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Angular Dependence of the Refractive Index in the Ionosphere

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The dispersion properties of an anisotropic medium, such as the ionosphere in presence of the earth's magnetic field, are conveniently represented by the variations of the refractive index with the direction of propagation. It is shown that, for a number of cases of practical interest, the variation of the index n with the angle θ takes a special form which can be expressed by a graphical construction; the result n is read on a circular scale, called the index circle, at the points where it is intersected by a straight line which depends in a simple manner on the direction of propagation θ. The same construction applies when the wave is evanescent and then it gives the imaginary value of the refractive index or extinction index.

The range of application of this construction includes the Appleton-Hartree equation, its generalization for any number of ions, the hydromagnetic waves in a medium with finite compressibility, and the various approximations corresponding to limiting or transitional cases—Aström waves, uniaxial media, and Booker's hydromagnetic approximation. It could also be applied to magnetized ferrites.

For most applications the index circle representation can take the place of the dispersion surface. The main features of this surface—the number of sheets, whether they are real or imaginary, and the direction of the asymptotes—are all displayed on the index circle. If the need arises, it is easy to trace the dispersion surface and to follow its deformations as the parameters of the medium are varied.

1. Introduction

In analyzing problems of propagation, diffraction, or radiation in a uniform anisotropic medium it is sometimes helpful to consider first the plane waves that can exist in that medium. These waves, called characteristic waves, can then be added to form solutions of the wave equations that satisfy desired boundary conditions. The characteristic waves have specific polarizations and their wave vectors cannot be arbitrary, but must have their extremities on a surface Σ, called the dispersion surface. This surface is also called the index surface because $k/k_0=n$ can be considered as the refractive index in the direction of vector $\mathbf{k}$. Another surface, related to the index surface by inversion, gives the phase velocity c/n as a function of direction.

The equation $f(n, \theta)=0$ of the surface Σ can be found for various ranges of frequency and various conditions in the ionosphere. Even when this equation is only of the second degree in n^2 the dependence of the roots on the parameters of the medium, and on the angle θ, is complicated enough to make it difficult to visualize the variations of n with the parameters. Even to obtain a sketch of the dispersion surface requires considerable work. The purpose of this paper is to point out some similarities between various dispersion relations, in particular in the angular dependence of n, and to present a graphical construction which simplifies the evaluation of the refractive index. This construction was introduced by Deschamps and Weeks [1962] for a single component magnetoplasma. It is now extended to any gyrotropic medium such as a multicomponent cold magnetoplasma or a magnetized ferrite. It is shown to apply also to the hydromagnetic waves in a compressible fluid.

2. General Form of the Dispersion Relation

We shall consider exclusively dispersion relations of the form

$$(L-L_+)(L-L_-)\cos^2\theta+(L-L_0)(L-L')\sin^2\theta=0 \tag{1}$$

where $L=n^{-2}$ and the coefficients L_+, L_-, L_0, L' are known functions of the frequency. The angle between the direction of propagation and the z-axis is θ. The variable L, when it is positive, is proportional to the square of the phase velocity in direction θ. The coefficients L_+ and L_- are the values of L for $\theta=0$, that is, for longitudinal propagation. The coefficients L_0 and L' are the values of L for $\theta=\pi/2$, that is, for transverse propagation. The reason for the peculiar notation for the coefficients will become clear when considering a gyrotropic medium. The four numbers

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L completely specify the surface Σ. Although the form of the relation (1) may seem very special, it does represent many cases of interest. Before proving this point by giving examples the construction which results from (1) will be described.

3. Description and Use of the Index-Circle

The first step is to construct a circular scale on which the values of n will be read. This is done as shown in figure 1; a linear scale in $L=n^{-2}$ is projected on the circle Γ tangent to it using the bottom point of the circle as a center of projection. Alternatively, a linear scale in $K=n^2$ can be projected from the top point, giving the same result. A graduation in n is shown in the figure, but a scale in phase velocity c/n could equally well be carried by the same circle. The points on the circle will be designated by the values of n, but sometimes by the corresponding values of K or L. The points for two reciprocal values of n are on the same vertical. The bottom point of Γ is $n=0$ and the top point $n=\infty$. Real values of n, corresponding to propagation, are represented to the right of the vertical axis. Imaginary values, corresponding to extinction,[2] are represented to the left.

