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Propagation of Vertically Polarized Electromagnetic Waves in a Horizontally Stratified Magnetoplasma

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When studying the propagation of VLF and ELF radio waves, Galejs and Row [1964] and Galejs [1964] took the ionosphere to be a planar stratified magnetoplasma medium. A wave equation was derived which describes the propagation of vertically polarized waves along the magnetic equator. It was shown that the wave equation could be solved exactly in closed form for a particular height variation of the elements of the permittivity tensor in the ionosphere. The purpose of the present paper is to give a method of treating the wave equation which enables other exact closed-form solutions to be obtained.

1. Introduction

In a planar continuously stratified isotropic medium, the general electromagnetic field consists of a superposition of two independent parts. These correspond to horizontally and vertically polarized waves, in which the electric and magnetic vectors respectively are parallel to the stratifications. The problem of finding the field distribution reduces to solving two second order ordinary differential equations. A great deal of attention has been given in the literature to obtaining exact closed-form solutions to these equations for particular profiles of the refractive index [e.g., Bremmer, 1958; Brekhovskikh, 1960; Budden, 1961; Ginzberg, 1961; Wait, 1962; Gould and Burman, 1964].

When a static magnetic field is imposed on an ionized medium, the medium becomes anisotropic. In an anisotropic stratified medium the electromagnetic field cannot, in general, be separated into two independent parts. In this case, the problem of finding the field distribution can be reduced to solving a pair of coupled second order ordinary differential equations [e.g., Budden, 1961]. These equations can also be written in the form of four first-order coupled differential equations or as a single fourth-order differential equation [Budden, 1961].

When studying the propagation of ELF radio waves, Galejs and Row [1964] considered propagation in a planar continuously stratified anisotropic ionosphere. The two special cases were dealt with in which the waves propagate along and perpendicularly across the magnetic equator. In the case of propagation along the magnetic equator, the earth's magnetic field is transverse to the direction of propagation, as well as being parallel to the stratifications in the ionosphere. Galejs and Row [1964] showed that, in this case, the general field is described by two uncoupled differential equations. Horizontally and vertically polarized waves then propagate independently. The present note will be concerned with this case of propagation along the magnetic equator.

Following Galejs and Row [1964], the x-axis is taken to be along the magnetic equator, running from east to west. The y-direction is vertically upwards and the z-direction is south to north. With the static magnetic field in the z-direction the tensor relative permittivity in the ionosphere can be written

$$[\epsilon]=\begin{bmatrix}\epsilon_1 & -\epsilon_2 & 0\\ \epsilon_2 & \epsilon_1 & 0\\ 0 & 0 & \epsilon_3\end{bmatrix} \tag{1}$$

where expressions for ϵ_1, ϵ_2, and ϵ_3 are given, for example, by Ratcliffe [1959] and Wait [1962]. For a horizontally stratified ionosphere ϵ_1, ϵ_2, and ϵ_3 depend on the y coordinate only. For waves propagating along the magnetic equator, the fields do not vary with z.

In the case of vertically polarized waves, the wave's magnetic field has a z component, H_z, only. This can be written

$$H_z=Y(y)\cdot\exp(ik_xx-i\omega t) \tag{2}$$

where k_x is a propagation constant, ω is the angular frequency, and t is the time. It is found that $Y(y)$ satisfies [Galejs and Row, 1964; Galejs, 1964]

$$\frac{d^2Y}{dy^2}+P(y)\frac{dY}{dy}+Q(y)Y=0. \tag{3}$$

In this equation,

$$P(y)=\frac{d}{dy}\left[\log\left(\frac{\epsilon_1}{\epsilon_1^2+\epsilon_2^2}\right)\right] \tag{4}$$

and

$$Q(y)=\left(\frac{\epsilon_1^2+\epsilon_2^2}{\epsilon_1}\right)k_0^2-k_x^2-ik_x\frac{\epsilon_2}{\epsilon_1}\frac{d}{dy}\left[\log\left(\frac{\epsilon_2}{\epsilon_1^2+\epsilon_2^2}\right)\right] \tag{5}$$

where $k_0=\omega/c$, c being the speed of light in free space.

