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An Approach to Improve Re-entry Communications by Suitable Orientations of Antenna and Static Magnetic Field

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In this paper analysis of two radiation problems which have practical application to improve radio communications during re-entry blackout period is presented. In both of these examples given here, the mutual orientation of the antenna and an external static magnetic field is chosen in such a way that the field components are independent of the component of the plasma-dielectric tensor parallel to the static magnetic field. This choice enables one to control more effectively the electromagnetic waves by controlling the applied d-c magnetic field.

Though the problems investigated here involved cylindrical body of infinite length, a knowledge of these results will throw more insight into the expected behavior of the radiation field from a uniform magnetic ring current around a conical space vehicle covered by a plasma sheath in the presence of a uniform azimuthal static magnetic field.

1. Introduction

Radio communication with all types of manned space vehicles is desirable during all phases of the flight mission. When a space vehicle enters the earth's atmosphere at hypersonic speed, a plasma sheath is formed around the vehicle and the antenna installed on it. The electron concentrations in the plasma sheath thus formed may be high enough to degrade communications with the vehicle.

There is also the familiar phenomenon in connection with radio wave propagation through the ionosphere that in highly ionized media (which is characterized by the plasma-electron angular frequency ω_p) a comparatively low frequency signal (i.e., $\omega < \omega_p$, ω being the radian signal frequency) cannot propagate. Use of high frequency ($\omega > \omega_p$), on the otherhand, does not always seem to be very practical, because the available power at such a high frequency is not sufficient. However, it is also a known fact that radio wave propagation in the ionosphere again becomes possible (even when $\omega < \omega_p$) whenever it is properly influenced by the earth's static magnetic field. Therefore, it is very natural for one to borrow this idea of somehow introducing a uniform static magnetic field in the plasma sheath surrounding the space vehicle during its re-entry phase. Actually, the idea of using a static magnetic field is not new, and there is a good deal of discussion on this subject in literature [Papa and Allis, 1961, and Hodara, 1961].

All of these discussions, however, pertain to simple geometry and analysis of plane wave propagation in magnetoplasma media. Therefore, effects of boundary of a space vehicle on the propagation of electromagnetic waves in a plasma were not taken into account. Unfortunately, most of the space vehicles of practical interest have complicated shapes and an exact analysis of electromagnetic wave propagation affected by such boundaries is beyond the scope of the present state-of-the-art. However, it is worthwhile to investigate the possibility of re-entry communication with a space vehicle having simple geometries, such as a semi-infinite conducting cone or an infinite conducting cylinder.

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Using such simple configurations, it was observed qualitatively in a recent communication [Samaddar, 1965] that a satisfactory elimination of the blackout depended considerably also on the mutual orientation of the static magnetic field in the plasma sheath and the antenna. In particular it was proposed that the orientation of the antenna and the static magnetic field should be chosen in such a way that the field components were independent of the component of the dielectric tensor parallel to the static magnetic field. This choice enables one to control more effectively the propagation characteristics by controlling the applied d-c magnetic field alone.

In this paper analytical results of the above mentioned proposed problems involving cylindrical geometry are presented. Since the special functions used in the analysis are not well tabulated, the numerical results are left to another paper in the near future.

The first problem considers the radiation from a magnetic current line source (an idealization of an axially slotted antenna) on an infinitely long metal cylinder, covered by a cold plasma in an axial static uniform magnetic field. In the second problem the study of wave propagation through a plasma-clad metal cylinder in a uniform, angular, static magnetic field is made. The source of excitation is a uniform magnetic ring current situated coaxially on the surface of the metal cylinder so that it will excite only the axially symmetric E-type modes.

It may be mentioned here that in both of the above mentioned problems, the electromagnetic fields do not depend on the component of the dielectric tensor (for the plasma) parallel to the static magnetic field [Samaddar, 1965]. Note that if this mutual orientation of the antenna and the static magnetic field is not preserved, a complete elimination of the blackout by simply increasing the strength of the static magnetic field cannot be attained in principle.

In passing it may be pointed out that though a theoretical study of the radiation of electromagnetic waves from a uniform magnetic ring current around a conical space vehicle covered by a plasma sheath in the presence of a uniform angular static magnetic field is very difficult, the results of the second problem mentioned in the preceding paragraphs will throw more light into the practical aspect of the conical configuration.

2. Problem Involving Axial Static Magnetic Field and Magnetic Current Line Source

2.1. Statement and the Formulation of the Problem

Consider a space vehicle which can be represented, for certain practical purposes, by a conducting infinite cylinder of radius a, which is surrounded uniformly by a plasma sheath of radius b in presence of a uniform axial static (d-c) magnetic field (see fig. 1). A magnetic current line source is situated axially along the surface of the metal cylinder. The region outside the plasma sheath is assumed to be free space. Due to the cylindrical symmetry of the problem, the cylindrical coordinate system ρ, θ, and z will be used. Thus, the coordinates of the source are given by a and θ_0.

The method of procedure to solve this problem is already known [Ohba, 1963; Samaddar, 1962; Wait, 1961]. As a matter of fact, the formal result is already given by Ohba [1963]. Therefore, we shall merely present the result here on the basis of the procedure adopted by Samaddar [1962]. In addition to the formal result, a few special cases will be derived using appropriate limits.

