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Waves Circulating Around a Rigid Cylindrical Obstacle in a Compressible Plasma

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Propagation of waves in a compressible plasma, bounded by a rigid convex surface, is considered in this paper. The situation is idealized to the extent that Maxwell's equations, when combined with continuum theory of fluid dynamics, are separable. Specifically, the model is a perfectly conducting cylinder of infinite length which is excited by a uniform voltage applied to an axial slot. It is shown that both electromagnetic and electroacoustic waves are excited in the plasma. Of particular interest is an azimuthal surface wave which circulates around the cylinder with exceptionally low attenuation. In the limiting case of a cold (incompressible) plasma, the surface wave is not excited.

1. Introduction

It is now known that certain significant differences exist between waves in cold plasma and waves in warm plasma [Stix, 1962; Denisse and Delcroix, 1963]. In the latter case, it is found, that in addition to the electromagnetic-type waves, one may also excite electroacoustic-type waves by a source. In homogeneous media, the propagation characteristics of these two wave types are determined bv uncoupled wave equations. The presence of either inhomogeneities or a d-c magnetic field will produce some coupling between the electromagnetic and electroacoustic waves. A particularly simple example of coupling occurs when a semi-infinite compressible plasma is bounded by a rigid dielectric with a plane interface [Hessel et al., 1962; Seshadri, 1964; Wait, 1964a]. A plane electromagnetic wave incident from either side of the interface will produce secondary waves of both electroacoustic and electromagnetic types in the plasma. A notable exception occurs when the electric field of the incident wave is purely parallel to the plane interface, in which case no coupling occurs.

An important parameter emerging in the analyses for electroacoustic-electromagnetic effects is the ratio (u/c) where u is the velocity of sound in the electron gas and c is the velocity of light in vacuum. Because this ratio is very small (e.g., 10^{-3} or 10^{-4}), in both laboratory and ionospheric plasmas, the electroacoustic effects are usually negligible insofar as they modify the electromagnetic characteristics of the waves. However, there appear to be certain configurations where the electroacoustic effects are particularly noticeable. Such an example considered in this paper is a compressible plasma medium bounded by a convex rigid surface. A source located on the surface will excite the usual creeping-type electromagnetic waves but, because of the compressibility, a strongly trapped surface wave will also be excited. On physical grounds, one may expect that deep in the "shadow" the trapped wave component would dominate the creeping wave components. This conjecture is verified by the analysis in the present paper.

2. Formulation

To simplify matters, a rather idealized model is considered. The d-c magnetic field is neglected. The convex surface is taken to be an infinite circular cylinder of radius a immersed in the plasma. The source is a narrow axial slot which is excited throughout the length of the cylinder by a voltage V_0. This configuration is almost completely analogous to the problem of a conducting sphere excited by an annular slot. Since the formal results for the sphere problem were given in some detail in an earlier paper [Wait, 1964b], the derivation of the corresponding formal equations for the cylinder problem will be omitted. The emphasis here will be on the transformation of the solution to a meaningful and possibly useful form.

The plasma medium is regarded as a one-component electron fluid. In other words, the ions are neglected in the equation of motion, yet their presence is required to neutralize the plasma. It is also assumed that the amplitude of the electromagnetic and the acoustic waves is sufficiently small that a linearized theory is valid [Oster, 1960]. The average density of the particles is n_0 and this is regarded as a constant in the plasma region. The pressure deviation of the electrons from the mean is p and their mean velocity is $\vec{v}$. As usual, the electric and magnetic fields are denoted $\vec{E}$ and $\vec{H}$, respectively. Collisions between particles and other forms of damping are consistently neglected in this paper.

The linearized hydrodynamic equation of motion is

$$mn_0 \frac{\partial \vec{v}}{\partial t} = n_0 e\vec{E} - \nabla p, \tag{1}$$

where e and m are the charge and the mass of the electron, respectively. The equation of continuity, when combined with the equation of state, leads readily to

$$u^2 mn_0 \nabla \cdot \vec{v} = -\partial p/\partial t, \tag{2}$$

where, as mentioned, u is the velocity of sound in the electron gas.