Writing

$$Y(y)=\left(\frac{\epsilon_1^2+\epsilon_2^2}{\epsilon_1}\right)^{1/2} f(y) \tag{6}$$

it is found [Galejs and Row, 1964; Galejs, 1964] that $f(y)$ satisfies

$$\frac{d^2f}{dy^2}+K(y)f=0 \tag{7}$$

where

$$K(y)=Q-\frac{1}{2}\frac{dP}{dy}-\frac{1}{4}P^2. \tag{8}$$

Galejs and Row [1964] showed that (3) could be solved exactly in closed form when ϵ_1 and ϵ_2 have the following exponential variations:

$$\epsilon_1=ae^{cy} \tag{9a}$$

and

$$\epsilon_2=be^{cy} \tag{9b}$$

where a, b, and c are independent of y. In this case ϵ_1 and ϵ_2 are proportional to the same exponential function of height. The purpose of the present note is to give a method of obtaining further exact solutions to (3).

2. Method

The procedure is to choose a form for $K(y)$ so that (7) can be solved exactly in closed form. Some suitable forms for $K(y)$ can be found in the work of Richards [1959] and Kamke [1948]. These forms will contain arbitrary constants.

The present method reverses the conventional procedure which would be to choose the profiles and then find the field distribution. Choosing $K(y)$ in this manner is equivalent to specifying the functions giving the field distribution. The problem is then to find profiles of ϵ_1 and ϵ_2 which give rise to the specified solutions. Such profiles of ϵ_1 and ϵ_2 are related by the equation

$$Q-\frac{1}{2}\frac{dP}{dy}-\frac{1}{4}P^2=K(y) \tag{10}$$

where $K(y)$ has the chosen form. Equation (10) is to be regarded as a differential equation which is to be solved in order to find appropriate profiles of ϵ_1 and ϵ_2. This equation can be integrated as follows.

First the functions $S(y)$ and $T(y)$ are defined by

$$S(y)=\frac{\epsilon_2}{\epsilon_1} \tag{11}$$

and

$$T(y)=\frac{\epsilon_1}{\epsilon_1^2+\epsilon_2^2}\cdot \tag{12}$$

In terms of these functions ϵ_1 and ϵ_2 are given by

$$\epsilon_1=\frac{1}{T(1+S^2)} \tag{13}$$

and

$$\epsilon_2=\frac{S}{T(1+S^2)} \tag{14}$$

where it is assumed that $T\neq 0$.

When S and T are substituted into the expressions for P and Q, (10) becomes

$$\frac{k_0^2}{T}-k_x^2-ik_x S\frac{d}{dy}[\log(ST)] -\frac{1}{2}\frac{d^2}{dy^2}(\log T)-\frac{1}{4}\left[\frac{d}{dy}(\log T)\right]^2=K(y). \tag{15}$$

This can be rearranged to give

$$\frac{-ik_x}{T}\frac{d}{dy}(ST)=K(y)+k_x^2-\frac{k_0^2}{T} +\frac{1}{2}\frac{d^2}{dy^2}(\log T)+\frac{1}{4}\left[\frac{d}{dy}(\log T)\right]^2\cdot \tag{16}$$

Hence, provided $k_x\neq 0$,

$$S(y)=\frac{i}{k_xT}\int T\left\{K(y)+k_x^2-\frac{k_0^2}{T}+\frac{1}{2}\frac{d^2}{dy^2}(\log T) +\frac{1}{4}\left[\frac{d}{dy}(\log T)\right]^2\right\}dy. \tag{17}$$

Thus the original problem of solving a rather complicated second order differential equation has been reduced to one of performing a single integration.

The function $T(y)$ can be taken to be any function of y. With $K(y)$ and $T(y)$ chosen, (17) gives $S(y)$ by a single integration. Then ϵ_1 and ϵ_2 can be obtained from (13) and (14); these expressions will involve the arbitrary constants contained in the forms chosen for $K(y)$ and $T(y)$. For some choices of $K(y)$ and $T(y)$ it will be possible to express the integral for $S(y)$ in closed form.

