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Geometrical Optics for Gyrotropic Bodies[1]

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The methods of geometrical optics are extended so that they may be applied to gyrotropic bodies. Various internal and external reflections are considered at nonparallel planar interfaces and means of determining the ray path or direction of energy flow are derived. Nonplanar geometries may be represented by the tangent planes at the various points of incidence. A method is given for computing the phases of the various fields. These values may be used to determine the reflected fields from such a gyrotropic body.

1. Introduction

A number of investigators have applied geometrical optics to anisotropic media, notably in ionospheric research [Budden, 1961] where ray tracing procedures have been carried out for anisotropic interfaces, and also of more recent studies where sources have been included [Arbel, 1960; Arbel and Felsen, 1963; Felsen, 1963a, and b] and scatterers embedded in anisotropic regions [Felsen, 1963a, b, 1964a, b; Rulf and Felsen, 1964]. While these studies have often involved special orientations of the gyrotropic axis (parallel or perpendicular to the interface) or special medium parameters (uniaxial case) to permit the derivation of explicit results for various radiation and diffraction problems, the relevance of these techniques to more general situations has been indicated.

A modified geometrical optics method also has been developed for approximating the electromagnetic scattering properties of isotropic bodies. This method has been successfully applied to a number of different cases including cylinders, spheres, and a prolate spheroid [Peters and Thomas, 1962; Thomas, 1962; Kawano and Peters, 1963; Kouyoumjian, Peters, and Thomas, 1963; Kawano and Peters, 1964]. These papers on the modified geometrical optics method have shown the manner in which diffraction of rays and caustics may be treated. In order to apply these same methods to gyrotropic media, geometrical optics for gyrotropic bodies is required. The purpose of this paper is to outline a geometrical optics technique for gyrotropic bodies. Much of the material is well known but has not been presented in concise form such that the modified geometrical optics method can be readily applied.

Wait [1961] has developed a boundary value solution for the infinite circular gyrotropic cylinder with an axial magnetic field. This is a special case in which the index of refraction is independent of the direction of propagation through the medium. The modified geometrical optics method [Lee, Peters, and Walter, 1964a, and b] has also been applied to this particular case and the results obtained are in excellent agreement with those obtained from the boundary value solution.

In all cases the static magnetic field is chosen so that it lies in the plane of incidence, i.e., the x-z plane (see fig. 1). Then this plane contains all refracted ray paths. If the static magnetic field had a y component, this would no longer be true. This does not represent a limitation on the methods but it does reduce the complications involved in obtaining numerical results.

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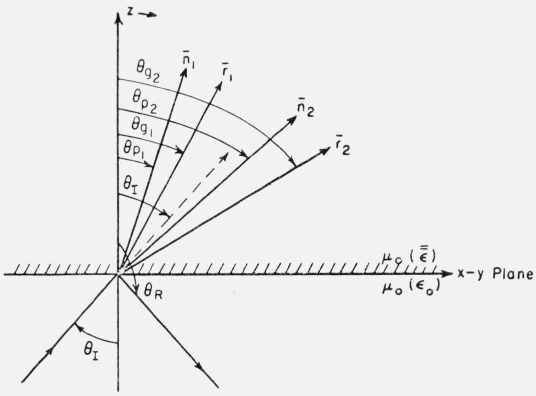

FIGURE 1. *Plane wave incident on a plasma gyrotropic interface.*

2. Ray Optics for Arbitrary Gyrotropic Boundaries

The fundamentals needed to trace a ray path through a finite, homogeneous, gyrotropic body and to compute the magnitude and phase of the emergent rays are developed in this paper by considering only planar interfaces. This might appear to be a severe restriction; however, modified geometrical optics as applied to isotropic bodies is also based on the planar interfaces. Snell's law and the various reflection and transmission coefficients are all derived under the assumption of infinite planar interfaces. Yet this modified geometrical optics method has been remarkably successful in computing the radar cross section for isotropic spheres and cylinders. This modified geometrical optics method also yields remarkably accurate radar cross sections for the plasma cylinder with an axial static magnetic field [Wait, 1961; Lee, Peters, and Walter, 1964 a, and b]. The reflection and transmission coefficients for the planar gyrotropic interfaces have been developed previously and are given in the appendix.