Maxwell's equations for the electromagnetic fields in the plasma are given by

$$\nabla \times \vec{E} = -\mu_0 \partial \vec{H}/\partial t, \tag{3}$$

and

$$\nabla \times H = \epsilon_0 \partial \vec{E}/\partial t + n_0 e\vec{v}, \tag{4}$$

where μ_0 and ϵ_0 are the magnetic permeability and dielectric constant of free space, respectively. [Note that for electrons, $e = -|e|$]

3. Formal Solution

Without subsequent loss of generality, all field quantities are assumed to vary as exp $(i\omega t)$ and, thus, the derivative $\partial/\partial t$ may be replaced everywhere by $i\omega$.

The conducting cylinder is now alined to be coaxial with the z-axis of a cylindrical coordinate system. Thus, the region $\rho > a$ is the compressible plasma. Also, because of the two-dimensional nature of the problem, the derivative $\partial/\partial z$ may be set equal to zero.

With the simplifications introduced above, it is now a simple matter to show that the electric and velocity field components may be found from the scalar pressure p and the magnetic field which has only a z component. Thus, for $\rho > a$,

$$E_\rho = \frac{1}{i\epsilon\omega\rho} \frac{\partial H_z}{\partial \phi} - \frac{e}{\epsilon\omega^2 m} \frac{\partial p}{\partial \rho}, \tag{5}$$

$$E_\phi = -\frac{1}{i\epsilon\omega} \frac{\partial H_z}{\partial \rho} - \frac{e}{\epsilon\omega^2 m} \frac{\partial p}{\rho \partial \phi}, \tag{6}$$

$$v_\rho = \frac{e}{-\epsilon\omega^2 m\rho} \frac{\partial H_z}{\partial \phi} - \frac{\epsilon_0}{i\epsilon\omega n_0 m} \frac{\partial p}{\partial \rho}, \tag{7}$$

$$v_\phi = \frac{e}{\epsilon\omega^2 m} \frac{\partial H_z}{\partial \rho} - \frac{\epsilon_0}{i\epsilon\omega n_0 m\rho} \frac{\partial p}{\partial \phi}, \tag{8}$$

where

$$\frac{\epsilon}{\epsilon_0} = 1 - \frac{\omega_0^2}{\omega^2} \text{ and } \omega_0^2 = \frac{n_0 e^2}{m\epsilon_0}.$$

The p and H_z satisfy

$$\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{k_p^2}{k_e^2}\right\}\frac{p}{H_z}=0, \tag{9}$$

where $k_e^2=\mu_0\epsilon\omega^2$ and $k_p^2=(\omega/u)^2(\epsilon/\epsilon_0)$.

Solutions of (9) may be written in the form

$$p=\sum_{n=-\infty}^{+\infty}A_nH_n^{(2)}(k_p\rho)e^{-in\phi}, \tag{10}$$

and

$$H_z=\sum_{n=-\infty}^{+\infty}C_nH_n^{(2)}(k_e\rho)e^{-in\phi}, \tag{11}$$

where $H_n^{(2)}(Z)$ is the Hankel function of the second kind of order n with argument Z, while A_n and C_n are constants yet to be determined. The use of the Hankel function of the second kind assures outgoing waves at infinity. As indicated, the summations are over integer values of n so that periodicity in ϕ is assured.

On the assumption that the cylinder is acting as a rigid body, the boundary condition on the velocity is $v_\rho=0$ at $\rho=a$ and $0<\phi<2\pi$. Thus, using (7), (10), and (11), it follows that

$$\frac{A_n}{C_n}=-\frac{n_0en}{\omega\epsilon_0}\frac{H_n^{(2)}(k_ea)}{(k_pa)H_n^{(2)\prime}(k_pa)}, \tag{12}$$

where the prime indicates a derivative with respect to the argument of the Hankel function.

The coefficient A_n may be found from the prescribed condition of the tangential electric field $E_\phi(a)$ on the cylinder. Explicitly, for $\rho=a$,

$$E_\phi=E_\phi(a)=\frac{V_0}{a\Delta}\text{ for }-(\Delta/2)<\phi<(\Delta/2) \tag{13}$$

$$=0\text{ for }(\Delta/2)<|\phi|\leqq\pi,$$

where Δ is the angular opening of the slot.