Since the cylindrical structure is independent of the z-coordinate (axial direction) and the source is a uniform magnetic current line source parallel to the z-axis, the field components ($E\rho$, E_θ, and H_z) will be independent of the z-coordinate, i.e., $\frac{\partial}{\partial z} \equiv 0$. In a straightforward manner, it can be shown that the longitudinal field H_z (assuming a suppressed harmonic time dependence $e^{i\omega t}$) has the following representations:[1]

[1] Note that the field components in this radiation problem are independent of ϵ_{zz}, the component of the dielectric tensor parallel to the d-c magnetic field B_0.

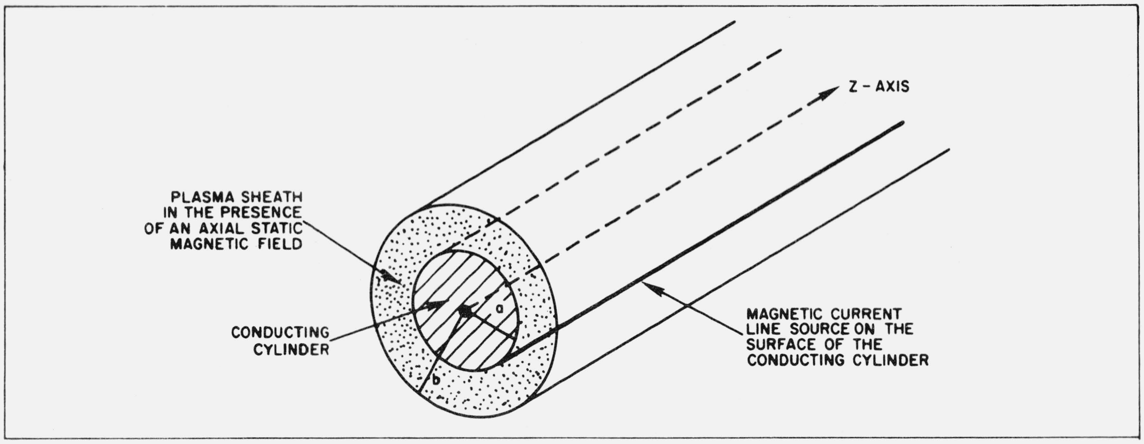

FIGURE 1. *A magnetic current line source on the surface of an infinitely long metal cylinder covered by a magnetoplasma sheath.*

$$H_z = \frac{\mathrm{im}\omega\epsilon_0\epsilon'}{2\pi \mathrm{a}\xi} \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\theta_0)}}{W_n} [\xi b\epsilon_1 H_n^{(2)}(K_0 b) R_n(\xi, b, \rho) - L_n Q_n(\xi; b, \rho)], \text{ for } a \leqslant \rho \leqslant b \tag{1a}$$

$$= \frac{\mathrm{im}\omega\epsilon_0\epsilon_1\epsilon'}{\pi^2 a\xi} \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\theta_0)}}{W_n} H_n^{(2)}(K_0\rho) \text{ for } \rho > b \tag{1b}$$

where,

$$W_n = L_n R_n(\xi; a, b) + \xi b \epsilon_1 S_n(\xi; a, b) H_n^{(2)}(K_0 b), \tag{2}$$

$$\epsilon' = [\epsilon_1^2 - \epsilon_2^2]/\epsilon_1 \tag{3i}$$

$$\xi = K_0 \sqrt{\epsilon'} \tag{3ii}$$

$$K_0 = \omega \sqrt{\mu_0 \epsilon_0} \tag{3iii}$$

$$\epsilon_1 = \epsilon_{\rho\rho} = \epsilon_{\theta\theta} = 1 + \frac{\omega_p^2(1 - i\nu/\omega)}{\omega_c^2 - \omega^2(1 - i\nu/\omega)^2} \tag{3iv}$$

$$\epsilon_{\rho\theta} = \epsilon_{\theta\rho} = \epsilon_2 = \frac{\omega_c}{\omega} \cdot \frac{\omega_p^2}{\omega_c^2 - \omega^2(1 - i\nu/\omega)^2} \tag{3v}$$

$$\omega_p = \text{angular electron-plasma frequency} = \sqrt{\frac{N_0 e^2}{\epsilon_0 m_e}}, \tag{3vi}$$

$$e = \text{magnitude of the charge of the electron} \tag{3vii}$$

$$N_0 = \text{number density of the electrons} \tag{3viii}$$

$$m_e = \text{mass of an electron} \tag{3ix}$$

$$\epsilon_0 = \text{free space permittivity} \tag{3x}$$

$$\omega_c = \frac{eB_0}{m_e} = \text{electron cyclotron frequency (angular)} \tag{3xi}$$

$$B_0 = \text{static (d-c) magnetic induction in the } z\text{-direction} \tag{3xii}$$

$$m = \text{strength of the magnetic current per unit length} \tag{3xiii}$$

$$\nu = \text{collision frequency} \tag{3xiv}$$

$$L_n = \xi b \sqrt{\epsilon'}\ \epsilon_1 H_n'^{(2)}(K_0 b) - n\epsilon_2 H_n^{(2)}(K_0 b) \tag{4i}$$

$$H_n'^{(2)}(K_0 b) = \frac{d}{d(K_0 b)} H_n^{(2)}(K_0 b) \tag{4ii}$$

$$R_n(\xi;\ a,\ b) = J_n(\xi b)N_n'(\xi a) - J_n'(\xi a)N_n(\xi b) \tag{4iii}$$

$$S_n(\xi;\ a,\ b) = J_n'(\xi a)N_n'(\xi b) - J_n'(\xi b)N_n'(\xi a) \tag{4iv}$$

$$Q_n(\xi;\ b,\ \rho) = J_n(\xi \rho)N_n(\xi b) - J_n(\xi b)N_n(\xi \rho) \tag{4v}$$

$$R_n(\xi;\ b,\ \rho) = J_n(\xi \rho)N_n'(\xi b) - J_n'(\xi b)N_n(\xi \rho). \tag{4vi}$$

