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Signal Degeneration in Laser Beams Propagated Through a Turbulent Atmosphere

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The statistical distributions of the angle of arrival, the spot location, the cross section, the amplitude, the carrier phase and the modulation phase of a laser beam traversing an anisotropically turbulent atmosphere are derived in terms of the space correlation function of the atmospheric index of refraction and the windspeed. The limitations imposed by the turbulent atmosphere and the loss of coherence on the depth and bandwidth of the modulation, on the length of the path and on the aperture of the receiving apparatus are analyzed. Experiments to obtain numerical parameters and to check functional dependencies are proposed.

1. Introduction

Consider receiving equipment in the form of a lens and a sensory organ (e.g., a photomultiplier) at its focus, the voltage across its output terminals being considered as the received signal. In a vacuum, or in a homogeneous atmosphere, the laser beam is directed along the optical axis and the sensory organ receives practically the full energy of the beam. A turbulent atmosphere, on the other hand, will make the parameters of the beam fluctuate at random and will distort the beam in one or more of the following ways:

(a) The angle of arrival will deviate from the direction of the optical axis and the beam will be focused at some other point in the focal plane ("image-dancing", "quivering"). If the signal is to be received, the area of the sensory organ has to be increased.

(b) The position of the beam will deviate from its central position ("illumination dancing", "spot dancing"); part of the beam will move out of the aperture. If this is to be prevented, the aperture has to be increased; part of it will necessarily have to remain idle and thus the full gain of the receiving apparatus is not realized.

(c) The cross section of the beam will fluctuate ("breathing"); since the total energy in the beam is very nearly constant (for practically no energy is lost by attenuation, and very little is scattered far out of the beam), this results in fluctuations of intensity ("scintillation"). If the cross section of the beam fluctuates beyond that of the aperture, there will be amplitude fluctuations at the output of the sensory organ.

(d) Within the beam the direction of the rays will fluctuate (crumbling of the wave front), causing a blurred image instead of focusing the rays. The effective sensitivity of the receiver will thus again be lowered.

(e) For the same reason, the illumination will not be uniform, but will fluctuate over the illuminated spot ("boiling"); partial focusing and defocusing will produce bright regions at the expense of other regions within the cross section of the beam.

(f) Fluctuations in transit time (due to fluctuations in the velocity of propagation), or phase fluctuations, will simulate a modulation; this will interfere with the true modulation, unless the modulation depth and bandwith of the latter are correctly chosen.

(g) Fluctuations as in (f) will not only take place for the entire beam, but will also vary across the cross section of the beam; since the sensory organ at any moment integrates over the entire aperture, the individual contributions from the elements of the aperture will, because of their differing phases, add up to less than their scalar sum. When the transit time difference across the beam reaches the period of the modulation frequency, the modulation may be obliterated.

The signal degeneration as in (a) to (e) could, in principle, be eliminated by sufficiently increasing the aperture and the active region of the sensory organ. Although the full potential gain of such an apparatus would not be realized, there would be no amplitude fluctuations, for the total energy of the beam would simply be always intercepted regardless of the direction, location, cross section or internal energy-distribution of the beam. However, not only would this make no difference to (f), but it would actually enhance the signal degeneration caused by (g). Thus too large an aperture will result in the modulation being averaged out, whilst too small an aperture will result in the temporary loss of the carrier altogether.

In the following sections the mean square values and, where possible, the entire distributions of the fluctuations (a) to (g) will be calculated in terms of the refractive-index fluctuations of the atmosphere. Experiments designed to measure these fluctuations and to verify the formulas derived below will be proposed; and possible countermeasures against particular types of signal degeneration will be suggested.

2. Characteristics of a Turbulent Atmosphere

The index of refraction n of a turbulent atmosphere is a random function $n(\mathbf{r})$ of the position (vector) $\mathbf{r}$. We define the random function $\mu(\mathbf{r})$ with mean value zero through

$$n = \langle n \rangle + \mu(\mathbf{r}) \approx 1 + \mu(\mathbf{r}) \tag{2.1}$$

where the angular brackets denote the mean value over the whole region; we also assume ergodicity so that this is also the mean value at any one point over a long period of time. As we shall be mainly interested only in the fluctuations $\mu(\mathbf{r})$, we commit only a negligible error (a few parts in 10^5) in setting $\langle n \rangle = 1$.

For briefness, let

$$\mu(\mathbf{r}_1) = \mu_1, \; \mu(\mathbf{r}_2) = \mu_2\,; \tag{2.2}$$

then the correlation function of μ is defined as

$$B(\mathbf{r}_1, \mathbf{r}_2) = \langle \mu_1, \mu_2 \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_1\mu_2 dt = \langle \mu_2 \rangle C(\mathbf{r}_1, \mathbf{r}_2) \tag{2.3}$$

where C is the autocorrelation coefficient.

Throughout the following it will be assumed that the turbulence is homogeneous (though not necessarily isotropic), i.e., that the statistical distribution of μ is the same at any point throughout the region traversed by the laser beam. In that case the correlation between two points will be determined only by the distance and direction between the two points, i.e., by $(\mathbf{r}_1 - \mathbf{r}_2) = \mathbf{r}$, but will be independent of the actual location of the two points, so that $B(\mathbf{r}_1, \mathbf{r}_2) = B(\mathbf{r})$.

Finally the turbulence will be isotropic if B depends only on the distance between the two considered points but not on the direction joining them, i.e., only on $r = |\mathbf{r}_1 - \mathbf{r}_2|$.

Theoretically derived correlation functions, such as those derived from the Obukhov-Kolmogorov "2/3 law", include empirical parameters hard to find by direct measurement, such as the inner and outer scale of turbulence, and are mathematically fairly unmanageable. Moreover, it is known from tropospheric scatter propagation (also from the elliptic shape of the laser spot) that atmospheric turbulence is in general anisotropic; but no practically useful correlation function describing anisotropic turbulence satisfactorily has so far been derived from aerodynamic considerations. We therefore prefer to introduce an arbitrarily chosen correlation function

$$B(\mathbf{r}) = \langle \mu^2 \rangle C(\mathbf{r}) = \langle \mu^2 \rangle e^{-x^2/X^2 - y^2/Y^2 - z^2/Z^2} \tag{2.4}$$

where x, y, z are the rectangular components of $\mathbf{r}$, and X, Y, Z are the correlation distances along these directions; for isotropic turbulence $(X=Y=Z=R)$ this reduces to

$$B(r) = \langle \mu^2 \rangle e^{-r^2/R^2}\,. \tag{2.5}$$

There is of course no physical reason for choosing this particular correlation function out of an indenumerable set of functions fulfilling the conditions given above; however, it is plausible to assume that (2.4) is sufficiently general to provide a satisfactory least-squares fit of measured data by a proper choice of the constants [1] $\langle \mu^2 \rangle$, X, Y, Z. As against theoretically derived correlation functions it has the advantage of being mathematically manageable and providing for anisotropic turbulence.

The structure function corresponding to (2.4) is

$$D(\mathbf{r}) = \langle (\mu_1 - \mu_2)^2 \rangle = 2B(0) - 2B(\mathbf{r})$$

$$= 2\langle \mu^2 \rangle [1 - \exp(-x^2/X^2 - y^2/Y^2 - z^2/Z^2)]. \tag{2.6}$$

3. Survey of Existing Methods

To solve the problem fully and exactly, one would have to solve the wave equation

$$\nabla^2 U + \frac{n^2}{c^2}\frac{\partial^2 U}{\partial t^2} = 0 \tag{3.1}$$

for the random function $n(\mathbf{r}, t)$. The variable U stands for any quantity satisfying the wave equation, e.g., a rectangular component of the field vectors $\mathbf{E}$ or $\mathbf{H}$. As at present no general and exact analytical method of solving (3.1) is known, it has to be solved by one of several approximate methods.

