_t^2. \tag{36}$$

If the magnetic ring source is very close to the surface of the spheroid then $r_1 \rightarrow b$, and (36) is evaluated to be

$$W = \frac{\pi \mu_0 \sigma^2 K^2 b^3}{2} \sum_{n=1}^{\infty} \frac{[P_n^1(0)]^2 [1 + A_n]^2}{n(n+1)(2n+1)(2n-1)}. \tag{37}$$

This series is dominated by its first term, and the factor $(1 + A_1)$ is related to the gain g given by (19). The series is therefore approximated to

$$W \doteq \frac{\pi \mu_0 \sigma^2 K^2 b^3 g^2}{12}. \tag{38}$$

It is now intended that (38) applies to prolate spheroidal antennas that do not depart too much from a sphere, and that are excited by a toroidal winding located very close to the surface of the spheroid.

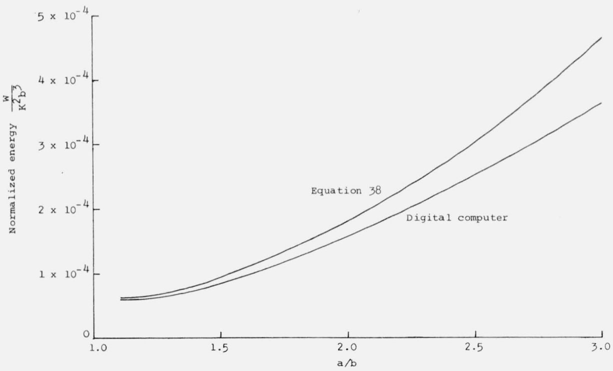

FIGURE 10. *Comparison of magnetic energy computed by two different methods.*

Independent computations involving prolate spheroidal functions performed on a digital computer show that the error of (39) is not too great if $a/b < 3$, see figure 10.

Substituting (38) into (34), and using the fact that $K = -V$, yields

$$L_p \doteq \frac{3}{\pi \omega^2 \mu_0 \sigma^2 g^2 b^3}. \tag{39}$$

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