The same assumption made in developing the modified geometrical optics method in these previous cases is also made here for this general gyrotropic case. That assumption is that the planar interfaces represent the tangent planes of any curved body at the point of incidence and the point of reflection. The ray technique for finite homogeneous gyrotropic bodies will now be summarized.

A ray is incident at the origin as shown in figure 1. The value of S is obtained from the angle of incidence, θ_I, as $S=\sin\theta_I$ and $C=\cos\theta_I$, where the incident plane wave is given by

$$U^I=U_0e^{-jk(Sx+Cz)}. \tag{1}$$

The parameters of the gyrotropic plasma medium are given by X and $\bar{Y}$ which are defined by

$$X=\frac{\omega_N^2}{\omega^2}=\frac{Ne^2}{\epsilon_0 m\omega^2} \tag{2}$$

and

$$\bar{Y}=\frac{\bar{\omega}_H}{\omega}=\frac{\mu_0 e\bar{H}_0}{m\omega}=Y(l_1\hat{\imath}+l_3\hat{k}), \tag{3}$$

where

ω_H is the cyclotron frequency, (4)

ω_N is the plasma frequency, (5)

ϵ_0 is the permittivity in free space,

μ_0 is the permeability in free space,
N is the number of electrons per unit volume,
e is the charge of electron,
m is the mass of electron,
H_0 is the applied static magnetic field,
l_1, l_2, l_3 are the direction cosines of the static magnetic field,

and

$\hat{i}$, $\hat{k}$ are the unit vectors for the coordinate system of figure 1.

These represent the physical parameters of the gyrotropic body to be treated. If that body has a curved surface it should be noted that S is a function of the point at which the ray enters the body for a particular plane wave incidence, or conversely that the coordinate system of figure 1 is always chosen so that the z-axis is normal to the surface at the point of incidence which is the origin. Thus the values of S, C, l_1, l_2, and l_3 are all functions of this angle of incidence, θ_I.

Once θ_I is found, values of S, C, l_1, l_2, and l_3 are readily obtained. These parameters uniquely determine the values of the coefficients α, β, γ, δ, and ϵ of Booker's quartic equation, which is given by

$$F(q)=\alpha q^4+\beta q^3+\gamma q^2+\delta q+\epsilon=0, \tag{6}$$

where the coefficients are functions of direction, θ_2. For the lossless case, the collision frequency is negligible, and

$$\alpha=(1-Y^2)+X(Y_z^2-1)=(1-Y^2)+X(l_1^2Y^2-1), \tag{7}$$

$$\beta=2XSY_zY_x=2l_1l_3SXY^2, \tag{8}$$

$$\gamma=-2(1-X)(C^2-X)+2Y^2(C^2-X)+X[Y^2-CY_z^2+S^2Y_x^2]$$

$$=-2(1-X)(C^2-X)+2Y^2(C^2-X)+XY^2[1-C^2l_3^2+S^2l_1^2], \tag{9}$$

$$\delta=-2C^2XSY_zY_x=-2l_1l_3SC^2XY^2, \tag{10}$$

and

$$\epsilon=(1-X)(C^2-X^2)^2-C^2Y^2(C^2-X)-C^2XS_1^2Y_x^2$$

$$=(1-X)(C^2-X)^2-C^2Y^2(C^2-X)-C^2S_1l_1^2XY^2. \tag{11}$$

Similar equations have been derived by [Johler and Walters, 1960] without neglecting the loss mechanism.

For the present case l_2 is set equal to zero to maintain the refracted angle in the plane of incidence. This is not an essential step but simply reduces the complexity of the solution. The more general case of $l_2\neq 0$ would follow the procedures given without any complication.

Once, α, β, γ, δ, and ϵ are obtained, Booker's quartic equation, (6), may be factored and the four values of q obtained. The two roots, q_1 and q_2, which make the signs of ray path angles θ_{g_1} and θ_{g_2} the same as θ_I are the ones being sought, since they represent waves traveling in the positive z-direction.

The angle θ_g is given by

$$\tan\theta_g = \frac{\dfrac{\partial\beta}{\partial S}q^3 + \dfrac{\partial\gamma}{\partial S}q^2 + \dfrac{\partial\delta}{\partial S}q + \dfrac{\partial\epsilon}{\partial S}}{\dfrac{\partial F(q)}{\partial q}}. \tag{12}$$

These angles are the directions of two ray paths associated with q_1 and q_2, respectively.