Any form for $K(y)$ can be chosen for which (7) can be solved in terms of known functions. By choosing different forms for $K(y)$ and $T(y)$ the method given above can be used to obtain exact solutions for the field distribution for many different profiles of ϵ_1 and ϵ_2. In general, the profiles of ϵ_1 and ϵ_2 obtained will depend on the propagation constant k_x. For waves incident from free space onto the anistropic stratified medium, $k_x=k_0\sin\theta$, where θ is the angle of incidence. Then ϵ_1 and ϵ_2 obtained above will depend, in general, on the angle of incidence. However, in the examples given in the next section it will be found that the dependance on k_x of the arbitrary parameters involved can be chosen so that ϵ_1 and ϵ_2 are independent of k_x.

3. Some Examples

In this section some particular examples of the method outlined above will be treated. The function $T(y)$ can be chosen to be any function of y. Here the choices $T=a$, $T=ae^{\alpha y}$, and $T=ay^n$ will be made where a, α, and n are independent of both y and k_x. In each of these cases the form of $K(y)$ as a function of y and k_x will be chosen so that $S(y)$ and hence ϵ_1 and ϵ_2 will be independent of k_x.

3.1. Case $T=a$

The case in which $T=a$, where a is a constant independent of y and k_x, will now be considered. Equation (17) now becomes

$$S(y)=\frac{i}{k_x}\int\left\{K(y)+k_x^2-\frac{k_0^2}{a}\right\}dy. \tag{18}$$

Any form for $K(y)$ can be chosen for which (7) can be solved in terms of known functions. The additional restriction that $K(y)$ can be written

$$K(y)=b_0+b_1g(y) \tag{19}$$

will now be imposed. In this equation b_0 and b_1 are independent of y, and $g(y)$ is taken to be independent of k_x. Also, b_0 and b_1 are taken to depend on k_x according to the equations

$$b_0=k_0^2a^{-1}-k_x^2-ik_xB \tag{20}$$

and

$$b_1=-ik_xC \tag{21}$$

where B and C are independent of k_x. Then (18) gives

$$S(y)=\int\{B+Cg(y)\}dy \tag{22}$$

which is independent of k_x. Then ϵ_1 and ϵ_2 will be independent of k_x.

First example: In this example

$$g(y)=0 \tag{23}$$

which gives

$$K(y)=b_0. \tag{24}$$

It is seen from (7) that

$$f(y)=\exp(\pm ib_0^{1/2}y). \tag{25}$$

Thus the fields are expressed in terms of simple exponential functions.

Equation (22) now gives

$$S(y)=A+By \tag{26}$$

where A is a constant of integration and can be taken to be independent of k_x. Hence

$$\epsilon_1=\frac{1}{a[1+(A+By)^2]} \tag{27a}$$

and

$$\epsilon_2=\frac{A+By}{a[1+(A+By)^2]} \tag{27b}$$

which are independent of k_x. For these profiles, the field distribution is given by (25) where b_0 is related to a and B by (20).

If $B=0$, then

$$\epsilon_1=\frac{1}{a(1+A^2)}$$

and

$$\epsilon_2=\frac{A}{a(1+A^2)}$$

corresponding to a homogeneous ionosphere. If further $A=0$, then $\epsilon_1=a^{-1}$ and $\epsilon_2=0$, corresponding to a homogeneous, isotropic ionosphere.

Second example: In this example

$$g(y)=y \tag{28}$$

which gives

$$K(y)=b_0+b_1y. \tag{29}$$

When $b_1=0$, this example reduces to the previous one. On writing

$$\zeta=-b_1^{-2/3}(b_0+b_1y) \tag{30}$$

(7) becomes

$$\frac{d^2f}{d\zeta^2}-\zeta f=0 \tag{31}$$

which is the Airy or Stokes equation. Thus

$$f(y)=Ai(\zeta),\qquad Bi(\zeta) \tag{32}$$

where $Ai(\zeta)$ and $Bi(\zeta)$ are Airy functions.