A simple exercise in Fourier analysis gives the alternate form, for $\rho=a$,

$$E_\phi(a)=\frac{V_0}{2\pi a}\sum_{n=-\infty}^{+\infty}\frac{\sin(n\Delta/2)}{(n\Delta/2)}e^{-in\phi} \tag{14}$$

for the whole interval $-\pi<\phi<\pi$. As indicated, this representation is appropriate for a uniform electric field of strength $V_0/(a\Delta)$ within the slot. A more realistic variation [Wait, 1959] of the electric field within the slot is given by

$$E_\phi(a)=\frac{V_0}{a\pi[(\Delta/2)^2-\phi^2]^{1/2}}\text{ for }-\Delta/2<\phi<\Delta/2, \tag{15}$$

where V_0 is still the voltage across the slot. The corresponding Fourier representation for this variation is readily found to be

$$E_\phi(a)=\frac{V_0}{2\pi a}\sum_{n=-\infty}^{+\infty}J_0(n\Delta/2)e^{-in\phi}, \tag{16}$$

where J_0 is the Bessel function of the first kind of order zero. In general, we may write

$$E_\phi(a)=\frac{V_0}{2\pi a}\sum_{n=-\infty}^{+\infty} f(n\Delta/2)e^{-in\phi}, \tag{17}$$

where $f(x)=(\sin x)/x$ for the uniform field assumption and $f(x)=J_0(x)$ for the nonuniform field in the slot. In both cases, $f(x)$ approaches unity as $n\Delta/2$ tends to zero. In this limiting case,

$$E_\phi(a)=\frac{V_0}{2\pi a}\sum_{n=-\infty}^{+\infty} e^{-in\phi}=\frac{V_0}{a}\,\delta(\phi), \tag{18}$$

where $\delta(\phi)$ is the unit impulse function at $\phi=0$. In what follows, the function $f(\Delta n/2)$ is retained in the analysis.

Using (6), along with (10), (11), and (12), it is now a straightforward matter to apply the prescribed field condition at $\rho=a$. This process yields

$$C_n=-\frac{V_0 e\epsilon\omega}{2\pi k_e a}\,\frac{f(n\Delta/2)}{H_n^{(2)\prime}(k_e a)\,(1-\delta_n)}, \tag{19}$$

where

$$\delta_n=\frac{n^2(\omega_0/\omega)^2}{(k_e a)\,(k_p a)}\,\frac{H_n^{(2)}(k_p a)}{H_n^{(2)\prime}(k_p a)}\,\frac{H_n^{(2)}(k_e a)}{H_n^{(2)\prime}(k_e a)}. \tag{20}$$

Thus,

$$H_z=-\frac{V_0 i\epsilon\omega}{2\pi k_e a}\sum_{n=-\infty}^{+\infty}\frac{f(n\Delta/2)}{(1-\delta_n)}\,\frac{H_n^{(2)}(k_e\rho)}{H_n^{(2)\prime}(k_e a)}\,e^{-in\phi}, \tag{21}$$

and

$$p=\frac{V_0 i\epsilon n_0 e}{2\pi\epsilon_0(k_e a)\,(k_p a)}\sum_{n=-\infty}^{+\infty}\frac{nf(n\Delta/2)}{(1-\delta_n)}\,\frac{H_n^{(2)}(k_e a)}{H_n^{(2)\prime}(k_e a)}\,\frac{H_n^{(2)}(k_p\rho)}{H_n^{(2)\prime}(k_p a)}\,e^{-in\phi}. \tag{22}$$

If the plasma were to become incompressible (i.e., cold), the parameter $(k_p a)$ would approach infinity. For this limiting case, the pressure $p\rightarrow 0$ and the parameter $\delta_n\rightarrow 0$. The resulting expression for H_z is then identical to the magnetic field of an axially slotted cylinder in a dielectric medium of dielectric constant ϵ and wave number k_e [Wait, 1959].