Consider now the relation (1) and use $t=\sin^2\theta$ as a variable instead of θ. If the roots of (1), denoted by $L_{\mathrm{I}}(t)$ and $L_{\mathrm{II}}(t)$ are plotted on the index circle, the line D_t joining the two representative points will depend on t. We note that L_{I} and L_{II} are in one-to-one correspondance because L_{I} defines t which defines L_{II}. This correspondance is obviously algebraic and reciprocal hence it is an involution. From a well-known theorem in projective geometry it results that the line

[2] The term "extinction" is preferred to "absorption", more commonly used, because the wave in that case is evanescent; its decrease in amplitude is not due to losses but to a distributed reflection.

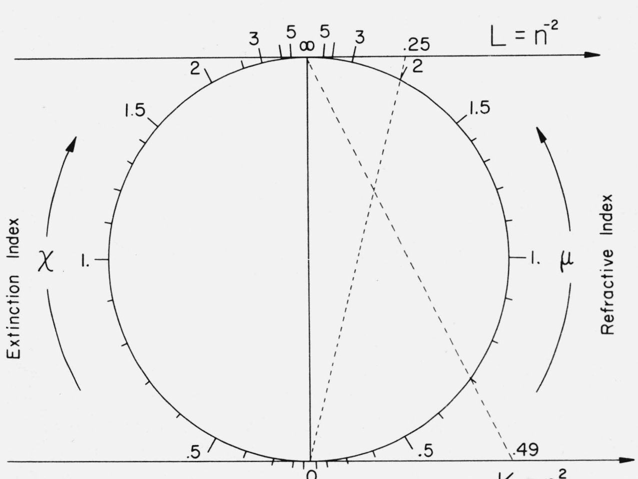

FIGURE 1. *Graduation of the index-circle.*
For illustration two projection lines are shown; from the top point to $K=0.49$ through $n=0.70$ and from the bottom point to $L=0.25$ through $n=2.00$. In this figure the refractive index is denoted by $n=\mu-j\chi$ while in the text μ denotes the permeability and χ the susceptibility.

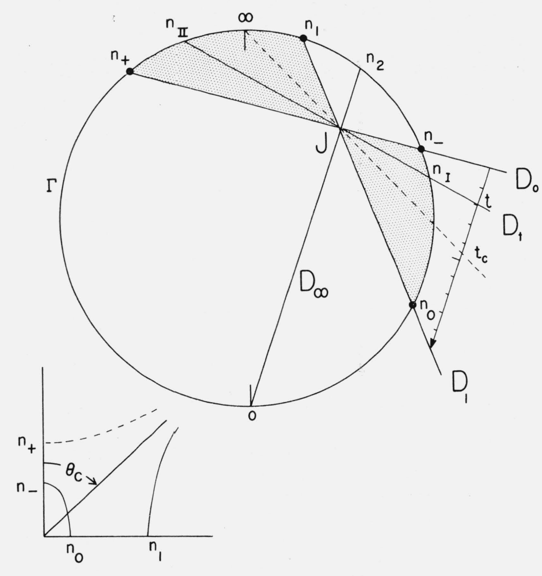

FIGURE 2. *Variation of the line* D_t *with* $t=\sin^2\theta$.
The squares of n_+, n_-, n_0, and n_1 are K_+, K_-, K_0, and K_1; they are the reciprocals of L_+, L_-, L_0, and L'. Because of the symmetry of the dispersion surface only one quarter of its meridional section is represented. The dotted line indicates an imaginary sheet of the surface corresponding to extinction.

D_t goes through a fixed point J. We know two positions of the line $D_t: D_0$ which joins n_+ to n_- and D_1 which joins n_0 to n_1. Thus the point J is known. Rewriting (1) in the form

$$(L-L_+)(L-L_-)+t(L_++L_--L_0-L')(L-L_2)=0 \quad (2)$$

shows that the roots for $t=\infty$ are $L=\infty$ and $L=L_2=(L_+L_--L_0L')/(L_++L_--L_0-L')$, corresponding to $n=0$ and $n=n_2$. These two points define D_∞, which also goes through the point J as shown in figure 2. Finally, the dependence of D_t on t is one-to-one and algebraic. Thus the intersection of D_t with a parallel to D_∞ will describe a linear scale in $t=\sin^2\theta$.