2.2. Special Cases

In absence of the static magnetic field, the plasma becomes isotropic. In this case, the expression for the field component H_z can be calculated from (1) by employing the following limits;

$$B_0 = 0,\ \epsilon_2 = 0$$

and

$$\epsilon' = \epsilon_1 = \epsilon_p = 1 - \frac{\omega_p^2}{\omega^2(1 - i\nu/\omega)}. \tag{5}$$

Using these limits of (5), the expressions for the field component H_z can be written in the following form:

$$H_z = \frac{\text{im}\ \omega\epsilon_0\epsilon_p^2}{2\pi a\xi_1} \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\theta_0)}}{W_{1n}} [\xi_1 b H_n^{(2)}(K_0 b) R_n(\xi_1;\ b,\ \rho)$$

$$- \xi_1 b \sqrt{\epsilon_p}\, H_n^{(2)}(K_0 b) Q_n(\xi_1;\ b \cdot \rho)] \qquad \text{for } a < \rho < b \tag{6a}$$

and

$$H_z = \frac{\text{im}\ \omega\epsilon_0\epsilon_p^2}{\pi^2 a\xi_1} \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\theta_0)}}{W_{1n}} H_n^{(2)}(K_0 \rho),\ \rho > b, \tag{6b}$$

where

$$\xi_1 = K_0 \sqrt{\epsilon_p}$$

$$W_{1n} = \xi_1 b \epsilon_p [\sqrt{\epsilon_p} H_n'^{(2)}(K_0 b) R_n(\xi_1;\ a,\ b) + S_n(\xi_1;\ a,\ b) H_n^{(2)}(K_0 b)]. \tag{7}$$

When there is no plasma around the conducting cylinder, the expression for the field can be obtained from (6) using the limit $a=b$ and $\epsilon_p=1$. Then one finds

$$H_z=\frac{im\sqrt{\epsilon_0/\mu_0}}{2\pi a}\sum_{n=-\infty}^{\infty}\frac{e^{in(\theta-\theta_0)}}{H_n'^{(2)}(K_0a)}H_n^{(2)}(K_0\rho),\ \rho>a. \tag{8}$$[2]

For a thin plasma, i.e., for $(b-a)\ll a$, $\xi(b-a)\ll 1$ and $K_0(b-a)\ll 1$ [Wait, 1959], the expressions of H_z for $\rho>b$ can be obtained by Taylor's series expansion of the cylindrical functions and their derivatives in the following manner:

$$Z_n(\eta a)\simeq Z_n(\eta b)+\eta(a-b)Z_n'(\eta b) \tag{9a}$$

and

$$Z_n'(\eta a)\simeq Z_n'(\eta b)+\eta(a-b)Z_n''(\eta b) \tag{9b}$$

where Z_n is any of the cylindrical functions and η may represent ξ, ξ_1, or K_0 as the case may be. In doing so, it is a simple matter to show that

$$R_n(\eta;\ a,\ b)\simeq\frac{2}{\pi\eta a} \tag{10a}$$

$$S_n(\eta;\ a,\ b)\simeq\frac{2(b-a)}{\pi b}\left(1-\frac{n^2}{\eta^2b^2}\right) \tag{10b}$$

$$W_n\simeq\frac{2}{\pi}\left[\frac{b}{a}\sqrt{\epsilon'}\,\epsilon_1H_n'^{(2)}(K_0b)+\left\{\epsilon_1\xi(b-a)\left(1-\frac{n^2}{\xi^2b^2}\right)-\frac{n\epsilon_2}{\xi a}\right\}H_n^{(2)}(K_0b)\right] \tag{11a}$$

and

$$W_{1n}\simeq\frac{2\sqrt{\epsilon_p}\epsilon_p}{\pi}\left[\frac{b}{a}H_n'^{(2)}(K_0b)+K_0(b-a)\left(1-\frac{n^2}{\xi_1^2b^2}\right)H_n^{(2)}(K_0b)\right]. \tag{11b}$$

Now substituting (11a) and (11b) into (1b) and (6b) respectively, we obtain the corresponding fields for the thin plasma sheath approximations.

From a knowledge of the numerical results obtained previously [Ohba, 1963; Samaddar, 1963–1964; Seshadri, 1964] for problems of this nature, some general remarks can be made here. Both the amplitude and phase of the far field behave asymmetrically with respect to the angle θ. This asymmetry is caused by the presence of the static magnetic field in the plasma. However, if the excitation is such that the electromagnetic fields do not experience [Samaddar, 1962] the anisotropic nature of the plasma, no asymmetry of the amplitude and phase as mentioned above will occur. For some given angle θ_1, the amplitude may increase and the phase decrease with the increase of the static magnetic field. However, this increase of the amplitude of the radiation field with the static magnetic field may not be true for all angles θ, $|\theta|\leq\pi$. This suggests that the application of an external magnetic field may change both the gain and the directivity of the radiation pattern.

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[2] Note that this result can also be obtained from (1) by taking the limit $\omega_c/\omega\gg 1$ and $\omega_c/\omega_p\gg 1$. In other words, in this limit, re-entry blackout disappears.