The angles θ_{p_1} and θ_{p_2} of the refractive index n will also be useful in the following calculations. They are given by

$$\tan\theta_p = \frac{S}{q}, \tag{13}$$

using the respective values q_1 and q_2.

The relative phase of the ray at any point r is given by

$$\phi = S_x + qz = \bar{n}\cdot\bar{r} = nr\cos(\theta_g - \theta_p), \tag{14}$$

and

$$n = \frac{S}{\sin\theta_p}. \tag{15}$$

It is necessary to determine this relative phase since the phasor sum of the various scattered field components is to be computed to find the total scattered field.

To specify the magnitude of the ray at the intersection point on the second interface, it is necessary to compute the transmission coefficients at the first interface. These are given by (A–1) and (A–2). One may also compute the reflection coefficient at the first interface at this time which would be associated with the directly reflected ray.

The fields associated with the two rays at their point of intersection with the second interface (shown in fig. 2) are given by

$$U^T_{(1A)} = U_{0||}T_1 e^{-jk[n_1 r_1 \cos(\theta_{p_1} - \theta_{g_1})]} \tag{16}$$

and

$$U^T_{(2A)} = U_{0||}T_2 e^{-jk[n_2 r_2 \cos(\theta_{p_2} - \theta_{g_2})]},$$

where r_1 and r_2 are the lengths of the ray paths. The directly reflected fields at the first interface are given by

$${}_{||}U^R_{||}(A) = U_{0||}R_{||0}e^{-jk(Sx - Cz)} \tag{17}$$

and

$${}_{||}U^R_{\perp}(A) = U_{0||}R_{\perp 0}e^{-jk(Sx - Cz)}.$$

It is now necessary to determine a new set of parameters associated with the transmission and reflection mechanism at the second interface. The first step is to transform the coordinate system such that the z_1' axis is now normal to this second interface, as illustrated by x_1', y_1', z_1',