The procedure to find the refractive index in any direction is, therefore, as follows. Mark on the index circle the values of the indexes for longitudinal and transverse propagation (fig. 2). Draw the lines D_0 and D_1 which intersect at some point J. Join this point to the bottom point $n=0$ to obtain D_∞. Place a scale, linear in $\sin^2\theta$, in such a way that it be parallel to D_∞ with its extremities 0 and 1 respectively on D_0 and D_1. Join the point J to the desired value of t on that scale. This gives the line D_t which intersects the circle Γ at n_{I} and n_{II}. As the angle θ varies from 0 to $\pi/2$, the line D_t sweeps through the shaded region in figure 2. In this instance one sheet of the dispersion surface is real while the other is partly imaginary. The surface is a surface of revolution about the direction of the constant magnetic field. A meridional cross section of this surface, deduced from the index-circle, is sketched in figure 2. The angle θ_c of the asymptote is obtained by joining $n=\infty$ to J and reading t_c on the scale; here $t_c=0.47$, hence $\theta_c=43°$.

4. Multicomponent Magnetoplasma

The first application of relation (1) is to a cold magnetoplasma with an arbitrary number of components. Each type of charged particle in the plasma is specified by a number density n_i, an electric charge q_i with sign ϵ_i, and a mass m_i. Two derived quantities are the plasma frequency $\Pi_i = (n_i q_i^2 / m_i \epsilon_0)^{1/2}$ and the gyrofrequency $\Omega_i = |\mu_0 H_0 q_i / m_i|$ where H_0 is the applied magnetic field in the direction of the z-axis. Collisions are neglected and the susceptibility of the medium is the sum of the susceptibilities of the components

$$\boldsymbol{\chi} = \sum_i \boldsymbol{\chi}^{(i)} \tag{3}$$

The susceptibility matrix for the ith component of the plasma has eigenvectors $\overrightarrow{\mathbf{e}_s}$ defined by

$$\mathbf{e}_{\pm} = \frac{1}{\sqrt{2}} (\mathbf{x} \mp j\mathbf{y}), \ \mathbf{e}_0 = \mathbf{z}$$

where $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are the reference unit vectors and the subscript s takes the values $+$, $-$, and 0. The corresponding eigenvalues are

$$\chi_s^{(i)} = -\frac{\Pi_i^2}{\omega(\omega + s\epsilon_i \Omega_i)} \tag{4}$$

The dielectric constant of the medium becomes a matrix $\mathbf{K} = 1 + \boldsymbol{\chi}$ which has also the e_s for eigenvectors and

$$K_s = 1 + \sum_i \chi_s^{(i)} \tag{5}$$

for eigenvalues. Thus $\mathbf{K}$ takes the special form

$$\mathbf{K} = \begin{bmatrix} K' & jK'' & 0 \\ -jK'' & K' & 0 \\ 0 & 0 & K_0 \end{bmatrix}, \tag{6}$$

with $K_{\pm} = K' \pm K''$. This particular form of the dielectric matrix characterizes a gyrotropic medium. It is convenient to introduce the matrix $\mathbf{L} = \mathbf{K}^{-1}$ which has the same eigenvectors $\mathbf{e}_s$ as $\mathbf{K}$ with eigenvalues $L_s = K_s^{-1}$. The matrix $\mathbf{L}$ takes the form

$$\mathbf{L} = \begin{bmatrix} L' & jL'' & 0 \\ -jL & L' & 0 \\ 0 & 0 & L_0 \end{bmatrix} \tag{7}$$

with $L_{\pm} = L' \pm L''$. Let us denote by K_1 the harmonic mean of K_+ and K_- and by L_1 that of L_+ and L_-; K_1 is the reciprocal of L' and L_1 that of K'.

With these notations, the dispersion relation takes precisely the form (1). However, while in (1) the four parameters L_+, L_-, L_0, L' were arbitrary, they must now be related since $\mathbf{K}$ depends only on the three numbers K_s. The relation is that L' is the average of L_+ and L_-, hence K_1 is the harmonic mean of K_+ and K_-. A construction of K_1 and K' on the index circle

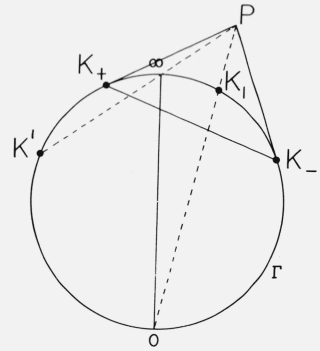

FIGURE 3. *Construction of the average* K′ *and of the harmonic mean* K_1 *of* K_+ *and* K_-.