Equations (21) and (22) are exact expressions for the magnetic field and the pressure, respectively, in the compressible plasma. Corresponding expressions for the electric and velocity field components may be obtained from (5) to (8). Some simplification is achieved in the far field when $\rho\rightarrow\infty$. Then, the Hankel functions of arguments $k_e\rho$ and $k_p\rho$ may be approximated in the manner

$$H_n^{(2)}(k_e\rho)\cong\left(\frac{2i}{\pi k_e\rho}\right)^{1/2} i^n \exp(-ik_e\rho) \tag{23}$$

and

$$H_n^{(2)}(k_p\rho)\cong\left(\frac{i}{\pi k_p\rho}\right)^{1/2} i^n \exp(-ik_p\rho). \tag{24}$$

4. Application of Watson Transformation

The subsequent discussion will deal with the behavior of the function H_z. In particular, we shall develop an expansion for H_z which should give some physical insight into the nature of the diffraction phenomena associated with the rigid cylinder.

For convenience, we write

$$H_z = \sum_{n=-\infty}^{+\infty} h_n e^{-in\phi}, \tag{25}$$

where

$$h_n = -\frac{i\epsilon\omega V_0}{2\pi x} \frac{H_n^{(2)}(x_1)}{H_n^{(2)\prime}(x)} \frac{f(n\Delta/2)}{1-\delta_n}, \tag{26}$$

with $x = k_e a$ and $x_1 = k_e \rho$.

Following the prescription of the Watson transformation [Sommerfeld, 1949; Wait, 1959], the summation in (25) is replaced by a contour integral in the complex ν plane, such that

$$H_z = \frac{3}{2i} \int_{C_1+C_2} h_\nu e^{-i\nu\phi} \frac{e^{i\nu\pi}}{\sin \nu\pi} d\nu, \tag{27}$$

where the poles of the integrand are at $\nu = n$. Here, C_1 is a straight line, just below the real axis, which runs from left to right; C_2 is a straight line, just above the real axis, which runs from right to left. It is a simple matter to verify that $2\pi i$ times the residues of the poles in (27) leads to the summation (25). Thus, the equivalence is established, provided that $C_1 + C_2$ enclosed only the poles at $\nu = n$.

It is demonstrated below that h_ν has a number of complex poles which lie in the fourth quadrant at $\nu = \nu_s$ where s indicates the number of the pole. Because of symmetry, poles also exist at $\nu = -\nu_s$ in the second quadrant. Furthermore, C_1 may be closed by an infinite semicircle in the *lower* half plane of ν without changing the value of the integral. Similarly, C_2 may be closed by an infinite semicircle in the *upper* half plane. Thus, H_z may now be replaced by a summation of the residues of the poles ν_s and $-\nu_s$. Therefore,

$$H_z = -2\pi \sum_s \frac{1}{\left[\frac{\partial}{\partial\nu}\frac{1}{h_\nu}\right]_{\nu=\nu_s}} \frac{\cos \nu_s(\pi - \phi)}{\sin \nu_s \pi}, \tag{28}$$

where the summation is over the poles ν_s which are solutions of

$$(1/h_\nu) = 0. \tag{29}$$

The latter pole-determining equation is exactly equivalent to the equation

$$\delta_\nu = 1, \tag{30}$$

where

$$\delta_\nu = \frac{\nu^2(\omega_0/\omega)^2}{xy} \frac{H_\nu^{(2)}(x)}{H_\nu^{(2)\prime}(x)} \frac{H_\nu^{(2)}(y)}{H_\nu^{(2)\prime}(y)}, \tag{31}$$

where $x=k_e a$ and $y=k_p a$. It is immediately evident that an evaluation of (31) leads to the propagation constant for the waves which "circulate" around the cylinder. Individually, they contain the factor $\exp(-i\nu_s\phi)$ for waves traveling in the positive ϕ direction or $\exp(+i\nu_s\phi)$ for waves traveling in the negative ϕ direction. If ϕ is measured in a positive sense it is convenient to write

$$\frac{\cos \nu_s(\pi-\phi)}{\sin \nu_s\pi}=ie^{-i\nu_s\phi}G_s(\phi), \tag{32}$$

where

$$G_s(\phi)=\frac{1+e^{-i2\nu_s(\pi-\phi)}}{1+e^{-2i\nu_s\pi}}. \tag{33}$$

It is evident that when $\phi \ll \pi$ and provided $-\mathrm{Im}\ \nu_s\pi \gg 1$, the function G_s may be replaced by unity. Under this condition, waves circulating around the cylinder are highly attenuated.