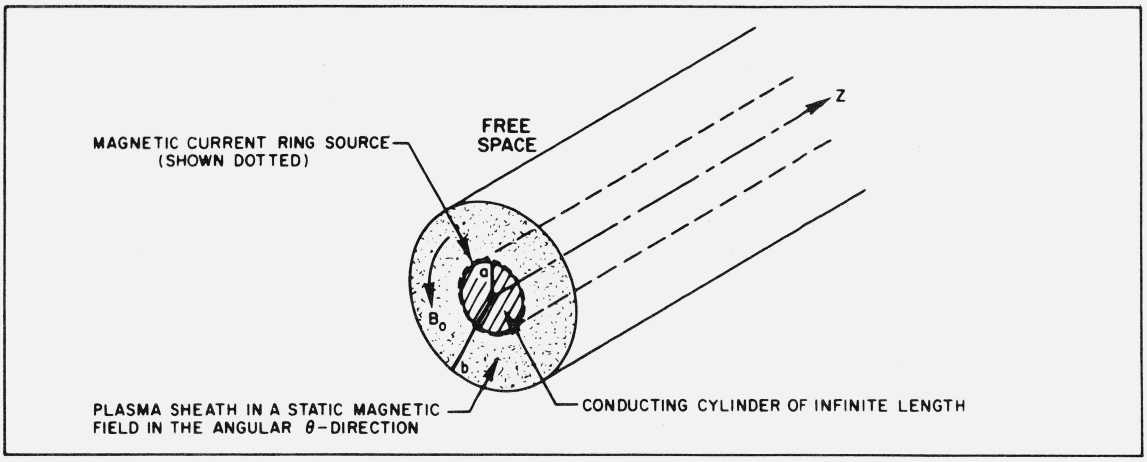

FIGURE 2. *A magnetic current ring source on the surface of an infinitely long metal cylinder covered by a magnetoplasma sheath.*

3. Problem Involving Angular Static Magnetic Field and Magnetic Current Ring Source

3.1. Statement and the Formulation of the Problem

In this problem we study the radiation of electromagnetic waves from a uniform magnetic current ring source (i.e., an idealization of a circumferential slot antenna) on the surface of an infinitely long conducting cylinder. The cylinder is covered by a plasma sheath concentric with the cylinder. An external static (d-c) magnetic field in the angular direction is assumed to exist in the plasma sheath (see fig. 2). The space outside the plasma is unbounded free space.

We further assume that the interaction of the plasma and the electromagnetic fields can be described by using the following linear hydrodynamic theory of a cold plasma together with the Maxwell's equations (with a harmonic time dependence $e^{i\omega t}$):

$$\nabla \times \mathbf{E} = -i\omega\mu_0 \mathbf{H} - \boldsymbol{\theta}_0 M \tag{12a}$$

$$\nabla \times \mathbf{H} = i\omega\epsilon_0 \mathbf{E} - eN_0 \mathbf{V} \tag{12b}$$

$$m_e(\nu + i\omega)\mathbf{V} = -e\mathbf{E} + eB_0\boldsymbol{\theta}_0 \times \mathbf{V}, \text{ where } \theta_0 \text{ is the unit vector in the angular direction.} \tag{12c}$$

From (12b) and (12c), it is evident that the motions of ions are neglected in comparison with those of the electrons. $\mathbf{V}$ is the a-c velocity of the electrons, and all other symbols have their usual meaning. The magnetic current ring source M can be represented by Dirac delta functions in the following manner:

$$M = \frac{m\delta(\rho - a)\delta(z)}{2\pi\rho}, \tag{13}$$

where m is the strength of the source and a is the radius of the cylinder.

Due to the choice of the source of excitation, the electromagnetic fields will be independent of the angular coordinate θ (i.e., $\frac{\partial}{\partial\theta} \equiv 0$). Now in a straightforward manner, it can be shown that only the components E_ρ, E_z, and H_θ will be excited and they obey the following partial differential equations:

$$\frac{\partial E_\rho}{\partial z}-\frac{\partial E_z}{\partial \rho}=-i\omega\mu_0 H_\theta-\frac{m\delta(\rho-a)\delta(z)}{2\pi\rho} \tag{14a}$$

$$\frac{\partial H_\theta}{\partial z}=-i\omega\epsilon_0(\epsilon_1 E_\rho+i\epsilon_3 E_z) \tag{14b}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\theta)=i\omega\epsilon_0(\epsilon_1 E_z-i\epsilon_3 E_\rho). \tag{14c}$$

where ϵ_1 and ϵ_3 are the components of the following dielectric tensor:

$$\underset{\sim}{\epsilon}=\begin{Vmatrix} \epsilon_1 & 0 & i\epsilon_3 \\ 0 & \epsilon_2 & 0 \\ -i\epsilon_3 & 0 & \epsilon_1 \end{Vmatrix} \tag{15}$$

$\epsilon_1=\epsilon_{\rho\rho}=\epsilon_{zz}$ has the same representation as given by (3iv), and $\epsilon_3=\epsilon_{\rho z}=\epsilon_{z\rho}$ has the same expression given by (3v), whereas $\epsilon_2=\epsilon_{\theta\theta}$ can be shown to equal to

$$1-\frac{\omega_p^2}{\omega^2(1-i\nu/\omega)}.$$

It may be noted here that the field equations (14) representing this radiation problem do not depend on the component, ϵ_2, of the dielectric tensor. Note that ϵ_2 is the component of $\underset{\sim}{\epsilon}$ parallel to B_0.