They are obtained by projecting respectively the points $n = \infty$ and $n = 0$ through the pole P of the line K_+K_-.

when K_+ and K_- are given is shown in figure 3. Its justification is that both cross-ratios $(K_+K_-K'\infty)$ and $(K_+K_-K_10)$ are equal to -1 and this relation is invariant under the projection that defines the circular scale. In figure 2 it has been assumed that $K_1 = n_1^2$ is the harmonic mean of $K_+ = n_+^2$ and $K_- = n_-^2$; thus it is an illustration valid for a gyrotropic medium.

A magnetized ferrite is also a gyrotropic medium because its permeability matrix has the form (6). The dispersion relation takes also the form (1), but with $K_0 = 1$. The resulting surfaces Σ will not be discussed here.

5. Appleton-Hartree Dispersion Relation

For frequencies much higher than the ion plasma frequency and the ion gyrofrequency, the properties of the ionosphere are well represented by the Appleton-Hartree equation. This equation assumes that the ions are at rest and takes into account only the motion of the electrons. The susceptibility $\boldsymbol{\chi}$ is that of the electrons. In this particular case there is one more relation between the three eigenvalues K_s, and it happens because χ_0 is the harmonic mean of χ_+ and χ_-. This implies

$$\frac{2}{K_0 - 1} = \frac{1}{K_+ - 1} + \frac{1}{K_- - 1} \tag{8}$$

Thus K_+ and K_- are sufficient to determine K_0 and K_1 and therefore the whole dispersion surface. It is easy to verify that (8) implies that $L_2 = K_2 = n_2 = 1$; the line D_∞ is now a fixed line joining the point $n = 0$ to the point $n = 1$. This particular relation was used by Deschamps and Weeks [1962], and the construction of D_0 and D_1 in terms of the magneto-ionic parameters X and Y was discussed in detail.

Figure 4 shows a sketch of the index-circle representation of the dispersion relation for various pairs of values of the parameters $X = \Pi^2/\omega^2$ and $Y^2 = \Omega^2/\omega^2$. This display is limited to a region in the vicinity of the plasma resonance and the gyroresonance of the

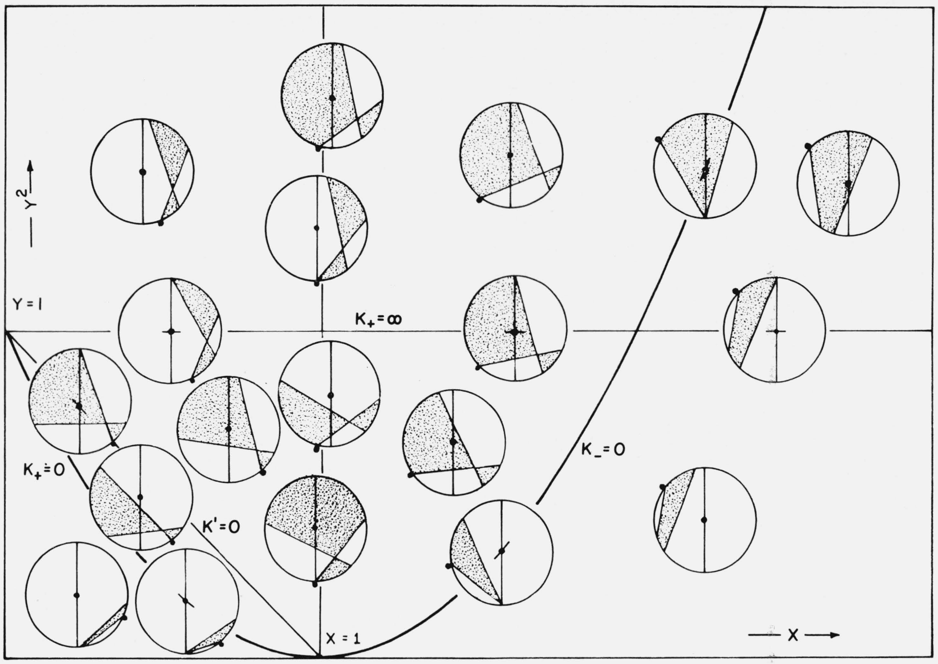

FIGURE 4. *Index-circle representation of dispersion properties in a magnetoplasma.*
The vertical and horizontal axes are $Y^2=\Omega^2/\omega^2$ and $X=\Pi^2/\omega^2$ respectively. The other lines in the $(X-Y^2)$ plane indicate where changes in the type of dispersion surface occur. The index n_0 is distinguished by a small dot.