A study of (30) indicates that there are two sets of poles. The first set which we shall describe as the electromagnetic or EM type, while the second set is called the acoustic or A type. The EM-type modes are located where $|\nu|$ is of the order of x, while the A-type modes are located where $|\nu|$ is of the order of y. Recognizing that $y \gg x$, it is evident that a different type of approximation is to be used in the two cases. With this point in mind it is desirable to rewrite (28) in the form

$$H_z=H_z^e+H_z^a, \tag{34}$$

where

$$H_z^e=-2\pi i\sum_s \frac{G_s^e(\phi)}{\left[\frac{\partial}{\partial\nu}\frac{1}{h_\nu}\right]_{\nu=\nu_s^e}}\exp(-i\nu_s^e\phi), \tag{35a}$$

and

$$H_z^a=-2\pi i\sum_s \frac{G_s^a(\phi)}{\left[\frac{\partial}{\partial\nu}\frac{1}{h_\nu}\right]_{\nu=\nu_s^a}}\exp(-i\nu_s^a\phi). \tag{35b}$$

Here, the superscript e refers to the contribution from the EM-type poles at ν_s^e, whereas the superscript a refers to the contribution from the A-type poles at ν_s^a.

5. Pole Determinations—EM Type

For the EM-type poles, we may use the third-order approximation for the Hankel functions such that

$$H_\nu^{(2)}(x)\cong\frac{i}{\pi^{1/2}}\left(\frac{2}{x}\right)^{1/3}w(t) \tag{36}$$

and

$$H_\nu^{(2)\prime}(x)\cong-\frac{i}{\pi^{1/2}}\left(\frac{2}{x}\right)^{2/3}w'(t), \tag{37}$$

where $t=(\nu-x)(2/x)^{1/3}$, $w(t)$ is the Airy function defined by

$$w(t)=\exp(-2\pi i/3)(-\pi t/3)^{1/2}H_{1/3}^{(2)}[(2/3)(-t)^{3/2}],$$

and $w'(t)=dw(t)/dt$. This third-order approximation is valid when ν is in the vicinity of x which itself is to be much larger than one [Wait, 1962]. Within this same domain the Hankel function $H_\nu^{(2)}(y)$ may be replaced by the first term of its asymptotic series. Thus, $H_\nu^{(2)\prime}(y)\cong -iH_\nu^{(2)}(y)$. Using the above approximations for the various Hankel functions, it follows without difficulty that

$$h_\nu^e \cong \frac{V_0 i\epsilon\omega}{2\pi x}\left(\frac{x}{2}\right)^{1/3}\frac{w(t-Y)}{w'(t)}\frac{f(\nu\Delta/2)}{1-\delta_\nu}, \tag{38}$$

where

$$\delta_\nu \cong -i\left(\frac{\omega_0}{\omega}\right)^2\frac{\nu^2}{xy}\left(\frac{x}{2}\right)^{1/3}\frac{w(t)}{w'(t)}, \tag{39}$$

and

$$Y=(2/x)^{1/3}k_e z_0, \qquad z_0=\rho-a.$$

Again, in order to simplify matters, the height z_0 of the observer has been assumed to be small compared with the radius of curvature a.

Using the third-order approximations described above, the residue series representation for EM-type waves is found to be

$$H_z^e = H_z^0 e^{i3\pi/4}(\pi\phi)^{1/2}(x/2)^{1/6}\sum_s \frac{w(t_s-Y)}{w(t_s)}G_s^e(\phi)f(\nu_s^e\Delta/2)\frac{\exp\left[-it_s(x/2)^{1/3}\phi\right]}{t_s-q_e^2-q_e(2/x)^{2/3}}, \tag{40}$$

where

$$q_e=-i\left(\frac{\omega_0}{\omega}\right)^2\frac{(\nu_s^e)^2}{xy}\left(\frac{x}{2}\right)^{1/3}\cong -i\,\frac{\omega_0^2}{\omega^2}\frac{x}{y}\left(\frac{x}{2}\right)^{1/3}, \tag{41}$$

and t_s are roots of the equation

$$w'(t)+i\frac{\omega_0^2}{\omega^2}\frac{\nu^2}{xy}\left(\frac{x}{2}\right)^{1/3}w(t)=0, \tag{42}$$

and

$$H_z^0=-\left(\frac{i\epsilon\omega V_0}{2\pi}\right)\left(\frac{2\pi}{ix\phi}\right)^{1/2}e^{-ix\phi}. \tag{43}$$

The quantity H_z^0 may be identified as the field of a "delta" slot at distance $a\phi$ on an equivalent flat ground plane.