In order to solve the partial differential equations in (14), it is convenient to take their Fourier transform with respect to z, i.e., if $P(\rho, z)$ is any component of the field, then its Fourier transform is defined by,

$$P(\rho, z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{P}(\rho, \zeta)e^{-i\zeta z}d\zeta \tag{16a}$$

and

$$\hat{P}(\rho, \zeta)=\int_{-\infty}^{\infty}P(\rho, z)e^{i\zeta z}dz. \tag{16b}$$

Now employing this operation of Fourier transform in each of the equations in (14), it can be shown after some rearrangement that the transformed field components obey the following ordinary differential equations:

$$\frac{d^2}{d\rho^2}\hat{H}_\theta+\frac{1}{\rho}\frac{d}{d\rho}\hat{H}_\theta+\left[K_0^2\frac{\epsilon_1^2-\epsilon_3^2}{\epsilon_1}-\zeta^2-\frac{1}{\rho^2}+\frac{\zeta\epsilon_3}{\epsilon_1\rho}\right]\hat{H}_\theta=i\omega\epsilon_0 m\frac{(\epsilon_1^2-\epsilon_3^2)}{2\pi\rho\epsilon_1}\delta(\rho-a) \tag{17a}$$

$$\hat{E}_z=\frac{1}{i\omega\epsilon_0(\epsilon_1^2-\epsilon_3^2)}\left[\epsilon_1\frac{1}{\rho}\frac{d}{d\rho}(\rho\hat{H}_\theta)-\zeta\epsilon_3\hat{H}_\theta\right]. \tag{17b}$$

It may be noted here that in absence of any static magnetic field, $\epsilon_3=0$, then the homogeneous differential equation corresponding to (17a) could have been satisfied by Bessel and Hankel functions. However, in this present problem, (17a) can be satisfied by two linearly independent confluent hypergeometric functions and the required solution for H_θ can be written in the following form:

$$\left.\begin{aligned}\hat{H}_\theta(\rho, \zeta) = A\left(\frac{\eta\rho}{2}\right) e^{-i\eta\rho}\Phi\left(\frac{3\eta+i\xi}{2\eta}, 3, 2i\eta\rho\right)\\ -B\frac{2i}{\sqrt{\pi}}(2\eta\rho)e^{-i\eta\rho}\Psi\left(\frac{3\eta+i\xi}{2\eta}, 3, 2i\eta\rho\right)\end{aligned}\right\} \quad \text{for } a \leqslant \rho \leqslant b \tag{18a}$$

$$= CH_1^{(2)}(\eta_0\rho), \text{ for } \rho \geqslant b, \tag{18b}$$

where A, B, and C are arbitrary constants and

$$\begin{aligned}\eta^2 &= K_0^2(\epsilon_1^2-\epsilon_3^2)/\epsilon_1-\zeta^2\\ \eta_0^2 &= K_0^2-\zeta^2\\ \xi &= \zeta\epsilon_3/\epsilon_1.\end{aligned} \tag{19}$$

this ξ should not be confused with that used in section 2.

The confluent hypergeometric functions Φ and Ψ used here are in accord with the definitions given in Bateman Manuscript Project, [1953] and Slater [1960]. The relation between the singular solution Ψ and the regular solution Φ can be expressed in the following way:

$$\begin{aligned}\Psi(\alpha, 3, 2i\eta\rho) = -\frac{1}{2\Gamma(\alpha-2)}\Bigg[\Phi(\alpha, 3, 2i\eta\rho)\log(2i\eta\rho)+\sum_{r=0}^{\infty}\frac{(\alpha)_r}{(3)_r}\cdot\\ \cdot\left\{\psi(\alpha+r)-\psi(1+r)-\psi(3+r)\right\}\frac{(2i\eta\rho)^r}{r!}\Bigg]+\frac{1}{\Gamma(\alpha)}\left[\left(\frac{1}{2i\eta\rho}\right)^2+\frac{2-\alpha}{2i\eta\rho}\right].\end{aligned} \tag{20}$$

where,

$$\alpha = \frac{3}{2}+\frac{i\xi}{2\eta}. \tag{21a}$$

$$\psi(y) = \frac{d}{dy}\Gamma(y)/\Gamma(y). \tag{21b}$$

$$(\alpha)_r = \Gamma(\alpha+r)/\Gamma(\alpha) = \alpha(\alpha+1)(\alpha+2)\ .\ .\ .\ (\alpha+r-1) \tag{21c}$$

$$(3)_r = \Gamma(3+r)/\Gamma(3) = 3.4.5\ .\ .\ .\ (3+r-1). \tag{21d}$$

The main reason for writing the equation (18a) in a little cumbersome way is to show that in the limit of $\xi=0$, (i.e., in absence of the d-c magnetic field), the functions Φ and Ψ reduce to Bessel and Hankel functions, respectively, as given below:

$$J_1(\eta, \rho) = \frac{(\eta\rho)}{2} e^{-i\eta\rho}\Phi\left(\frac{3}{2}, 3, 2i\eta\rho\right) \tag{22a}$$

and

$$H_1^{(2)}(\eta\rho) = -\frac{2i}{\sqrt{\pi}}(2\eta\rho)e^{-i\eta\rho}\Psi\left(\frac{3}{2}, 3, 2i\eta\rho\right). \tag{22b}$$