The simplest approximation is that of geometric optics or ray tracing. It can be derived directly from (3.1) [Landau and Lifshits, 1959] for sufficiently small ratios λ/a with λ the wavelength and a the dimensions of the inhomogenities (or roughly the correlation distance). This leads to Fermat's principle, according to which the ray path is the curve for which the transit time is minimum:

$$\nabla t = \frac{1}{c}\int_0^L n(x, y, z) dl = \text{minimum} \tag{3.2}$$

with L the length and dl an element of length of the path. Using variational methods, the quantities of interest and their fluctuations may be found. The simplicity of the method permits more complicated phenomena, such as anisotropic turbulence, to be tackled more easily. Unfortunately the method breaks down for small inhomogenities on very long paths, as it neglects diffraction effects.

[1] If this is not so, one may use (2.4) with different constants for large $\mathbf{r}$ and for $\mathbf{r} \to 0$, or any other useful correlation function. The resulting formulas in this paper will be expressed in terms of a general function $B(\mathbf{r})$ for which (2.4) will be substituted as an example.

The latter are respected in the "Method of small perturbations", which assumes that the amplitude and phase fluctuations of the wave in the random medium are small compared to the unperturbed values in a homogeneous medium. A further refinement of this method is due to Rytov [Tatarski, 1961; Chernov, 1960]. The method leads to tedious integrations, making investigations beyond well-known results difficult. It has also recently been pointed out that the method may in certain cases not be superior to that of geometric optics [Hufnagel and Stanley, 1964].

A third method developed in the past few years by Wolf, Beran, Parrent, Zucker, and others [Beran and Parrent, 1964] is the "theory of partial coherence", which in essence by-passes solutions in terms of the usual electromagnetic field vectors, but rather directly investigates the "mutual coherence function", a correlation function of an analytic signal that can be shown itself to obey a wave equation. This approach has been used in connection with the present problem by Hufnagel and Stanley [1964]; however, their derivation includes the assumption of very small scattering angles—apparently so small that no widening of the beam sets in, for setting $\mathbf{p}_1 = \mathbf{p}_2$ in eq (4.6) of their paper, one finds the amplitude undiminished for any length of path (z), which does not agree with the experiment and seems to be a coarser approximation than that of geometric optics. Nevertheless, since the method aims directly at the function of immediate interest, it evidently holds much promise for the future.

4. Choice of Method

Confronted with the lack of a general method of solution, we make the following compromise:

We first investigate the distortions caused by large inhomogenities (outer scale of turbulence). If they are much larger than the cross section of the beam, they are evidently responsible for modifying the beam as a whole, i.e., displacing it, changing its angle of arrival and varying the transit time of the wave front as a whole; they are also responsible for amplitude fluctuations, as far as these are caused by refractive broadening of the beam cross section (cf. below). These inhomogenities may safely be assumed large enough to satisfy the criterion for the applicability of geometric optics, and the calculation of the resulting distortions is simple enough to permit investigation of the more complicated problem of anisotropic turbulence.

This method neglects diffraction effects and therefore the results obtained (as above) are then checked against those obtained by wave optics. This comparison seems to indicate that geometric optics have been unduly spurned (possibly by extrapolating from conditions in the microwave band, although this concerns frequencies 10^5 times smaller): the results differ insignificantly for amplitude fluctuations; for phase fluctuations they lead to practically the same result, whilst for other cases wave optics have not led to any explicit results at all (particularly where anisotropic turbulence is concerned). In what follows we therefore primarily use geometric optics and only check the results against those that can be obtained by wave optics.

5. Basic Relations

To apply geometric optics, the dimensions of the inhomogenities l_0 have to be sufficiently large to satisfy the criterion [Tatarski, 1961; pp. 120–121]

$$l_0 >> \sqrt{L\lambda} \tag{5.1}$$

where L is the pathlength and λ the wavelength. Starting with (3.2) one may then use the Euler-Lagrange equation to derive the "ray equation" [Chernov, 1960; pp. 12–15]

$$\frac{d(n\mathbf{s})}{dl} - \nabla n = 0 \tag{5.2}$$

where $\mathbf{s}$ is a unit vector tangent to the ray.

Since

$$dl = \sqrt{dx^2 + dy^2 + dz^2} = dx\,\sqrt{1 + (dy/dx)^2 + (dz/dx)^2},$$

then for a ray starting out in, and never strongly deviating from, the x-direction, $(dy/dx)^2$ and $(dz/dx)^2$ will be small compared to unity, so that we may set $dl \approx dx$ and (5.2) becomes

$$\frac{d(n\mathbf{s})}{dx} - \nabla\, n = 0.$$

Integration yields

$$n(\mathbf{r})\mathbf{s}(\mathbf{r})\Big|_0^L = \int_0^L \nabla\, n dx. \tag{5.3}$$

Since the value of n at the end points of a sufficiently long path obviously cannot affect its direction, we set it equal to its mean value unity. Also $\mathbf{s}(0, 0, 0) = \mathbf{x}_0$, a unit vector along the x-axis; hence

$$\mathbf{s}(L, 0, 0) = \mathbf{x}_0 + \int_0^L \nabla \mu dx \tag{5.4}$$

or decomposing $\mathbf{s}$ into its components,

$$s_x = 1 + \int_0^L \frac{\partial \mu}{\partial x} dx; \qquad s_y = \int_0^L \frac{\partial \mu}{\partial y} dx; \qquad s_z = \int_0^L \frac{\partial \mu}{\partial z} dx. \tag{5.5}$$

The random values of the three components are not quite independent, as they are related by the condition $s_x^2 + s_y^2 + s_z^2 = 1$.

If L is long enough to traverse very many inhomogenities (as we assume), then it follows from the Central Limit Theorem and (5.5) that, regardless of the distributions of $\partial\mu/\partial x$, $\partial\mu/\partial y$, $\partial\mu/\partial z$, the components s_x, s_y, s_z are all normally distributed. Obviously $\langle s_y \rangle = \langle s_z \rangle = 0$.

The small perturbation method consists in replacing the solution of (3.1)

$$U_0 = A_0 e^{-i(\omega t - kx)}$$

which is valid for a plane wave propagating along the x-axis through a medium with $n = \text{const}$, by

$$U(r) = A(r) e^{-i[\omega t - F(r)]} \tag{5.6}$$

where A and F are unknown functions whose standard deviation is assumed small compared to the unperturbed values A_0 and kx. Rytov's refinement of this method essentially consists in writing (5.6) as

$$U(r) = A_0 e^{-i[\omega t - Q(r)]},$$

making

$$Q(r) = F - i \log [A(r)/A_0].$$

It may then be shown [Chernov, 1960] that under certain conditions that are well satisfied for optical frequencies, the fluctuating part of A, i.e., $\Psi = Q - kx$, is given by

$$\Psi = -\frac{ik^2}{2\pi} \int_v \frac{1}{r} e^{ik[r-(x-\xi)]} \mu(\xi, \eta, \zeta) dv \tag{5.7}$$

where r is the distance from an element dv with coordinates ξ, η, ζ to the observation point x, y, z and the integration is carried out over the entire volume from which waves arrive at the receiving point.