Equation (22) now gives

$$S(y)=A+By+\tfrac{1}{2}Cy^2. \tag{33}$$

Hence

$$\epsilon_1=\frac{1}{a[1+(A+By+\frac{1}{2}Cy^2)^2]} \tag{34a}$$

and

$$\epsilon_2=\frac{A+By+\frac{1}{2}Cy^2}{a[1+(A+By+\frac{1}{2}Cy^2)^2]}. \tag{34b}$$

Third example: In this example

$$g(y)=y+\frac{b_2}{b_1}y^2 \tag{35}$$

or

$$K(y)=b_0+b_1y+b_2y^2 \tag{36}$$

where

$$b_2=-ik_xD, \tag{37}$$

D being independent of k_x. Thus

$$Cg(y)=Cy+Dy^2. \tag{38}$$

When $b_2=0$, this example reduces to the previous one. After writing

$$y=\frac{u}{(-4b_2)^{1/4}}-\frac{b_1}{2b_2} \tag{39}$$

equation (7) becomes

$$\frac{d^2f}{du^2}+\left(\nu+\frac{1}{2}-\frac{u^2}{4}\right)f=0 \tag{40}$$

where

$$\nu+\frac{1}{2}=\frac{1}{(-4b_2)^{1/2}}\left(b_0-\frac{b_1^2}{4b_2}\right). \tag{41}$$

Equation (40) is Weber's equation. Thus [Magnus and Oberhettinger, 1949]

$$f(y)=D_\nu(u),\ D_{-\nu-1}(iu) \tag{42}$$

where the D's are Weber parabolic cylinder functions. Equation (22) now gives

$$S(y)=A+By+\frac{1}{2}Cy^2+\frac{1}{3}Dy^3. \tag{43}$$

Hence

$$\epsilon_1=\frac{1}{a\left[1+\left(A+By+\frac{1}{2}Cy^2+\frac{1}{3}Dy^3\right)^2\right]}. \tag{44a}$$

and

$$\epsilon_2=\frac{A+By+\frac{1}{2}Cy^2+\frac{1}{3}Dy^3}{a\left[1+\left(A+By+\frac{1}{2}Cy^2+\frac{1}{3}Dy^3\right)^2\right]}. \tag{44b}$$

Fourth example: In this example

$$g(y)=e^{\alpha y} \tag{45}$$

or

$$K(y)=b_0+b_1e^{\alpha y} \tag{46}$$

where α is independent of y and k_x. On writing

$$u=\frac{2b_1^{1/2}}{\alpha}\exp\left(\frac{\alpha y}{2}\right) \tag{47}$$

(7) becomes

$$\frac{d^2f}{du^2}+\frac{1}{u}\frac{df}{du}+\left(1-\frac{\nu^2}{u^2}\right)f=0 \tag{48}$$

where

$$\nu^2=-(4b_0)/\alpha^2. \tag{49}$$

Equation (48) is Bessel's equation. Thus

$$f(y)=Z_\nu(u) \tag{50}$$

where Z_ν represents any Bessel function of order ν. In this case

$$S(y)=A+By+(C/\alpha)e^{\alpha y}. \tag{51}$$

Hence

$$\epsilon_1=\frac{1}{a[1+\{A+By+(C/\alpha)e^{\alpha y}\}^2]} \tag{52a}$$

and

$$\epsilon_2=\frac{A+By+(C/\alpha)e^{\alpha y}}{a[1+\{A+By+(C/\alpha)e^{\alpha y}\}^2]}. \tag{52b}$$

Fifth example: In this example

$$g(y)=\cos(\alpha y) \tag{53}$$

or

$$K(y)=b_0+b_1\cos(\alpha y). \tag{54}$$

In this case (7) can be solved in terms of Mathieu functions. Equation (22) gives

$$S(y)=A+By+(C/\alpha)\sin(\alpha y). \tag{55}$$

Hence

$$\epsilon_1=\frac{1}{a[1+\{A+By+(C/\alpha)\sin(\alpha y)\}^2]} \tag{56a}$$

and

$$\epsilon_2=\frac{A+By+(C/\alpha)\sin(\alpha y)}{a[1+\{A+By+(C/\alpha)\sin(\alpha y)\}^2]}. \tag{56b}$$