Though it looks elegant to use the known confluent hypergeometric functions Φ and Ψ as the solutions of (17a), for numerical computations this representation may not be convenient, since the functions Φ and Ψ with complex arguments are not tabulated. Therefore, it may be worthwhile to express Φ in terms of the Bessel function $J_1(\eta\rho)$ and a series as given by (23). Once Φ is calculated from (23), the singular solution Ψ can be computed by using (20) and (23):

$$\Phi\left(\frac{3}{2}+\frac{i\xi}{2\eta}, 3, 2i\eta\rho\right)=\frac{2}{\eta\rho}e^{i\eta\rho}\left[J_1(\eta\rho)+\sum_{n=0}^{\infty}d_n(\eta\rho)^{n+2}\right], \tag{23}$$

where

$$d_0=-\xi/(6\eta), \tag{24a}$$

$$d_1=\xi^2/(48\eta^2), \tag{24b}$$

$$d_2=\xi(11\eta^2-\xi^2)/(48.15\eta^3) \tag{24c}$$

and for $n\geqslant 2$

$$d_{2n-1}\eta\{(2n+1)^2-1\}+d_{2n-2}\xi+d_{2n-3}\eta=0 \tag{24d}$$

$$d_{2n}\eta\{4(n+1)^2-1\}+d_{2n-1}\xi+d_{2n-2}\eta=\frac{(-1)^{n+1}\xi}{2^{2n+1}n!(n+1)!}. \tag{24e}$$

Note that $d_n=0$ for $\xi=0$. Before calculating the unknown coefficients A, B, and C by using the appropriate boundary conditions, we present the following relations which will be found very useful in the sequel:

$$\text{The Wronskian, } \Phi\frac{d}{dy}\Psi-\Psi\frac{d}{dy}\Phi=-2(2i\eta y)^{-3}e^{2i\eta y}/\Gamma(\alpha) \tag{25a}$$

$$\Psi'(y)=\frac{1}{2i\eta}\frac{d}{dy}\Psi(\alpha, 3, 2i\eta y)=-\alpha\Psi(\alpha+1, 4, 2i\eta y), \tag{25b}$$

$$\Phi'(y)=\frac{1}{2i\eta}\frac{d}{dy}\Phi(\alpha, 3, 2i\eta y)=\frac{\alpha}{3}\Phi(\alpha+1, 4, 2i\eta y). \tag{25c}$$

The transformed field $\hat{H}_\theta(\rho, \zeta)$ given by (18) satisfies the following boundary conditions:

$$\hat{H}_\theta(b^-, \zeta)=\hat{H}_\theta(b^+, \zeta), \tag{26a}$$

$$\frac{1}{(\epsilon_1^2-\epsilon_3^2)}\left[\epsilon_1\frac{1}{\rho}\frac{d}{d\rho}(\rho\hat{H}_\theta)-\zeta\epsilon_3\hat{H}_\theta\right]_{\rho=b^-}=\frac{1}{\rho}\frac{d}{d\rho}(\rho\hat{H}_\theta)\bigg|_{\rho=b^+} \tag{26b}$$

$$\frac{d}{d\rho}(\rho\hat{H}_\theta)\bigg|_{\rho=a}=\frac{\mathrm{i}m\omega\epsilon_0(\epsilon_1^2-\epsilon_3^2)}{2\pi\epsilon_1}. \tag{26c}$$

The relation (26c) specifies the source condition. Now applying the boundary conditions (26) on the solution (18) together with the relations (25), the transformed field component $\hat{H}_\theta(\rho, \zeta)$

can be shown to be equivalent to the following expressions:

$$\hat{H}_\theta(\rho,\ \zeta)=\frac{\text{im}\ \omega\epsilon_0(\epsilon_1^2-\epsilon_3^2)}{2\pi\eta a\epsilon_1[Q_1-Q_2]}\left[\Phi(\rho)\left\{\Gamma(\alpha)Q\Psi(b)-\frac{e^{i2\eta b}}{2(\eta b)^2}\right\}-\Gamma(\alpha)Q\Phi(b)\Psi(\rho)\right] \tag{27a}$$

for $a\leqslant\rho\leqslant b$

and

$$\hat{H}_\theta(\rho,\ \zeta)=-\frac{\text{im}\ \omega\epsilon_0(\epsilon_1^2-\epsilon_3^2)\Phi(b)e^{i\eta(a+b)}}{4\pi(\eta a)(\eta b)\epsilon_1 H_1^{(2)}(\eta_0 b)[Q_1-Q_2]}\cdot H_1^{(2)}(\eta_0\rho),\ \rho\geqslant b \tag{27b}$$

where,

$$\Phi(\rho)=\Phi(\alpha,\ 3,\ 2i\eta\rho),\ \Psi(\rho)=\Psi(\alpha,\ 3,\ 2i\eta\rho),$$

and

$$Q_1=[(2-i\eta a)\Phi(a)+2i\eta a\Phi'(a)]\left[\Gamma(\alpha)\Psi(b)Q-\frac{e^{i2\eta b}}{2(\eta b)^2}\right]. \tag{28a}$$

$$Q_2=[(2-i\eta a)\Psi(a)+2i\eta a\Psi'(a)]\Phi(b)\Gamma(\alpha)Q. \tag{28b}$$

$$Q=\Phi(b)\left[\eta_0 b\left(\frac{\epsilon_1^2-\epsilon_3^2}{\epsilon_1}\right)\frac{H_0^{(2)}(\eta_0 b)}{H_1^{(2)}(\eta_0 b)}-(2-\xi b-i\eta b)\right]-2i\eta b\Phi'(b), \tag{28c}$$

$\Psi'(a)$ and $\Phi'(b)$ are defined by (25b) and (25c) with $y=a$ or b as the case may be.