Equation (40), which is a fairly general result, may be simplified somewhat when ϕ is not near π and provided the width $a\Delta$ of the slot satisfies $k_e a\Delta \ll 1$. In this case, both $G_s^e(\phi)$ and $f(\nu_s^e\Delta/2)$ may be replaced by unity. Furthermore, under the assumption $y\gg x$, the dimensionless factor q_e has a magnitude which is small compared with unity. Thus, the factor $t_s-q_e^2-q_e(2/x)^{1/3}$ in (40) may be replaced by t_s to within a good approximation. Thus, the simplified version of (40) reads

$$H_z^e \cong \frac{\epsilon\omega V_0}{2}\left(\frac{2}{x}\right)^{1/3}\sum_s\frac{w(t_s-Y)}{t_s w(t_s)}\exp\left[-ix\phi-i(x/2)^{1/3}t_s\phi\right]. \tag{44}$$

To determine the solution of (42), an iterative method may be adopted. For the first-order iteration, ν^2 is replaced by x^2 in the factor multiplying $w(t)$. Thus, (42) may be written

$$w'(t)-q_e w(t)=0, \tag{45a}$$

where q_e is given by (41). A solution of (45a) appropriate for small values of q_e, is written

$$t_s = \tau_s + \frac{1}{\tau_s} q_e - \frac{1}{2(\tau_s)^3} q_e^2 + \text{terms containing high powers of } q_e,$$

where τ_s are solutions of

$$w'(\tau) = 0. \tag{45b}$$

Values of τ_s are well known from the theory of Airy functions [e.g., Fock, 1945]. Thus,

$$\tau_1 = 1.0188 \exp(-i\pi/3),$$
$$\tau_2 = 3.2482 \exp(-i\pi/3),$$
$$\tau_3 = 4.8201 \exp(-i\pi/3),$$

etc.

A higher order solution of (42) may be obtained by replacing ν in (42) by $x + (x/2)^{1/3} t_s$ where t_s is given by (45b). Then, a higher order solution T_s is obtained from

$$w'(T) - Q_e w(T) = 0, \tag{46}$$

where

$$Q_e = -i\left(\frac{\omega_0}{\omega}\right)^2 \frac{[x + (x/2)^{1/3} t_s]^2}{xy} \left(\frac{x}{2}\right)^{1/3},$$

where, as usual, $x = k_e a$ and $y = k_p a$.
Thus, the higher order approximation is

$$T_s = \tau_s + \frac{1}{\tau_s} Q_e - \frac{1}{2(\tau_s)^3} Q_e^2 + \ldots.$$

If one wished, further iterations could be effected. However, as a practical matter, the zero-order solution given by (45b) should be quite adequate.

In the limiting case of an incompressible (or cold) plasma where $y = k_p a \to \infty$, the parameter q_e vanishes completely. The root t_s occurring in (40) is then identical to τ_s. The residues series representation then has the same form as the groundwave field for radio propagation over a spherical earth in an atmosphere of dielectric constant ϵ [e.g., ch. V in Wait, 1962].

The preceding development indicates that the EM-type residue waves, circulating around the rigid cylinder, are only slightly modified by the compressibility of the ambient plasma.

6. Pole Determination—A Type

We now turn our attention to the contribution from the electro-acoustic or A-type poles. As indicated, the poles occur when ν is of the order of y and since $y \gg x$, the Debye approximation [Sommerfeld, 1949] for the Hankel function $H_\nu^{(2)}(x)$ is appropriate. Therefore, it is found that

$$\frac{H_\nu^{(2)}(x_1)}{H_\nu^{(2)\prime}(x)} \cong -\frac{1}{[(\nu/x)^2 - 1]^{1/2}} \exp\left[-\int_x^{x_1} [(\nu/x')^2 - 1]^{1/2}\, dx'\right], \tag{47}$$

where $x = k_e a$ and $x_1 = k_e \rho = k_e(a + z_0)$.