3.2. Case $T(y)=ae^{\alpha y}$

The case in which

$$T(y)=ae^{\alpha y} \tag{57}$$

where a and α are independent of both y and k_x will now be considered. In this case (17) becomes

$$S(y)=\frac{ie^{-\alpha y}}{k_x}\int e^{\alpha y}\left\{K(y)-\frac{k_0^2}{a}e^{-\alpha y}+k_x^2+\frac{\alpha^2}{4}\right\}dy. \tag{58}$$

$K(y)$ can be chosen to be any function of y for which (7) can be solved in terms of known functions. Here the form of $K(y)$ as a function of y and k_x will be restricted so as to obtain a function $S(y)$ which is independent of k_x. Thus, $K(y)$ will be chosen to have the form

$$K(y)=b_0+b_1e^{-\alpha y}+b_2g(y) \tag{59}$$

where $g(y)$ is independent of k_x. Also b_0, b_1, and b_2 are taken to depend on k_x in such a manner as to satisfy the equations

$$b_0=-k_x^2-\frac{\alpha^2}{4}-ik_xB, \tag{60}$$

$$b_1=\frac{k_0^2}{a}-ik_xC, \tag{61}$$

and

$$b_2=-ik_xD \tag{62}$$

where B, C, and D are independent of k_x. Then (58) becomes

$$S(y)=e^{-\alpha y}\int\{Be^{\alpha y}+C+De^{\alpha y}g(y)\}\,dy \tag{63}$$

which is independent of k_x.

First example: In this example

$$g(y)=0 \tag{64}$$

which gives

$$K(y)=b_0+b_1e^{-\alpha y}. \tag{65}$$

On writing

$$u=\frac{2b_1^{1/2}}{\alpha}\exp\left(-\frac{\alpha y}{2}\right) \tag{66}$$

(7) becomes

$$\frac{d^2f}{du^2}+\frac{1}{u}\frac{df}{du}+\left(1-\frac{\nu^2}{u^2}\right)f=0 \tag{67}$$

where

$$\nu^2=-(4b_0)/\alpha^2. \tag{68}$$

Hence

$$f(y)=Z_\nu(u) \tag{69}$$

where Z_ν represents any Bessel function of order ν.

In this case (63) gives

$$S(y)=(B/\alpha)+(A+Cy)e^{-\alpha y} \tag{70}$$

where A is a constant of integration, which can be taken to be independent of k_x. Thus

$$\epsilon_1=\frac{1}{ae^{\alpha y}[1+\{(B/\alpha)+(A+Cy)e^{-\alpha y}\}^2]} \tag{71a}$$

and

$$\epsilon_2=\frac{(B/\alpha)+(A+Cy)e^{-\alpha y}}{ae^{\alpha y}[1+\{(B/\alpha)+(A+Cy)e^{-\alpha y}\}^2]}. \tag{71b}$$

When $A=0$ and $C=0$, the profiles (71) become

$$\epsilon_1=\frac{1}{ae^{\alpha y}[1+(B/\alpha)^2]}$$

and

$$\epsilon_2=\frac{(B/\alpha)}{ae^{\alpha y}[1+(B/\alpha)^2]}.$$

These have the same form as the profiles (9) considered by Galejs and Row [1964].

Second example: In this example

$$g(y)=e^{-2\alpha y} \tag{72}$$

or

$$K(y)=b_0+b_1e^{-\alpha y}+b_2e^{-2\alpha y}. \tag{73}$$

After writing

$$u=\frac{2ib_2^{1/2}}{\alpha}e^{-\alpha y} \tag{74}$$

and

$$f(y)=\exp\left(\frac{\alpha y}{2}\right)\cdot W(u) \tag{75}$$

equation (7) gives

$$\frac{d^2W}{du^2}+\left[\frac{-1}{4}+\frac{b_1}{2ib_2^{1/2}\alpha u}+\left(\frac{b_0}{\alpha^2}+\frac{1}{4}\right)\frac{1}{u^2}\right]W=0. \tag{76}$$

This may be compared with Whittaker's confluent hypergeometric equation [e.g., Richards, 1959]

$$\frac{d^2W}{dx^2}+\left[\frac{-a^2}{4}+\frac{ak}{x}+\frac{1-4m^2}{4x^2}\right]W=0 \tag{77}$$

which has linearly independent solutions $W_{k,m}(ax)$ and $W_{-k,m}(-ax)$. Hence

$$f(y)=\exp\left(\frac{\alpha y}{2}\right)\cdot W_{\pm k,m}(\pm u) \tag{78}$$

where

$$k=\frac{b_1}{2ib_2^{1/2}\alpha} \tag{79}$$

and

$$m^2=-\frac{b_0}{\alpha^2}. \tag{80}$$

Either sign can be taken for m since $W_{k,+m}(x)=W_{k,-m}(x)$ [Magnus and Oberhettinger, 1949].