The following investigation will be based on (5.5); the results will then be checked against those obtained from (5.7).

At this point we also insert formulas for certain double integrals that very frequently occur in propagation through a random medium.

The formula

$$\int_0^L dx_1 \int_0^L f(x_1 - x_2) dx_2 = 2 \int_0^L (L - x) f(x) dx \tag{5.8}$$

where $f(x)$ is an even function, is obtained by partial integration; quite similarly one obtains the more general relation

$$\int_0^{L_1} dx_1 \int_0^{L_2} f(x) dx_2 = \int_0^{L_1} (L_1 - x) f(x) dx$$
$$+ \int_0^{L_2} (L_2 - x) f(x) dx - \int_0^{L_1 - L_2} (L_1 + L_2 - x) f(x) dx, \tag{5.9}$$

of which (5.8) is a special case for $L_1 = L_2$.

In all cases to be considered, $f(x)$ is a correlation function having significant values only for $x << L_1, L_2$; we may therefore replace the upper limits by ∞ and neglect x compared with L; this leads to

$$\int_0^{L_1} dx_1 \int_0^{L_2} f(x) dx_2$$

$$\approx (L_1 + L_2) \int_0^{\infty} f(x) dx - (L_1 + L_2) \int_0^{L_1 - L_2} f(x) dx$$

$$= (L_1 + L_2) \int_{L_1 - L_2}^{\infty} f(x) dx. \tag{5.10}$$

6. Fluctuations of the Angle of Arrival ("Quivering")

Let θ and ϕ be the (small) deviations of the beam from its original direction x_0, so that by transforming to spherical coordinates

$$s_x = \cos\theta \cos\phi \approx (1 - \tfrac{1}{2}\theta^2)(1 - \tfrac{1}{2}\phi^2) \tag{6.1}$$

$$s_y = \sin\theta \cos\phi \approx \theta (1 - \tfrac{1}{2}\phi^2) \tag{6.2}$$

$$s_z = \sin\phi \approx \phi \cdot \tag{6.3}$$

Neglecting second-order terms and recalling the remarks following (5.5), the probability densities of θ and ϕ are therefore

$$p_\theta(\theta) = \frac{1}{\sqrt{2\pi\langle\theta^2\rangle}} \exp(-\theta^2/2\langle\theta^2\rangle) \tag{6.4}$$

$$p_\phi(\phi) = \frac{}{\sqrt{2\pi\langle\phi^2\rangle}} \exp(-\phi^2/2\langle\phi^2\rangle) \tag{6.5}$$

where $\langle\theta^2\rangle \approx \langle s_y^2\rangle$ and $\langle\phi^2\rangle \approx \langle s_z^2\rangle$ are determined from (5.5):

$$\langle\theta^2\rangle = \langle s_y^2\rangle = \left\langle \int_0^L \frac{\partial\mu(x_1, y_1, z_1)}{\partial y_1} dx_1 \int_0^L \frac{\partial\mu(x_2, y_2, z_2)}{\partial y_2} dx_2 \right\rangle$$

$$= \int_0^L \int_0^L \frac{\partial^2}{\partial y_1 \partial y_2} \langle \mu_1 \mu_2 \rangle \, dx_1 \, dx_2. \tag{6.6}$$

Introducing relative coordinates $x = x_1 - x_2$, $y = y_1 - y_2$, $z = z_1 - z_2$, we have

$$\frac{\partial^2}{\partial y_1 \partial y_2} = -\frac{\partial^2}{\partial y^2}; \ \langle \mu_1 \mu_2 \rangle = \langle \mu^2 \rangle \, C(r) \tag{6.7}$$

and applying (5.10) we find

$$\langle\theta^2\rangle = -2L\langle\mu^2\rangle \int_0^\infty \left(\frac{\partial^2 C}{\partial y^2}\right)_{y=z=0} dx. \tag{6.8}$$

For C given by (2.4) we have

$$\left(\frac{\partial^2 C}{\partial y^2}\right)_{y=z=0} = -\frac{2}{Y^2} \, e^{-x^2/X^2}$$

so that

$$\langle \theta^2 \rangle = \frac{2L\langle \mu^2 \rangle X\sqrt{\pi}}{Y^2}. \tag{6.9}$$

Similarly,

$$\langle \phi^2 \rangle = \frac{2L\langle \mu^2 \rangle X\sqrt{\pi}}{Z^2}. \tag{6.10}$$

The mean square of the total deviation is

$$\langle \epsilon^2 \rangle = \langle \theta^2 \rangle + \langle \phi^2 \rangle = 2L\langle \mu^2 \rangle X\sqrt{\pi}\left(\frac{1}{Y^2}+\frac{1}{Z^2}\right) \tag{6.11}$$

which for isotropic turbulence ($X=Y=Z=R$) reduces to

$$\langle \epsilon^2 \rangle = \frac{4L\langle \mu^2 \rangle \sqrt{\pi}}{R} \tag{6.12}$$

a formula already derived by Chernov [1960; p. 17, eq (33) and (35)]. The distributions of θ and ϕ are both normal as already explained after (5.5); their mean values are zero and their respective variances are given by (6.9) and (6.10) in terms of the fluctuations of the atmosphere and the path length. It follows that there must be a mutually perpendicular pair of axes that will make θ and ϕ statistically independent; it is these two axes that we choose as the y- and z-axes. It should be noted that their directions need not necessarily be horizontal and vertical, although for physical reasons this is probably their most common orientation. It also follows that the total deviation ϵ from the unperturbed angle of arrival, related to θ and ϕ by $\epsilon^2 = \theta^2 + \phi^2$, is Hoyt-distributed: [2]

$$p_\epsilon(\epsilon) = \frac{\epsilon}{\sqrt{AB}} I_0\left(\frac{A-B}{4AB}\epsilon\right)\exp\left(-\frac{A+B}{4AB}\epsilon^2\right) \qquad (\epsilon > 0) \tag{6.13}$$

where for the sake of briefness we have put $\langle \theta^2 \rangle = A$, $\langle \phi^2 \rangle = B$. For isotropic turbulence, $A=B$, and (6.13) reduces to the Rayleigh distribution:

$$p_\epsilon(\epsilon) = \frac{2\epsilon}{\langle \epsilon^2 \rangle}\exp\left(-\epsilon^2/\langle \epsilon^2 \rangle\right). \tag{6.14}$$

From (6.9) and (6.10) the anisotropy ratio is

$$\frac{Y}{Z} = \sqrt{\frac{\langle \phi^2 \rangle}{\langle \theta^2 \rangle}}. \tag{6.15}$$

To measure $\langle \theta^2 \rangle$ and $\langle \phi^2 \rangle$ the following experiment is proposed. Let the beam pass through a telescope focused at infinity and take a time exposure of the image long enough to ensure appropriate averaging. Deviations of the image from the center corresponding to the optical axis are proportional to deviations of the angle of arrival and may easily be calibrated. The density of the photographic record will then be determined by the distribution $p(\theta, \phi) = p_\theta(\theta)p_\phi(\phi)$ as given by (6.4) and (6.5). The distributions may then be evaluated densitometrically; a particularly convenient method is to process the equiprobability curves directly.[3] For isotropic turbulence, the equiprobability curves should be circles; for anisotropic turbulence they will be ellipses, and the ratio of major to minor axes directly measures (6.15). The values of $\langle \theta^2 \rangle$ and $\langle \phi^2 \rangle$ may again be obtained by densitometry along the major and minor axes. This yields the distributions $p(\theta, 0)$ and $p(0, \phi)$, from which the mean squares are easily derived by standard methods.