Since we shall be interested in the far field, we shall now take the inverse transform of $\hat{H}_\theta(\rho,\ \zeta)$ only for $\rho>b$. Therefore, taking the inverse transform of (27b), we have

$$H_\theta(\rho,\ z)=\frac{-\text{im}\ \omega\epsilon_0(\epsilon_1^2-\epsilon_3^2)}{8\pi^2\epsilon_1 ab}\int_{-\infty}^{\infty}\Phi(b)\frac{e^{i\eta(a+b)}H_1^{(2)}(\eta_0\rho)e^{-i\zeta z}d\zeta}{\eta^2 H_1^{(2)}(\eta_0 b)[Q_1-Q_2]}. \tag{29}$$

3.2. Evaluation of the Integral for the Radiated Field

Now defining a new function $F(\zeta)$ by

$$F(\zeta)=\frac{-\text{im}\ \omega\epsilon_0(\epsilon_1^2-\epsilon_3^2)\Phi(b)e^{i\eta(a+b)}}{8\pi^2\epsilon_1 ab\eta^2 H_1^{(2)}(\eta_0 b)[Q_1-Q_2]}, \tag{30}$$

the integral in (29) can be written in the following way:

$$H_\theta(\rho,z)=\int_\gamma F(\zeta)H_1^{(2)}(\eta_0\rho)e^{-i\zeta z}d\zeta, \tag{31}$$

where the path of integration γ runs along the real axis of the complex ζ plane with identations at the branch points $\zeta=\pm K_0$ and at the real poles (if any) in such a way that the far field satisfies the radiation condition at $r=(\rho^2+z^2)^{1/2}\rightarrow\infty$. In other words, the integral converges (fig. 3).

The poles of $F(\zeta)$, which corresponds to surface waves are given by

$$Q_1=Q_2. \tag{32}$$

Since these surface waves, unless they are encountered with any discontinuity along the direction of propagation, have no appreciable influence on the radiated field, they will not be considered here. A detailed study of the surface waves for this problem is left to a subsequent paper.

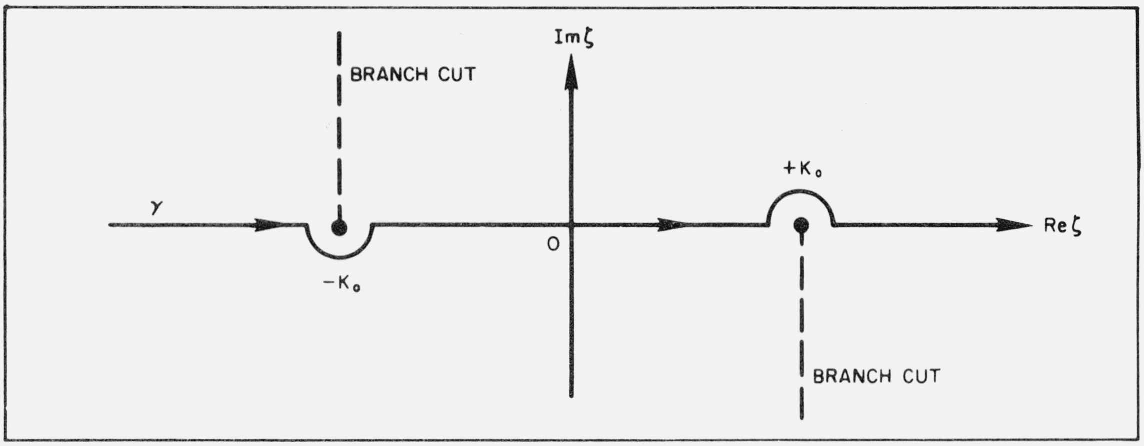

FIGURE 3. *Path of integration, γ, and branch cuts in the complex ζ plane.*

Though η which appears in the integrand, is a multiple-valued function of ζ in the neighborhood of $\zeta=\pm K_0\left(\frac{\epsilon_1^2-\epsilon_3^2}{\epsilon_1}\right)^{1/2}$ it can be shown that the integrand is an even function[3] of η. Therefore, $\eta=0$, i.e., $\zeta=\pm K_0\left(\frac{\epsilon_1^2-\epsilon_3^2}{\epsilon_1}\right)^{1/2}$ are not branch points of the integrand. However, at $\eta_0=0$, the Hankel functions have logarithmic singularity. Therefore, $\zeta=\pm K_0$(i.e., $\eta_0=0$) are branch points of the integrand. The figure 3 shows the choice of the branch cuts.

Once the singularities of the integrand are identified, the evaluation of the integral (31) by the steepest descent method for $\eta_0\rho \gg 1$ is a straightforward procedure (assuming that there is no pole near the saddle point, nor on the steepest descent path). Therefore, without going into detail the first order asymptotic value can be given by

$$H_\theta(\rho, z) \sim -2F(\zeta=K_0 \cos \beta)\frac{e^{-iK_0 r}}{r}, \text{ for } \eta_0\rho \gg 1. \tag{33}$$

where

$$\begin{aligned} z &= r \cos \beta \\ \rho &= r \sin \beta \\ \zeta &= K_0 \cos \beta \\ \eta_0 &= K_0 \sin \beta \end{aligned}$$

and r, β, θ define a spherical coordinate system. The function $F(\zeta)$ of ζ is replaced by $F(K_0 \cos \beta)$.