electrons and to higher frequencies. The lines in the $(X-Y^2)$ plane where the type of dispersion surface changes are indicated. This should be compared with plots given by Clemmow [1955] and Allis [1962] for the index surface and the phase velocity surface as a function of X and Y^2.

6. Hydromagnetic Approximation

Consider a plasma made up of electrons and only one type of positive ions. The electrons are characterized by the frequencies Π_e and Ω_e, the ions by Π_i and Ω_i. Because of the neutrality of the plasma,

$$\frac{\Pi_e^2}{\Omega_e}=\frac{\Pi_i^2}{\Omega_i}. \tag{9}$$

When the frequency ω is much larger than Π_i and Ω_i the Appleton-Hartree equation is justified. When ω is much smaller than Π_e and Ω_e another simplification becomes possible [Booker, 1963]. This simplification can be justified by finding approximations for the three eigenvalues K_s. Two of these, K_+ and K_-, are

$$K_\pm=1-\frac{\Pi_e^2+\Pi_i^2}{(\omega\pm\Omega_i)(\omega\mp\Omega_e)}. \tag{10}$$

In the numerator Π_i can be neglected compared to Π_e and in the denominator ω can be neglected compared to Ω_e hence,

$$K_\pm\simeq 1-\frac{\Pi_e^2}{\mp\Omega_e(\omega\pm\Omega_i)}$$

using (9)

$$K_\pm\simeq 1+\frac{\Pi_i^2}{\Omega_i(\Omega_i\pm\omega)} \tag{11}$$

the third eigenvalue

$$K_0=1-\frac{\Pi_i^2}{\omega^2}-\frac{\Pi_e^2}{\omega^2}\simeq\infty. \tag{12}$$

Thus the electron characteristics are eliminated and the subscript i will be dropped in what follows.

Introducing

$$K_A=n_A^2=1+\frac{\Pi^2}{\Omega^2}, \tag{13}$$

we see that the cross-ratio $(K_+K_-K_A1)$ equals -1, hence the four corresponding points on the index-circle form an harmonic division. Consequently the tangents to Γ at point K_A and at point $n=1$ intersect on the line D_0 joining K_+ to K_- (fig. 5). The line D_1 joins $K_0=\infty$ to the point K_1 constructed as in figure 3. The lines D_0 and D_1 intersect at J, and D_∞ is the line from $n=0$ to J. The variation of D_t with $t=\sin^2\theta$ is obtained as in section 3 by drawing a parallel to D_∞ and marking $t=0$ and $t=1$ on D_0 and D_1 respectively.

7. Aström Waves

If the frequency ω is much lower than Ω_i, a further simplification results: $K_+=K_-=K_A$. The line D_0 becomes tangent to Γ at K_A and D_1 joins K_A to $K_0=\infty$ while D_∞ joins K_A to 0 (fig. 6). The velocity corresponding to n_A reduces, when $\Pi \gg \Omega$, to

$$\frac{c}{n_A} \simeq c\frac{\Omega}{\Pi} = H_0\left(\frac{\mu_0}{\rho_i}\right)^{1/2} \tag{14}$$

where ρ_i is the ion density $n_i m_i$. This is the ionic Alfvén velocity.

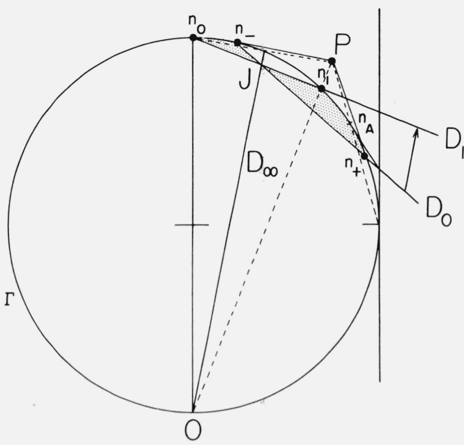

FIGURE 5. *Hydromagnetic approximation.*
The construction is for the case where Π_i and Ω_i have the same order of magnitude; n_A is neither equal to 1 nor to ∞. The frequency $\omega < \Omega_i$ hence n_- is to the right of the vertical axis, otherwise it would be to the left of it. All points labeled n with some subscript could have been labeled K with the same subscript.