7. Displacement of the Spot ("Dancing")

The displacements η and ζ of the beam along the y and z axes from its unperturbed position are obviously given by

$$\eta = \int_0^L s_y\, dx, \qquad \zeta = \int_0^L s_z dx, \tag{7.1}$$

where the random quantities s_y and s_z are given by (5.5). They are, as we have seen, normally distributed and hence η and ζ are distributed normally also. It follows from $\langle s_y \rangle = \langle s_z \rangle = 0$ that

$$\langle \eta \rangle = \langle \zeta \rangle = 0. \tag{7.2}$$

To find the mean square of η we first find the correlation function for $\theta \approx s_y$ at distances L_1, L_2 from the source. In analogy to (6.6) and using (6.7) we now have

$$B_\theta(L_1, L_2) = \langle \theta(L_1)\theta(L_2) \rangle = -\langle \mu^2 \rangle \int_0^{L_1}\int_0^{L_2} \frac{\partial^2 C}{\partial y^2} dx_1 dx_2. \tag{7.3}$$

Applying (5.10), this yields

$$B_\theta(L_1, L_2) = -\langle \mu^2 \rangle (L_1+L_2)\int_{L_1-L_2}^{\infty}\left(\frac{\partial^2 C}{\partial y^2}\right)_{x=y=0} dx. \tag{7.4}$$

Substituting from (2.4) and integrating we obtain the correlation function

$$\langle \theta(L_1)\theta(L_2) \rangle = \frac{(L_1+L_2)X\langle \mu^2 \rangle \sqrt{\pi}}{Y^2}\,\mathrm{erfc}\,\frac{L_1-L_2}{X}. \tag{7.5}$$

Similarly,

$$\langle \phi(L_1)\phi(L_2) \rangle = \frac{(L_1+L_2)X\langle \mu^2 \rangle \sqrt{\pi}}{Z^2}\,\mathrm{erfc}\,\frac{L_1-L_2}{X}. \tag{7.6}$$

[2] Curves of (6.13) will be found in [Hoyt, 1947]. Curves of the integral distribution of (6.13) are given by Beckmann [1963], pp. 208–212; substitute ϵ for r, set $B=0$ and k equal to the anisotropy ratio (6.15).

[3] By reversing negative A onto negative B; the print made through the combined negatives A and B will then (owing to the logarithmic sensitivity of the emulsion) show up the curves of equal density, i.e., the equiprobability curves. Details of the procedure may be found in [Lau and Krug, 1957].

In the last two formulas erfc is the error function complement,

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

From (7.1) we have, for path length L,

$$\langle \eta^2 \rangle = \int_0^L \int_0^L \langle \theta(L_1)\theta(L_2) \rangle dL_1 dL_2$$

$$= \frac{X\langle \mu^2 \rangle \sqrt{\pi}}{2Y^2} \int_0^L \int_0^L (L_1 + L_2) \operatorname{erfc} \frac{L_1 - L_2}{X} dL_1 dL_2. \quad (7.8)$$

By a procedure analogous to obtaining (5.10), this leads to

$$\langle \eta^2 \rangle = \frac{L^2 X^2 \langle \mu^2 \rangle \sqrt{\pi}}{2Y^2} \int_{-L/X}^{L/X} \operatorname{erfc} t dt. \quad (7.9)$$

The integral equals twice the upper limit, so that finally

$$\langle \eta^2 \rangle = \frac{L^3 X \langle \mu^2 \rangle \sqrt{\pi}}{Y^2}. \quad (7.10)$$

Similarly,

$$\langle \zeta^2 \rangle = \frac{L^3 X \langle \mu^2 \rangle \sqrt{\pi}}{Z^2}. \quad (7.11)$$

For isotropic turbulence, $X = Y = Z = R$, so that in this case the last two relations reduce to

$$\langle \rho^2 \rangle = \langle \eta^2 \rangle + \langle \zeta^2 \rangle = \frac{2L^3 \langle \mu^2 \rangle \sqrt{\pi}}{R}. \quad (7.12)$$

A similar formula was derived for this special case by a more complicated method by Chernov [1960; p. 26, eq (67)—square and substitute from eq (35) on p. 17.]; it differs from (7.12) in the numerical factor, which is 4/3 and not 2 as above.

Since η and ζ are normally distributed, ρ will be Hoyt-distributed as ϵ was in (6.13), and for the same reasons; only the mean square values along the axes now depend on the path-length L in a strikingly different manner (proportional to L^3).

The proposed measurements are therefore also similar to the ones in the previous section. For measuring the distributions of η and ζ, however, the time exposures should be made by photographing a screen (reflecting or translucent) illuminated by the laser beam.

The anisotropy parameters may be measured analogously to (6.15); the inner scale of turbulence may be estimated by finding the distance up to which the 3d-power dependence on L remains valid and the result may be compared with (6.16).

It is probable that in this measurement the displacement of the beam ("dancing") will be partially masked by expansions and contractions of its cross section ("breathing"). To separate the two effects, it is suggested to use two pairs of photocells, one for each of the two (η and ζ) axes. If the two cells are mounted straddling the central (unperturbed) point of illumination and connected to a differential amplifier, the output will be affected by deflections of the beam, but the arrangement should not react to simultaneous changes of intensity caused by "breathing". It should thus be possible to investigate the distributions of η and ζ.

8. Fluctuations of the Beam Cross Section and Amplitude

Consider a slice of the laser beam of width dx, with the front and back surfaces (possibly curved) of area S and $S + dS$ normal to the rays, i.e., to the unit vectors $\mathbf{s}$. Then it follows from the Divergence Theorem that

$$\nabla \cdot \mathbf{s} S dx = dS. \quad (8.1)$$

Rewriting (5.5) as

$$s_x = 1 + \int_0^x \frac{\partial \mu}{\partial x'} dx' \approx 1, \; s_y = \int_0^x \frac{\partial \mu}{\partial y} dx',$$

$$s_z = \int_0^z \frac{\partial \mu}{\partial z} dx' \quad (8.2)$$

we find div $\mathbf{s}$ and substitute in (8.1), obtaining

$$\frac{dS}{S} = -\frac{dI}{I} = dx \int_0^x \left(\frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2} \right) dx', \quad (8.3)$$

where I, the energy flow per unit area, is the intensity or power density of the beam. Integrating, we find

$$\log \frac{S}{S_0} = -\log \frac{I}{I_0} = -2 \log \frac{A}{A_0} = \int_0^L dx \int_0^x \nabla_T^2 \mu dx'$$

where A is the amplitude and ∇_T^2 denotes the transverse Laplacian as applied in the integrand of (8.3). Since by assumption $L \gg R$, it follows again that $\log A$ is normally distributed, and hence A is lognormally distributed.