3.3. A Few Special Cases

In absence of any static magnetic field in the angular direction, $B_0=0$, $\epsilon_3=0$, $\xi=0$, $\epsilon_1=\epsilon_p=1-\omega_p^2/[\omega^2(1-i\nu/\omega)]$, and using these limits together with the identities (22), the following rela-

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[3] To show this, it is useful to know that $e^{-i\eta\rho}\,\Phi\left(\frac{3\eta+i\xi}{2\eta}, 3, 2i\eta\rho\right)$ is also an even function of η, which follows from (23). Furthermore, the behavior of $\eta\rho\, e^{-i\eta\rho}\,\Psi\left(\frac{3\eta+i\xi}{2\eta}, 3, 2i\eta\rho\right)$ as η approaches zero is similar to that of $H_1^{(2)}(\eta\rho)$.

tions can be established:

$$\lim_{B_0\to 0} Q_1 = \frac{i\pi e^{i\eta(a+2b)} J_1(\eta b) J_0(\eta a)}{2(\eta b)^2 H_1^{(2)}(\eta_0 b)} \left[\eta_0 b \epsilon_p H_0^{(2)}(\eta_0 b) H_1^{(2)}(\eta b) - \eta b H_0^{(2)}(\eta b) H_1^{(2)}(\eta_0 b)\right] \tag{34a}$$

$$\lim_{B_0\to 0} Q_2 = \frac{i\pi e^{i\eta(a+2b)} J_1(\eta b) H_0^{(2)}(\eta a)}{2(\eta b)^2 H_1^{(2)}(\eta_0 b)} \left[\eta_0 b \epsilon_p J_1(\eta b) H_0^{(2)}(\eta_0 b) - \eta b J_0(\eta b) H_1^{(2)}(\eta_0 b)\right] \tag{34b}$$

$$\lim_{B_0\to 0} [Q_1 - Q_2] = -\frac{\pi e^{i\eta(a+2b)} J_1(\eta b)}{2(\eta b)^2 H_1^{(2)}(\eta_0 b)} [L_1 - L_2]\cdot \tag{34c}$$

$$L_1 = \eta b H_1^{(2)}(\eta_0 b) \left[J_0(\eta a) N_0(\eta b) - J_0(\eta b) N_0(\eta a)\right]. \tag{34d}$$

$$L_2 = \eta_0 b \epsilon_p H_0^{(2)}(\eta_0 b) \left[J_0(\eta a) N_1(\eta b) - J_1(\eta b) N_0(\eta a)\right]. \tag{34e}$$

with

$$\eta = \sqrt{K_0^2 \epsilon_p - \zeta^2}.$$

Now, using these limiting results, we have

$$\lim_{B_0\to 0} F(\zeta) = F_1(\zeta) = \frac{i m \omega \epsilon_0 \epsilon_p}{2\pi^3 \eta a (L_1 - L_2)}\cdot \tag{35}$$

Therefore, the far field expression in this case becomes

$$H_\theta(\rho, z, B_0 = 0) = H_{1\theta}(\rho, z) \sim -2 F_1(\zeta = K_0 \cos\beta) \frac{e^{-iK_0 r}}{r}, \text{ for } \eta_0 \rho \gg 1. \tag{36}$$

In the limit of a thin plasma sheath, i.e., $\eta(b-a) \ll 1$, with $B_0 = 0$, $F_1(\zeta)$ becomes,

$$F_1(\zeta) = \frac{\mathrm{i} m \omega \epsilon_0 \epsilon_p}{4\pi^2 \eta a \left[\eta(b-a) H_1^{(2)}(\eta_0 b) + \frac{\eta_0}{\eta} \epsilon_p H_0^{(2)}(\eta_0 b)\right]}\cdot \tag{37}$$

For the situation where there is no plasma sheath surrounding the conducting cylinder, one finds:

$$a = b,\ \epsilon_p = 1,\ L_1 = 0,\ \eta = \eta_0$$

$$L_2 = -\frac{2}{\pi} H_0^{(2)}(\eta_0 a)$$

$$\therefore \lim_{\substack{B_0\to 0\\ \epsilon_p \to 1}} F(\zeta) = \lim_{\epsilon_p\to 1} F_1(\zeta) = F_2(\zeta) = \frac{\mathrm{i} m \omega \epsilon_0}{4\pi^2 \eta_0 a H_0^{(2)}(\eta_0 a)}\cdot \tag{38}$$

Therefore, the far field is given by

$$H_\theta(\rho, z, B_0 = 0, \epsilon_p = 1) = H_{2\theta}(\rho, z) \sim -2 F_2(\zeta = K_0 \cos\beta) \frac{e^{-iK_0 r}}{r}\cdot \tag{39}$$

In the limit of a very thin plasma sheath, $F_1(\zeta)$ can be expressed in terms of $F_2(\zeta)$ in the following way:

$$F_1(\zeta) \approx F_2(\zeta) \left[1 + \frac{\Delta \eta^2 (b-a) H_1^{(2)}(\eta_0 b)}{\eta_0 \epsilon_p H_0^{(2)}(\eta_0 b)}\right]^{-1}, \text{ for } \eta(b-a) \ll 1, \tag{40}$$

where $\Delta = 1 - \eta_0^2 \epsilon_p / \eta^2$.

It may be noted here that if one uses the limits $\omega_c/\omega \gg 1$ $\omega_c/\omega_p \gg 1$ carefully in (27) and (33), one can show that (33) reduces to (39). This implies that (even without performing numerical computation) in the limit of high d-c magnetic flux, the plasma behaves as the free space and thus the blackout hazard disappears.

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