In the present problem, (63) gives

$$S(y)=(B/\alpha)+(A+Cy)e^{-\alpha y}-(D/\alpha)e^{-2\alpha y} \tag{81}$$

where A is a constant of integration. Thus

$$\epsilon_1=\frac{1}{ae^{\alpha y}[1+\{(B/\alpha)+(A+Cy)e^{-\alpha y}-(D/\alpha)e^{-2\alpha y}\}^2]} \tag{82a}$$

and

$$\epsilon_2=\frac{(B/\alpha)+(A+Cy)e^{-\alpha y}-(D/\alpha)e^{-2\alpha y}}{ae^{\alpha y}[1+\{(B/\alpha)+(A+Cy)e^{-\alpha y}-(D/\alpha)e^{-2\alpha y}\}^2]}. \tag{82b}$$

When $b_2=0$ and hence $D=0$, these profiles reduce to those of the previous example.

3.3. Case $T(y)=ay^n$

When $T(y)=ay^n$, (17) becomes

$$S(y)=\frac{i}{k_xy^n}\int y^n\left\{K(y)-\frac{k_0^2}{ay^n}+k_x^2+\frac{n(n-2)}{4y^2}\right\}dy. \tag{83}$$

The quantities a and n are taken to be independent of both y and k_x. When $n=0$ the function $T(y)$ considered here reduces to $T(y)=a$, which case was dealt with in section 3.1.

$K(y)$ is now chosen to be of the form

$$K(y)=b_0+\frac{b_1}{y^n}+\frac{b_2}{y^2}+b_3g(y) \tag{84}$$

where $g(y)$ is independent of k_x. Also b_0, b_1, b_2, and b_3 are taken to depend on k_x through the equations

$$b_0=-k_x^2-ik_xB, \tag{85}$$

$$b_1=k_0^2a^{-1}-ik_xC, \tag{86}$$

$$b_2=-\frac{1}{4}n(n-2)-ik_xD, \tag{87}$$

and

$$b_3=-ik_xE \tag{88}$$

where B, C, D, and E are independent of k_x. Hence

$$S(y)=y^{-n}\int\{By^n+C+Dy^{n-2}+Ey^ng(y)\}\,dy \tag{89}$$

which is independent of k_x.

First example: In this example $g(y)=0$ and $n=2$. Thus

$$K(y)=b_0+\frac{b_1+b_2}{y^2}. \tag{90}$$

On writing

$$u=b_0^{1/2}y \tag{91}$$

and

$$f(y)=y^{1/2}h(u) \tag{92}$$

equation (7) gives

$$\frac{d^2h}{du^2}+\frac{1}{u}\frac{dh}{du}+\left(1-\frac{\nu^2}{u^2}\right)h=0 \tag{93}$$

where

$$\nu^2=\frac{1}{4}-(b_1+b_2). \tag{94}$$

Equation (93) is Bessel's equation. Hence

$$f(y)=y^{1/2}Z_\nu(b_0^{1/2}y) \tag{95}$$

where ν is given by (94).