The mean square value of $\log (A/A_0)$ is derived by the usual integrations as before and leads to

$$\left\langle \left(\log \frac{A}{A_0} \right)^2 \right\rangle = \frac{L^3}{6} \int_0^\infty (\nabla_T^2 \nabla_T^2 B)_{y=z=0} dx. \quad (8.4)$$

For the correlation function (2.4) we have

$$[\nabla_T^2 \nabla_T^2 B(x, y, z)]_{y=z=0} = 4\langle \mu^2 \rangle \left(\frac{3}{Y^4} + \frac{2}{Y^2 Z^2} + \frac{3}{Z^4} \right) e^{-x^2/X^2}$$

and the integration yields

$$\left\langle \left(\log \frac{A}{A_0} \right)^2 \right\rangle = \langle \mu^2 \rangle L^3 X \frac{\sqrt{\pi}}{3} \left(\frac{3}{Y^4} + \frac{2}{Y^2 Z^2} + \frac{3}{Z^4} \right). \quad (8.5)$$

For isotropic turbulence this reduces to

$$\left\langle \left(\log \frac{A}{A_0}\right)^2 \right\rangle = \frac{8\sqrt{\pi}\langle\mu^2\rangle L^3}{3R^3} \tag{8.6}$$

in agreement with Chernov [1960; p. 34]. For simplicity we keep to isotropic turbulence for the rest of this section, but the results are easily generalized to anisotropic turbulence by using (8.5) rather than (8.6). Taking the mean value of (8.4) we find, since obviously $\langle \nabla_T^2 \mu \rangle = 0$,

$$\left\langle \log \frac{S}{S_0} \right\rangle = \left\langle \log \frac{A_0^2}{A^2} \right\rangle = \tfrac{1}{2} \langle \nabla_T^2 \mu \rangle L^2 = 0. \tag{8.7}$$

It is easily shown that if $y = \log x$ is distributed normally with mean value a and variance σ^2, then x is lognormally distributed with mean value

$$\langle x \rangle = \exp\,(a + \tfrac{1}{2}\,\sigma^2). \tag{8.8}$$

Hence it follows from (8.7) that

$$\left\langle \frac{A}{A_0} \right\rangle = \exp\left[-\frac{1}{2} \langle \log^2 (A/A_0) \rangle \right] \tag{8.9}$$

and

$$\left\langle \frac{S}{S_0} \right\rangle = \exp\left[\frac{1}{2} \langle \log^2 (S/S_0) \rangle \right]. \tag{8.10}$$

Substituting from (8.4), this yields

$$\left\langle \frac{S}{S_0} \right\rangle = \exp\left[\frac{L^3}{3} \int_0^\infty (\nabla_T^2 \nabla_T^2 B)_{y=z=0} dx \right] \tag{8.11}$$

and

$$\left\langle \frac{A}{A_0} \right\rangle = \exp\left[-\frac{L^3}{12} \int_0^\infty (\nabla_T^2 \nabla_T^2 B)_{y=z=0} dx \right]. \tag{8.12}$$

Specifically for the correlation function (2.4) we obtain

$$\langle S \rangle = S_0 \exp\left(\frac{16\sqrt{\pi}\langle\mu^2\rangle L^3}{3R^3} \right) \tag{8.13}$$

$$\langle A \rangle = A_0 \exp\left(-\frac{4\sqrt{\pi}\langle\mu^2\rangle L^3}{3R^3} \right) \tag{8.14}$$

$$\langle I \rangle = I_0 \exp\left(-\frac{16\sqrt{\pi}\langle\mu^2\rangle L^3}{3R^3} \right) \tag{8.15}$$

with obvious modifications for anisotropic turbulence based on (8.5).

Equation (8.11) or (8.13) allows us to define an effective angle of divergence α_{eff}: a laser beam of equal power propagated in an homogeneous atmosphere would produce the same mean cross section, amplitude and intensity at a distance L if instead of being parallel, it had a divergence α_{eff}, obviously given by

$$\alpha_{\text{eff}} = \frac{\sqrt{\langle S \rangle} - \sqrt{S_0}}{\sqrt{\pi} L}. \tag{8.16}$$

Judging from measurements in the microwave spectrum, $\langle \mu^2 \rangle$ is at most of the order of 10^{-10}, therefore the exponents of (8.11) through (8.15) will all be small against unity, so that we may expand the exponentials, obtaining generally and specifically

$$\alpha_{\text{eff}} = \frac{L^2\langle\mu^2\rangle}{6} \sqrt{\frac{S_0}{\pi}} \int_0^\infty (\nabla_T^2 \nabla_T^2 C)_{y=z=0} dx = \frac{16\sqrt{S_0} L^2 \langle\mu^2\rangle}{6R^3}. \tag{8.17}$$

Note that α_{eff} was defined in terms of equivalent intensity; the equivalent "divergence of arrival", defined as the vertex angle of the cone tangent to the surface of the tube of rays at the receiving end of the path L, is obviously given by

$$\beta_{\text{eff}} = \frac{1}{\sqrt{\pi}} \frac{d}{dL} \sqrt{\langle S(L) \rangle} = \frac{\langle\mu^2\rangle L^2 \sqrt{S_0}}{2\sqrt{\pi}} \int_0^\infty (\nabla_T^2 \nabla_T^2 C)_0 dx = 3\alpha_{\text{eff}}. \tag{8.18}$$

Any of the relations (8.11) through (8.15) may again be used to estimate the important parameters $\langle \mu^2 \rangle$ and l_0, which are costly and difficult to measure by refractometer runs (airborne or otherwise). This is again done by measuring the dependencies of one or more of the left-hand sides of (8.11) through (8.15) on L and comparing them to the theoretical formulas.

Measurements of $\langle S/S_0 \rangle$ and $\langle \log^2 (S/S_0) \rangle$ and, if desired, of the entire distribution of S/S_0 may be made by photographing the laser spot on a screen (translucent or by illumination). The photographs should be instantaneous and made in rapid succession (filming) to prevent averaging effects. The corresponding measurements of I and A must be made through an optical instrument with aperture much smaller than $\langle S \rangle$ to prevent integrating effects [4] and amplitude fluctuations due to perturbed phases over the aperture. A photomultiplier with no or small-aperture optics will achieve this; but measurements of S, though more exacting, will probably be more reliable.

Further discussion of amplitude fluctuations, in particular, degeneration of amplitude modulation, is delayed to section 10.

9. Phase Fluctuations

All of the distortions hitherto considered may be overcome by a sufficiently large aperture collecting

[4] A typical integrating effect sets in when viewing stars through large telescopes: they will twinkle less (but quiver more) than when observed by naked eye.

the beam as explained in section 1; unfortunately, the phase fluctuations lead to the opposite requirement, for their detrimental effect increases with growing aperture.

In a homogeneous atmosphere a plane wave arriving in the direction of the optical axis will at any moment have the same phase over the entire beam cross section; this phase will change linearly with time. The phase fluctuations caused by a turbulent atmosphere are of three different kinds.

First, the phase may change its value, at any one time, linearly over the aperture. This case is equivalent to, and in fact indistinguishable from, a plane wave arriving at a finite angle with respect to the optical axis. This case has already been dealt with in section 6; as a result the beam will be focused elsewhere in the focal plane of the lens. The resulting "quivering" of the image can be overcome by increasing the area of the sensory organ; the size of the aperture is immaterial.

Second, the phase may at any instant be constant over the aperture, but may not vary linearly with time. This is caused by the fluctuations in transit time of the individual crests of the wave. The result is a parasitic phase-modulation that cannot be eliminated by choice of the aperture or size of the sensory organ.

Third, the phase may change irregularly over the aperture at any one instant. This is caused by fluctuating differences between the transit times of the individual rays within the beam (crumbling of the phase front). Since the phase is random, but continuous, small phase changes over the aperture can be ensured only by a small aperture.