In view of the large magnitude of ν, (47) may be further approximated to

$$\frac{H_\nu^{(2)}(x_1)}{H_\nu^{(2)\prime}(x)} \cong -\frac{x}{\nu}\exp\left[-\frac{\nu}{x}(x_1-x)\right] = -\frac{k_e a}{\nu}\exp\left[-\nu\frac{z_0}{a}\right]. \tag{48}$$

Within the same approximation the function δ_ν, defined by (31), is represented by

$$\delta_\nu \cong -\left(\frac{\omega_0}{\omega}\right)^2 \frac{\nu}{y}\frac{H_\nu^{(2)}(y)}{H_\nu^{(2)\prime}(y)}. \tag{49}$$

If it is assumed, provisionally, that the second order or Debye approximation is valid for $H_\nu^{(2)}(y)$, (49) may be simplified to

$$\delta_\nu = +\left(\frac{\omega_0}{\omega}\right)^2 \frac{\nu}{y}\frac{1}{[(\nu/y)^2-1]^{1/2}}.$$

The pole condition $\delta_\nu = 1$ then leads to the simple result that

$$\nu = y\left(1-\frac{\omega_0^4}{\omega^4}\right)^{-1/2} = \frac{\omega a}{u}\left(1+\frac{\omega_0^2}{\omega^2}\right)^{-1/2}. \tag{50}$$

This corresponds to an unattenuated surface wave propagation along the rigid surface with a propagation constant

$$k_p[1-(\omega_0/\omega)^4]^{-1/2} \text{ or } \frac{\omega}{u}\left(1+\frac{\omega_0^2}{\omega^2}\right)^{-1/2}.$$

This is precisely what one would expect for a compressible plasma medium bounded by a plane rigid surface. The applicability of the result to a curved surface rests on the validity of the Debye approximation for the Hankel function $H_\nu^{(2)}(y)$. Certainly, when $\omega_0^4/\omega^4 \ll 1$, this situation would be violated. Therefore, it is necessary to use a third-order type approximation for $H_\nu^{(2)}(y)$.

Using the Airy function representations for the Hankel functions, as given by (36) and (37), it is now found that for the A-type poles,

$$\delta_\nu \cong \left(\frac{y}{2}\right)^{1/3}\left(\frac{\omega_0}{\omega}\right)^2 \frac{w(\hat{t})}{w'(\hat{t})}, \tag{51}$$

where

$$\hat{t} = (\nu - y)(2/y)^{1/3}.$$

The contribution from the A-type poles may now be written

$$H_z^a \cong -\frac{\epsilon\omega V_0}{y}\left(\frac{y}{2}\right)^{1/3} \sum_s H_s G_s^a(\phi) f(\nu_s^a \Delta/2)\frac{\exp[-i\nu_s^a\phi]}{\hat{t}_s - q_a^2}, \tag{52}$$

where

$$H_s = \left.\frac{H_\nu^{(2)}(x_1)}{H_\nu^{(2)}(x)}\right]_{\nu=\nu_s^a} \cong -\frac{k_e a}{\nu_s^a}\exp\left(-\nu_s^a\frac{z_0}{a}\right), \tag{53}$$

and

$$\nu_s^a = y + (y/2)^{1/3}\hat{t}_s. \tag{54}$$

Here, $\hat{t}_s$ are solutions of

$$w'(\hat{t}) - q_a w(\hat{t}) = 0, \tag{55}$$

where

$$q_a = (y/2)^{1/3}(\omega_0/\omega)^2. \tag{56}$$

The root of (55) of greatest interest is the one which has a surface wave character. Since q_a is essentially real, it may be shown that [Wait, 1964c] the surface wave root of (55) is

$$\hat{t}_0 = \mathrm{Re}\hat{t}_0 + i\mathrm{Im}\hat{t}_0, \tag{57}$$

where

$$\mathrm{Re}\hat{t}_0 \sim q_a^2 + \frac{1}{2q_a} + \frac{1}{8q_a^4} + \frac{5}{32q_a^7} + \ldots, \tag{58}$$

and

$$-\mathrm{Im}\hat{t}_0 \sim 2q_a^2 \exp\left[-\frac{4}{3}q_a^3 - 1 - \frac{7}{12q_a^3} - \frac{31}{48q_a^6} - \ldots\right]. \tag{59}$$