Equation (89) gives

$$S(y)=\frac{1}{3}By+(C+D)y^{-1}+Ay^{-2} \tag{96}$$

where A is a constant of integration. Hence

$$\epsilon_1=\frac{1}{ay^2\left[1+\left\{\frac{1}{3}By+(C+D)y^{-1}+Ay^{-2}\right\}^2\right]} \tag{97a}$$

and

$$\epsilon_2=\frac{\frac{1}{3}By+(C+D)y^{-1}+Ay^{-2}}{ay^2\left[1+\left\{\frac{1}{3}By+(C+D)y^{-1}+Ay^{-2}\right\}^2\right]}. \tag{97b}$$

Second example: In this example $g(y)=0$ and $n=1$. Thus

$$K(y)=b_0+\frac{b_1}{y}+\frac{b_2}{y^2}. \tag{98}$$

Equation (7) can now be compared with Whittaker's equation (77). Thus

$$f(y)=W_{\pm k,\,m}(\pm 2ib_0^{1/2}y) \tag{99}$$

where

$$k=\frac{b_1}{2ib_0^{1/2}} \tag{100}$$

and

$$m^2=\frac{1}{4}-b_2. \tag{101}$$

Equation (89) gives

$$S(y)=\frac{1}{2}By+C+(A+D\log y)y^{-1} \tag{102}$$

where A is a constant of integration. Hence

$$\epsilon_1=\frac{1}{ay\left[1+\left\{\frac{1}{2}By+C+(A+D\log y)y^{-1}\right\}^2\right]} \tag{103a}$$

and

$$\epsilon_2=\frac{\frac{1}{2}By+C+(A+D\log y)y^{-1}}{ay\left[1+\left\{\frac{1}{2}By+C+(A+D\log y)y^{-1}\right\}^2\right]}\cdot \tag{103b}$$

Third example: In this example $g(y)=0$ and $n=-2$. Thus

$$K(y)=b_0+b_1y^2+\frac{b_2}{y^2}\cdot \tag{104}$$

After writing

$$u=ib_1^{1/2}y^2 \tag{105}$$

and

$$f(y)=y^{-1/2}h(u) \tag{106}$$

equation (7) gives

$$\frac{d^2h}{du^2}+\left[\frac{-1}{4}+\frac{b_0}{4ib_1^{1/2}u}+\frac{3+4b_2}{16u^2}\right]h=0. \tag{107}$$

This may be compared with Whittaker's equation (77). Thus

$$f(y)=y^{-1/2}W_{\pm k,\,m}(\pm ib_1^{1/2}y^2) \tag{108}$$

where

$$k=b_0(4ib_1^{1/2})^{-1} \tag{109}$$

and

$$m^2=\frac{1}{4}\left(\frac{1}{4}-b_2\right)\cdot \tag{110}$$

Equation (89) gives

$$S(y)=Cy^3+Ay^2-By-\frac{1}{3}Dy^{-1}. \tag{111}$$

Hence

$$\epsilon_1=\frac{1}{ay^{-2}\left[1+\left(Cy^3+Ay^2-By-\frac{1}{3}Dy^{-1}\right)^2\right]} \tag{112a}$$

and

$$\epsilon_2=\frac{Cy^3+Ay^2-By-\frac{1}{3}Dy^{-1}}{ay^{-2}\left[1+\left(Cy^3+Ay^2-By-\frac{1}{3}Dy^{-1}\right)^2\right]}\cdot \tag{112b}$$

4. Conclusion

When studying the propagation of ELF and VLF radio waves Galejs and Row [1964] and Galejs [1964] took the ionosphere to be a continuously stratified magnetoplasma with the imposed magnetic field parallel to the stratifications. A wave equation was derived which describes the propagation of vertically polarized waves for the case of propagation transverse to the imposed magnetic field. This case corresponds to waves traveling along the magnetic equator. The present paper is concerned with obtaining solutions to the wave equation.

By the method given in section 2 of this paper, the problem of solving the wave equation has been reduced to one of performing a single integration. In section 3 a number of examples have been given showing how the method can be used to obtain exact closed-form wave functions for various profiles of the elements ϵ_1 and ϵ_2 of the permittivity tensor in the magnetoplasma.

Most of the wave functions given in these examples were obtained by using the table given by Richards [1959, pp. 349–350]. This table gives solutions to differential equations of the form $y''+I(x)y=0$. Further wave functions can easily be found by using the same general procedure.

Finally it will be noted that results for the case of an isotropic stratified medium can be obtained from the examples given by choosing the arbitrary parameters concerned so that the function $S(y)=0$. Then (13) and (14) give $\epsilon_1=1/T(y)$ and $\epsilon_2=0$.

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