The transit time of the signal is

$$t=\frac{1}{c}\int_0^L n\,(x, y, z)\,dx \tag{9.1}$$

and hence the deviation from the mean is

$$\Delta t=t-\langle t\rangle=\frac{1}{c}\int_0^L \mu\,(x, y, z)\,dx\,. \tag{9.2}$$

If $\psi'=\omega t+\psi$ is the total phase, the phase fluctuations of the carrier and modulation are given respectively by

$$\psi=\omega\Delta t,\ \Psi=\Omega\Delta t \tag{9.3}$$

where ω is the carrier, and Ω the modulation frequency. It again follows from the Central Limit Theorem (for $L\gg R$) that the phases of both the carrier and the modulation are distributed normally with mean zero. The variance will then be

$$\langle\psi^2\rangle=\langle\mu^2\rangle k^2\int_0^L dx_1\int_0^L C(x_1-x_2, y_1-y_2, z_1-z_2)\,dx_2 \tag{9.4}$$

where $k=\omega/c$ is the phase constant. A similar expression will hold for Ψ with $K=\Omega/c$ substituted for k.

Applying (5.10) for $L\gg R$, this yields

$$\langle\psi^2\rangle=2\langle\mu^2\rangle k^2L\int_0^\infty C(x, 0, 0)dx \tag{9.5}$$

where we have gone over to relative coordinates $x=x_1-x_2$, $y=y_1-y_2$, $z=z_1-z_2$. In particular, for the correlation function (2.4), this yields

$$\langle\psi^2\rangle=\langle\mu^2\rangle\,k^2L\,X\sqrt{\pi}\,\cdot \tag{9.6}$$

The relation will also hold for the modulation phase if we replace ψ and k by Ψ and K.

Formula (9.6) describes the second of the above effects at the point $(L, 0, 0)$; for the third effect we have to find the correlation of the phase fluctuations in the plane $x=L$;

$$\begin{aligned}\langle\psi(L, y_1, z_1)\psi(L, y_2, z_2)\rangle&=\langle\psi_1\psi_2\rangle\\&=\langle\mu^2\rangle k^2\int_0^L dx_1\int_0^L C(x, y, z)dx_2\\&=2\langle\mu^2\rangle k^2L\int_0^\infty C(x, y, z)dx.\end{aligned} \tag{9.7}$$

In particular, for C given by (2.4), this becomes

$$\langle\psi_1\psi_2\rangle=\langle\mu^2\rangle k^2L\,X\sqrt{\pi}\exp\left(-\frac{y^2}{Y^2}-\frac{z^2}{Z^2}\right), \tag{9.8}$$

reducing to (9.6) for $\psi_1=\psi_2$, i.e., for $y=z=0$. For isotropic turbulence, i.e., $X=Y=Z=R$, this reduces to

$$\langle\psi_1\psi_2\rangle=\langle\mu^2\rangle k^2\,L\,R\sqrt{\pi}\exp\left(-\rho^2/R^2\right), \tag{9.9}$$

where

$$\rho^2=y^2+z^2. \tag{9.10}$$

This checks with the result obtained in this special case by using a formula derived by Tatarski [1961; p. 100, (6.28)] in terms of structure functions.

It may be seen from (9.8) that $\langle\psi^2\rangle$ is proportional to the correlation distance X. The latter is a measure of the size of the inhomogeneities that are mainly responsible for the phase fluctuations, which leads one to believe that for calculating phase fluctuations, the validity of geometric optics may be extended to long paths L. This belief is further strengthened by the fact that calculations of $\langle\psi^2\rangle$ by physical optics lead to the same dependence (except for a numerical factor $1/2$); more evidence still is provided by Fried and Cloud [1964b], who find small-scale turbulence relatively insignificant for phase fluctuations.

Formula (9.8) solves the problem for space fluctuations; however, for time fluctuations the interesting quantity is not ψ, but $d\psi/dt$. We therefore introduce the instantaneous frequency [Downing, 1964, p. 60]

$$\frac{d\psi'}{dt}=\omega+u \tag{9.11}$$

where $u=d\psi/dt$ is a new random variable with mean zero. Let $U(\tau)$ and $F(\tau)$ be the time-correlation functions of u and ψ respectively; then

$$U(\tau)=\langle u(r,\, t)u(r,\, t+\tau)\rangle=-\frac{d^2F(\tau)}{d\tau^2} \tag{9.12}$$

hence

$$\langle u^2\rangle=\left\langle\frac{d\psi^2}{dt}\right\rangle=U(0)=-\frac{d^2F(0)}{d\tau^2}\cdot \tag{9.13}$$

The problem thus reduces to finding $F(\tau)$. Now if it is assumed that the time variation of μ at any point is primarily due to drift, and only to a negligible degree to turbulence, diffusion, etc., so that the field of inhomogenities is "frozen" and carried across the path by the wind as a whole, then using the principle of relativity it is easily shown that

$$B(\mathbf{r})=B(\mathbf{v}\tau), \tag{9.14}$$

where $\mathbf{v}$ is the velocity of the wind. Thus for the crosswind components v_y and v_z we have directly from (9.8)

$$\langle\psi(0)\psi(\tau)\rangle_{\text{crosswind}}=F_c(\tau)$$

$$=\langle\mu^2\rangle k^2LX\sqrt{\pi}\,\exp\left[-\left(\frac{v_y^2}{Y^2}+\frac{v_z^2}{Z^2}\right)\tau^2\right]\cdot \tag{9.15}$$

To apply the same method to upwind or downwind, we first find by applying (5.10)

$$\langle\psi(L,\,0,\,0)\psi(L+\Delta L,\,0,\,0)\rangle$$

$$=(2L+\Delta L)\langle\mu^2\rangle k^2\,X\,\frac{\sqrt{\pi}}{2}\,\operatorname{erfc}\frac{\Delta L}{X}\cdot \tag{9.16}$$

Hence, setting $\Delta L=v_x\tau$

$$\langle\psi(0)\psi(\tau)\rangle_{\text{upwind}}=F_u(\tau)$$

$$=(2L+v_x\tau)\langle\mu^2\rangle k^2\,X\,\frac{\sqrt{\pi}}{2}\,\operatorname{erfc}\left(\frac{v_x\tau}{\mathrm{X}}\right)\cdot \tag{9.17}$$

From (9.15) and (9.13)

$$\left\langle\left(\frac{d\psi}{dt}\right)\right\rangle_{\text{crosswind}}=2\langle\mu^2\rangle k^2\,LX\,\sqrt{\pi}\left(\frac{v_y^2}{Y^2}+\frac{v_z^2}{Z^2}\right), \tag{9.18}$$

and from (9.17)

$$\left\langle\left(\frac{d\Psi}{dt}\right)^2\right\rangle_{\text{upwind}}=2\langle\mu^2\rangle k^2\,Xv_x. \tag{9.19}$$

As might be expected from physical reasoning, the fluctuations caused by upwind or downwind drift are negligible compared to the crosswind fluctuations; (9.17) and (9.19) should be used only if no crosswind is present.

The standard deviation of u is, from (9.18)

$$\sigma_u=k\sqrt{2LX\sqrt{\pi}W\langle\mu^2\rangle} \tag{9.20}$$

where W stands for the round bracket in (9.18).

As may be seen from (9.11), this kind of fluctuation is in effect a parasitic modulation of the instantaneous frequency of the carrier. This modulation is normal with mean zero and variance (9.18).