When q_a becomes greater than about three, the imaginary part of $\hat{t}_0$ is negligible. Then,

$$\hat{t}_0 \cong q_a^2, \tag{60}$$

which corresponds to

$$\nu_0^a \cong y + (y/2)^{1/3}\hat{t}_0 \cong y[1 + (\omega_0/\omega)^4/2]. \tag{61}$$

Provided $(\omega_0/\omega)^4$ is sufficiently small, this result agrees with (50). The height-gain function H_0 for this surface wave mode is given approximately by

$$H_0 = \exp\,[-\nu_0^a z_0/a] \cong \exp\,(-k_p z_0), \tag{62}$$

which indicates the strong trapping or "clinging" effect.

When q_a becomes small the attenuation, which is proportional to $-\mathrm{Im}\,\hat{t}_0$, is appreciable. For example, if q_a is 1.00, $-\mathrm{Im}\,\hat{t}_0 = 0.1551$, and if q_a is reduced to zero, $-\mathrm{Im}\,\hat{t}_0$ approaches 0.5094. This corresponds to the cold plasma limit.

There are other roots of (55) which are somewhat analogous to the conventional EM poles discussed in connection with (44). The height-gain functions for these modes are given by (53) with $s = 1, 2, 3, \ldots$. However, it would appear that for all practical purposes, contributions from these roots are negligible.

In (52), the function f for the A-type poles is approximately represented by

$$\begin{aligned} f(\nu_s^a\Delta/2) &\cong f(k_p a\Delta/2) \\ &\cong \frac{\sin(k_p a\Delta/2)}{(k_p a\Delta/2)} \text{ for the uniformly excited slot} \\ &\cong J_0(k_p a\Delta/2) \text{ for the nonuniformly excited slot.} \end{aligned}$$

Because $k_p a$ is a very large parameter, the function $|f|$ will be somewhat less than unity unless

$a\Delta$, the slot width, was made exceptionally small. This points up the difficulty in exciting the electroacoustic-type waves by a finite source.

In the preceding discussion, it has been assumed implicitly that $\omega_0^2/\omega^2 < 1$. Thus, the propagation constants k_e and k_p are real. However, if $\omega_0^2/\omega^2 > 1$, k_e and k_p become purely imaginary. The analysis given in this paper is applicable to this case if the factor $(1-\omega_0^2/\omega^2)^{1/2}$ is replaced everywhere by $-i(\omega_0^2/\omega^2-1)^{1/2}$. An interesting consequence is that the modes ν_s^e and ν_s^a have large imaginary parts except the surface wave mode ν_0^a which is of the acoustic type. As seen by (50), this particular mode does not become highly attenuated as ω_0^2/ω^2 increases beyond unity.

7. Concluding Remarks

From the analysis given in this paper, it is evident that the finite compressibility of the plasma will modify the characteristics of the electromagnetic field in the vicinity of the rigid cylinder. In fact, for excitation by an axial slot of infinitesimal width, a strongly trapped surface wave is excited which travels around the cylinder with exceptionally low attenuation, provided the plasma is a non-dissipative medium.[1] However, the excitation of the surface waves is quite weak compared with the excitation of the electromagnetic-type modes. (As seen by (40) and (52), the ratio of the excitation factors is approximately $(u/c)^{2/3}$ where u is the velocity of sound waves in the electron fluid.) Nevertheless, if the field is measured within a distance k_p^{-1} from the surface, the field amplitude of the surface wave may be comparable with, or even much greater than that of the electromagnetic-type modes. Of course, in the limiting case of an incompressible (or cold) plasma, the surface wave is no longer excited.

[1] This means that collisionless damping is also neglected.

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