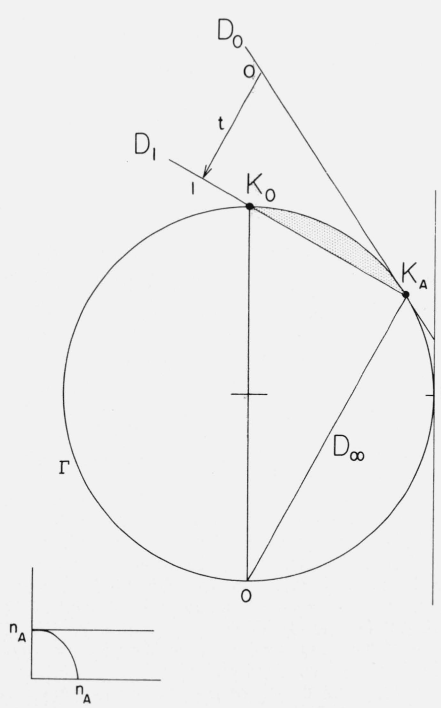

FIGURE 6. *Aström waves.*
Index-circle representation and dispersion surface when $\omega \ll \Omega$.

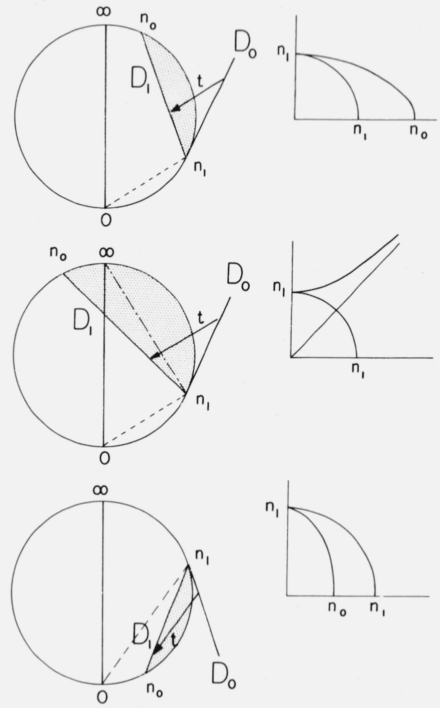

FIGURE 7. *Uniaxial medium.*
Three possible cases are shown with the corresponding dispersion surfaces: n_0 real n_1, n_0 real $< n_1$, and n_0 imaginary. The transition $n_0 = \infty$ is the case of figure 6.

The dispersion relation can be factorized. As D_t rotates about K_A one of the solutions is $n = n_A$ and the other is $n = n_A/\cos\,\theta$. This can be deduced from the general construction for D_t. The dispersion surface is made up of a sphere of radius n_A, corresponding to a wave with transverse polarization of the E vector, and of two planes $z = \pm n_A$, corresponding to a wave with transverse polarization of the H vector.

8. Uniaxial Medium

For a gyrotropic medium the dispersion relation is invariant under rotation about the z-axis and the matrix **K** has three distinct eigenvalues as explained above. When the two eigenvalues K_+ and K_- are equal, the matrix **K** becomes diagonal, $K'' = 0$, $K_1 = K'$, and the medium is said to be uniaxial. The dispersion surface decomposes (fig. 7) into a sphere $n = n_1$ and a quadric of revolution with semi-axes n_1 in the z-direction and n_0 in any direction of the xy plane. The Alfvén-Aström dispersion relation (fig. 6) is the special case where $n_0 = \infty$.