If the signal is amplitude modulated with frequency Ω, the carrier with the two sidebands will float in accordance with $u(t)$. The receiver bandwidth (optical filters) will therefore have to be widened from 2Ω to $2\Omega+\Delta$, where Δ must be large enough to accommodate $u(t)$ for most of the time. Its value for a given probability of accommodation is given by the Laplace function, e.g., for the signal to be accommodated for 99.7 percent of the time, Δ must equal 6 times the standard deviation (9.20). The resulting widening of the bandwidth increases the noise in proportion, so that the signal-to-noise ratio will decrease by

$$(S/N)_{\text{homog}}-(S/N)_{\text{turbu}}$$

$$=20\log_{10}\left(1+\frac{6k\sqrt{\sqrt{\pi}\,LXW\,\langle\mu^2\rangle}}{\Omega}\right)\text{dB} \tag{9.21}$$

where we have taken $\Delta=6\sigma_u$. Alternatively, one may calculate the bandwidth limitation for a given signal-to-noise ratio; a system with bandwidth 2Ω will demodulate frequencies up to Ω in a homogeneous atmosphere, but this value will decrease to

$$\Omega'=\Omega-\frac{\Delta}{2}=\Omega-6k\sqrt{\pi LXW\langle\mu^2\rangle} \tag{9.22}$$

in a turbulent atmosphere.

If the signal is frequency-modulated,

$$V=A\exp\left\{i\int_0^t[\omega+\Delta\omega\cos\Omega\tau+u(\tau)]d\tau\right\} \tag{9.23}$$

where $\Delta\omega$ is the frequency deviation. The demodulator will recover the instantaneous frequency $u(t)+\Delta\omega\cos\Omega t$; if $u(t)$ is to produce no appreciable distortion, we must have

$$\langle u^2\rangle\ll\langle(\Delta\omega\cos\Omega t)^2\rangle=\frac{(\Delta\omega)^2}{2} \tag{9.24}$$

or from (9.6),

$$\Delta\omega\gg k\sqrt{2LXW\sqrt{\pi}\,\langle\mu^2\rangle} \tag{9.25}$$

leading once more to the requirement of a wide-band system.

It should be pointed out that for technical reasons it is almost impossible to keep ω in a contemporary laser constant; the thermal fluctuations alone cause a frequency drift which is probably much greater than (9.18). Also, the bandwidth of contemporary optical filters is still so wide that the reduction (9.22) will have no practical effect. Thus at the present the time-phase fluctuations are hardly dangerous.

As for phase fluctuations in space, i.e., over the aperture, which may be considered rectangular (area $4yz$), elliptic (πyz) or circular (πp^2), if no significant distortion is to set in, we must have

$$\langle[\psi(y)-\psi(0)]^2\rangle \ll \pi^2$$

where y may be replaced by z or ρ. The windspeed is immaterial for this type of fluctuation (except in as far as it determines X, Y, and Z). From (9.8) and (2.6) we therefore have

$$2\langle\mu^2\rangle k^2LX\sqrt{\pi}\,(1-e^{-y^2/Y^2}) \ll \pi^2 \tag{9.26}$$

where y and Y may be replaced by z and Z, and for isotropic turbulence one may set $X=Y=Z=R$. This determines the size of the aperture or else the permissible bandwidth of the modulation: assuming $y \ll Y$ and expanding the exponential, we find the condition on the aperture size for coherence of the carrier phase fronts:

$$y \ll \frac{\lambda Y}{2\sqrt{\langle\mu^2\rangle\sqrt{\pi}\,LX}};\; z \ll \frac{\lambda Z}{2\sqrt{\langle\mu^2\rangle\sqrt{\pi}\,LX}}. \tag{9.27}$$

For $\lambda=6\times10^{-7}$ m, $X=Y=Z=60$ m, $L=10$ km, $\langle\mu^2\rangle=10^{-10}$ and permitting a standard phase deviation of 0.1π, we find an aperture radius of only 0.32 mm, so that unless the path is short and the atmosphere stable, there is not much hope of keeping the carrier wave fronts coherent.

For the modulation frequency $\Omega=cK$ the situation is much better. Without assuming $y<<Y$, and assessing a permissible standard phase deviation $\Delta\Psi$ (e.g., 0.1π), we have from (9.26) the maximum aperture dimension

$$y=\sqrt{-\log\left(1-\frac{(\Delta\Psi)^2}{2\langle\mu^2\rangle K^2LX\sqrt{\pi}}\right)} \tag{9.28}$$

or alternatively, for a given aperture the maximum modulation frequency is given by

$$\Omega^2=\frac{c^2(\Delta\Psi)^2}{2\langle\mu^2\rangle LX\sqrt{\pi}\,[1-\exp(-y^2/Y^2)]} \tag{9.29}$$

with obvious modifications for z or ρ.

Formula (9.29) shows that even if the aperture diameter far exceeds the correlation distance of the turbulence ($y>>Y$) the permissible modulation bandwidth remains finite:

$$\Omega=\frac{c\Delta\Psi}{\sqrt{2\langle\mu^2\rangle LX\sqrt{\pi}}}. \tag{9.30}$$

Even under extreme conditions ($L=100$ km, $\langle\mu^2\rangle=10^{-10}$, $X=10$ m, $\Delta\Psi=0.1\pi$) this still provides for a modulation frequency of the order of 700 kc/s. It would therefore appear that in designing a laser communication system, one should start with the largest aperture technically or economically feasible (so as to reduce amplitude fluctuations to a minimum) and then compromise on modulation bandwidth versus path length. On the other hand, if the path length and bandwidth are prespecified, the aperture may be determined from (9.28). In all cases the result must be checked against the spectrum of the parasitic amplitude modulation given by (10.10).

To measure the phase fluctuations in time and space it is evidently best to use the apparatus described by Read [1964], in which the laser beam is divided by a beam splitter; one beam goes through the atmosphere and returns to the receiver after being reflected by a corner reflector, the other goes to the receiver directly and provides a standard. The two beams are reunited by a second beam splitter before striking the receiver the beam is split into two parallel beams with varying distance between them and after traversing the atmosphere, the phase difference between the two beams is measured at the receiving end of the path for various separations. These experiments should be repeated with a modulated beam and the measurements should be performed on the modulation phase; this will enable the measurements to be carried out also on long paths (Read worked with path lengths of only up to 270 m).

10. Comparison With Wave-Equation Solutions

To take account of the effects of small inhomogeneities, wave optics as mentioned in section 5 must be used. This has been done by Tatarski [1961], Chernov [1960] and others; recently the present problem has been investigated by further development of the Rytov method by Fried and Cloud [1964a, b, c] and Fried [1964]. The reader is referred to these authors for derivations; only the results will be stated here (unfortunately they are confined to isotropic turbulence).

Angle of arrival: No explicit solution by wave optics is known to the author; however, this reduces in essence to finding the geometrical shape of the perturbed phase front, the normal to which determines this angle at any point. The geometry of the perturbed phase front is considered by Fried and Cloud [1964b, c], and Fried [1964].

Spot location: Again, as far as the author is aware, no formulas equivalent to (7.10) and (7.11) have been derived by wave optics.

Beam cross section: From the wave optics point of view the "spot" can no longer be regarded as illuminated by a tube of rays; its diameter is given by the closed curve of equal mean amplitudes and therefore (as in geometric optics) tied up with the amplitude fluctuations. Since no significant energy is diverted from the beam, the beam cross section must vary inversely as the square of the amplitude.