9. Hydromagnetic Wave in a Compressible Fluid

Let us consider a conducting fluid with finite compressibility in the presence of a constant magnetic field $\mathbf{H}_0$. The field quantities are the electric vector $\mathbf{E}$, the magnetic vector $\mathbf{H}$, and the velocity $\mathbf{v}$. These are related by field equations as described in any work on magnetohydrodynamics. See, for example Jackson [1962] or Stix [1962]. Parameters describing the medium are the acoustic speed $a=(\gamma p_0/\rho_0)^{1/2}$ and the Alfvén speed $A=H_0(\mu_0/\rho_0)^{1/2}$. In these formulas ρ_0 is the density, p_0 the pressure, μ_0 the permeability, and γ the ratio of specific heats. The formula for A has the same form as (14) with ρ_0, the fluid density, replacing the ion density ρ_i. To the two characteristic speeds we can associate two refractive indexes $n_a=c/a$ and $n_A=c/A$ and the numbers $L_a=n_a^{-2}$ and $L_A=n_A^{-2}$.

The dispersion relation is shown to decompose into two equations. The first one

$$L=L_A\cos^2\theta \tag{15}$$

or

$$n={}_A/\cos\theta,$$

applies to the Alfvén wave where the velocity $\mathbf{v}$ is transverse to the wave vector $\mathbf{k}$ and to the magnetic field $\mathbf{H}_0$ (shear mode). The corresponding dispersion surface is made up of the two planes $z=\pm n_A$. The second equation takes the special form (1)

$$(L-L_a)\,(L-L_A)\cos^2\theta+L(L-L_A-L_a)\sin^2\theta=0 \tag{16}$$

or

$$(L-L_a)\,(L-L_A)-L_aL_A\sin^2\theta=0. \tag{17}$$

The index-circle representation can be applied to this situation as shown in figure 8. Because the two pairs of roots L_a and L_A for $\theta=0$, and 0 and L_a+L_A for $\theta=\pi/2$ have the same average the lines D_0 and D_1 intersect on the tangent to the circle Γ at the bottom point. This tangent is in fact D_∞ because (17) shows that the roots for $\sin^2\theta=\infty$ are both $L=\infty$. Figure 8 is for $a>A$ and the resulting dispersion surface is drawn below the circle Γ. If $a>A$ the surface would have the same shape but the planes $z=\pm n_A$ would touch the open sheet instead of the closed one.

When a is very small compared to A and to c, n_a can be considered infinite and the compressibility can be neglected. The open sheet recedes to infinity and the closed sheet becomes a sphere. The situation is quite similar to that of the Åström waves shown in figure 6.

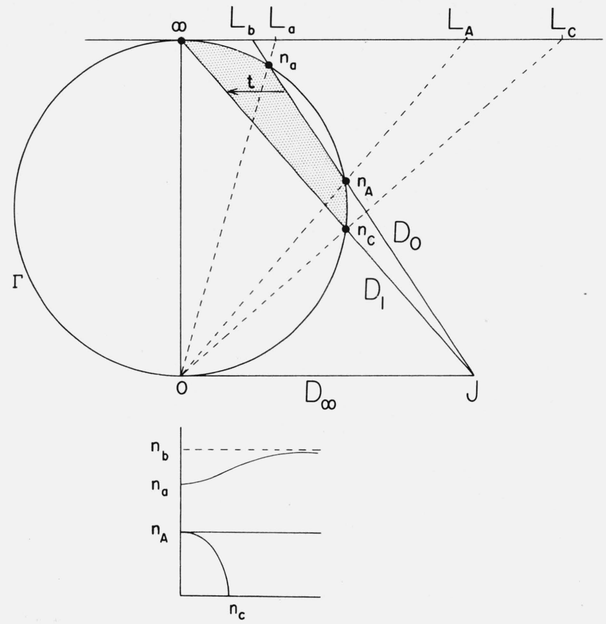

FIGURE 8. *Hydromagnetic waves in a compressible fluid.*
The horizontal asymptote to the open sheet is given by $K_b=n_b^2=n_a^2+n_A^2$ and the finite value of the transverse refractive index by $L_c=n_c^{+2}=n_a^{-2}+n_A^{-2}$.

10. Conclusions

The method described applies only to a special form of the dispersion relation. The medium must be lossless and have symmetry of revolution. The equation must be, at most, of the second degree in n^2 or decompose into factors having this property. These conditions, however, are met in a number of cases: magnetoionic theory of cold plasmas, hydromagnetic theory of compressible fluids, and theory of magnetized ferrites.

The index-circle representation gives the imaginary values of n (extinction index) as well as the real ones for any direction of the wave normal. It shows at a glance the type of the dispersion surface; whether it has one or two real sheets, or none. It shows the zeros and infinities of the refractive index, cutoff, and resonances, and gives the critical angles for which they occur.

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