Amplitude fluctuations: For $4L/kR^2 \ll 1$, a condition always fulfilled in the optical band, wave optics yield [Chernov 1960]

$$\left\langle \left(\log \frac{A}{A_0} \right)^2 \right\rangle = \frac{L^3}{6} \int_0^\infty [\nabla_T^2 \nabla_T^2 B]_{y=z=0} dx \qquad (10.1)$$

which is identical with (8.4); hence for the correlation function (2.4), the expressions (8.5) or (8.6) will hold as before.

Let us now set $A = \langle A \rangle (1+\alpha)$, where $\alpha(t)$ is a random variable with mean zero that will be detected by an AM demodulator. It is easily shown that

$$\frac{\langle A^2 \rangle}{\langle A \rangle^2} = 1 + \langle \alpha^2 \rangle. \qquad (10.2)$$

An AM signal orginially modulated with modulation depth m will thus be distorted by the turbulent atmosphere to

$$V = A_0[1 + m \cos \Omega t + \alpha(t)] e^{i\omega t}. \qquad (10.3)$$

The mean signal power to mean noise power is thus $m^2/2\langle \alpha^2 \rangle$; to prevent the modulation being distorted by the amplitude fluctuations, we must therefore have

$$m^2 \gg 2\langle \alpha^2 \rangle = 2 \left(\frac{\langle A^2 \rangle}{\langle A \rangle^2} - 1 \right) \qquad (10.4)$$

where we have substituted for $\langle \alpha^2 \rangle$ from (10.2). Since A is, as in section 8, lognormally distributed with $\langle \log A \rangle = 0$ (we have for simplicity set $A_0 = 1$), we have

$$\langle A^2 \rangle = e^{2\langle \log^2 A \rangle}, \qquad \langle A \rangle = e^{1/2 \langle \log^2 A \rangle}. \qquad (10.5)$$

Substituting this in (10.4) and using (8.6), we obtain after elementary manipulations the condition for the signal to be high above the noise:

$$m^2 \gg 2 \left[\exp \left(\frac{8L^3}{3R^3} \sqrt{\pi} \langle \mu^2 \rangle \right) - 1 \right]. \qquad (10.6)$$

Since $m^2 \leq 1$, the signal will be above the noise only for

$$\langle \mu^2 \rangle \frac{L^3}{R^3} < \frac{3\sqrt{\pi}}{8} \, ln \, \frac{3}{2} = 0.2696. \qquad (10.7)$$

It may be seen from (10.6) that the mean S/N ratio (more precisely the ratio of mean signal power to mean noise power) depends sensitively on $\langle \mu^2 \rangle$, i.e., on the intensity of atmospheric turbulence. For $L = 200$ km, $R = 100$ m, $\lambda = 6{,}000$ Å, and (an improbably high value of) $\langle \mu^2 \rangle = 10^{-10}$, we find (10.7) satisfied with a safety factor of over 200, but the S/N ratio is only ($20 \log_{10} m - 17.72$) dB, i.e., the modulation depth must be at least 81.5 percent for the mean signal to be above the noise; whereas for the more likely value of $\langle \mu^2 \rangle = 10^{-12}$ a modulation depth of only 30 percent will keep the signal about 8 dB above the noise.

Apart from overcoming the parasitic amplitude modulation by sufficient modulation depth, it can also be discriminated against on the basis of its spectrum. The time correlation function of the fluctuations, assuming (2.5), is found by Chernov [1960] to be

$$\langle \log A(0) \log A(\tau) \rangle = \langle \log^2 A(0) \rangle e^{-\tau^2/T^2} \qquad (10.8)$$

where $T = R/v_c$ with v_c the crosswind velocity. The spectral density of the fluctuations is the Fourier transform of this time-correlation; performing the integration, we obtain

$$F(\Omega_\alpha) = \frac{16\pi\sqrt{2}}{3} \langle \mu^2 \rangle \frac{R^3}{L^3} \exp(-\Omega_\alpha^2 R^2/4v_c^2) \qquad (10.9)$$

where Ω_α is the angular frequency of the parasitic modulation. Integrating (10.9) over Ω_α, we find (from tables of the error function) that 99 percent of the parasitic spectrum lies below the frequency

$$(\Omega_\alpha)_{99\%} = \frac{3.64 v_c}{R}. \qquad (10.10)$$

Even assuming an improbably small correlation distance of $R = 2$ mm and a hurricane windspeed of 75 mph (33 m/sec), this gives $(\Omega_\alpha)_{99\%} = 0.6$ Mc/s, which is still well below the value of an intercarrier frequency likely to be adopted in a wide-band communication system, although approaching the limit set by (9.30). Frequency discrimination thus appears more effective than fortifying the signal by increased modulation depth. For more reasonable values of R and v_c we find $(\Omega_\alpha)_{99\%}$ of the order of tens or hundreds of cycles; similar results may be derived from Tatarski's analysis [1961].

The frequency spectrum of the amplitude fluctuations does not depend on the frequency of the carrier (unless μ and R should themselves be frequency-dependent); hence the limitations imposed by turbulence-induced amplitude fluctuations in laser communications should be no worse than in the case of microwave links. On the contrary, since a significant part of, or even the entire cross section of the beam may be intercepted by the receiving aperture (which is not the case for microwave links), amplitude fluctuations may be further decreased by increasing the aperture, for in the limit the whole energy of the beam will be intercepted so that only the amplitude fluctuations correlated with the phase fluctuations (and due to "compression" and "expansion" of the carrier by the varying transit time) remain. Also, "boiling" will be averaged out at the receiver output if the aperture is large and will not be further considered.

Thus, in general, the signal degeneration due to amplitude fluctuations may be counteracted by high modulation depth, frequency discrimination and a large receiving aperture.

Phase fluctuations: For conditions in the optical frequency band, wave optics yield [Chernov, 1960,

p. 75, eq (141)]

$$\langle \psi^2 \rangle = \frac{\sqrt{\pi}}{2} \langle \mu^2 \rangle k^2 LX \tag{10.11}$$

which differs from (9.6) only by a factor of 1/2. Thus in the case of phase fluctuations, by using wave optics one gains a meager 3 dB in accuracy[5] and loses the advantage of considering anisotropic turbulence without unduly cumbersome mathematics. When, in addition the arguments following (9.10) are considered, it therefore seems that geometric optics are preferable in this case and all others following from it (sec. 9).

11. Conclusion

The statistical distributions of the angle of arrival, the spot location, the cross section, the amplitude, the carrier phase and the modulation phase of a laser beam traversing a turbulent atmosphere have been derived in terms of the space correlation function of the atmospheric index of refraction and the windspeed. The fluctuations of these quantities impose limits on the depth and bandwidth of the modulation, on the length of the path and on the aperture of the receiving apparatus; these limits are interdependent and may be individually improved at the expense of the others. Experiments are proposed to find more exact numerical values affecting the above limits. Unless these experiments yield data far off the order assumed in this report, it appears that turbulence-induced signal degeneration can in general be effectively counteracted by choosing the aperture and the band and depth of the modulation as dictated by the path length and the atmospheric parameters to keep the modulation phase constant over the aperture. On the other hand, the coherence of the carrier wave fronts is, for reasonable apertures and path lengths, entirely determined by the turbulence of the atmosphere and thus out of the designer's hands.

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[5] Even this is not certain, for the derivation of (10.11) by wave optics is marred by many assumptions, neglections and approximations.

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(Paper 69D4–498)