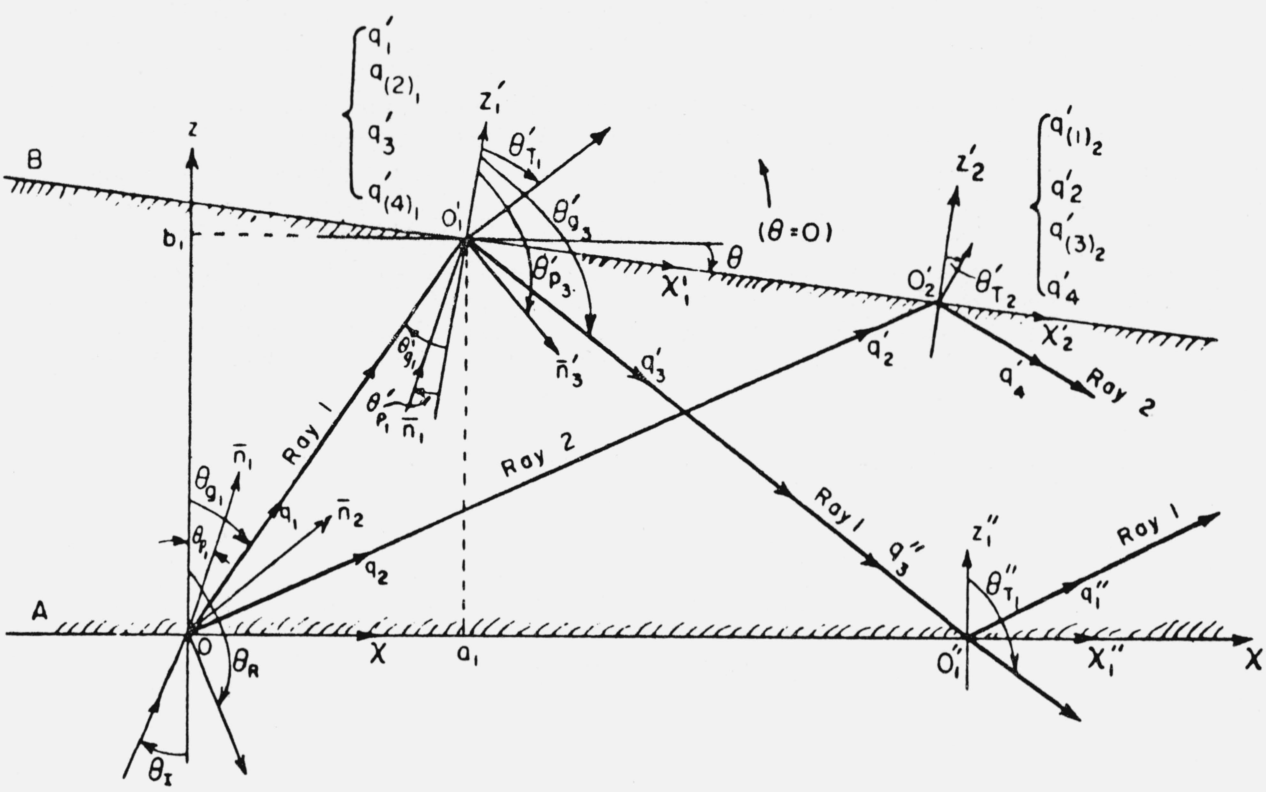

FIGURE 2. *Ray tracing for nonparallel interfaces.*

coordinate system for ray 1 and shown in figure 2. The values q_1' and S_1' of the ray incident on this second interface in this primed coordinate system are [Lee, Peters, and Walters, 1964 a and b]

$$q' = q\frac{\cos(\theta_p + \theta)}{\cos\theta_p} \tag{18}$$

and

$$S' = S\frac{\sin(\theta_p + \theta)}{\sin\theta_p}, \tag{19}$$

The subscript (1) on the coordinate axes refers to the system associated with q_1. A second coordinate system, designated by x_2', y_2', z_2', is associated with q_2 for ray 2. This is necessary because the phase angle θ_p differs for the two ray paths r_1 and r_2.

The value $C_1'^2 = 1 - S_1'^2$ and the values of the direction cosines l_1' and l_3' of the magnetic field are readily obtained in the new coordinate system. The coefficients of Booker's quartic equation (α', β', γ', δ', ϵ') now may be obtained from (7–11).

For ray 1, these coefficients are a function of S_1' at point O_1'. Thus Booker's equation must be factored to obtain the roots q_1', $q_{(2)1}'$, q_3', $q_{(4)1}'$. However q_1' is already known and Booker's equation reduces to the cubic equation

$$\alpha_1' q'^3 + (\beta_1' + \alpha_1' q_1') q'^2 + (\gamma_1^2 + \beta_1' q_1' + \alpha_1' q_1'^2) q' + (\delta_1' + \gamma_1' q_1' + \beta_1' q_1'^2 + \alpha_1' q_1'^3) = 0, \tag{20}$$

where α_1', β_1', γ_1', δ_1' are associated with S_1'. There are three solutions to (20) which are designated by $q_{(2)1}'$, q_3', and $q_{(4)1}'$. It has been shown [Lee, 1963] that the desired value of q_3' should be chosen so that if $q_1' > q_{(2)1}'$, then $q_3' < q_{(4)1}'$, when the associated pair (q_1', q_3') is being sought. Now (for ray 1) $\theta_{g_3'}$, shown in figure 2, is found from (12), and $\theta_{p_3'}$ from (13). The index of refraction, n_3', is found from (15). The same technique can be applied to find q_4' from q_2' at point O_2' for ray 2.

The point of intersection of the ray path with the next interface may then be found for the geometry being treated. In the example of figure 2, this is the original interface. Calculation of phase at this intersection follows the method described above. The reflection and transmission coefficients at this boundary, i.e., at the point O' of figure 2 are given by (A–3) and (A–4). At point O_1', q_1', q_3', and $q_{(4)1}'$ are used to calculate ${}_1'R_3'$, ${}_1'T_{||}$, and ${}_1'T_{\perp}$. At point O_2', q_2', q_4', and $q_{(3)2}'$ are used to calculate ${}_2'R_4'$, ${}_2'T_{||}$, and ${}_2'T_{\perp}$.

Thus only reflection coefficients ${}_1'R_3'$ and ${}_2'R_4'$ are considered in the following treatment. For cases where ${}_1'R_{(4)1}'$ and ${}_2'R_{(3)2}'$ are not negligible the additional computation [Lee, 1963] would parallel those used in this paper. Reflections described by coefficients ${}_1'R_{(4)1}'$ and ${}_2'R_{(3)2}'$ are interpreted as coupling between waves.

The fields transmitted into free space at the second interface are given by

$$_{||}U^T_{||(1B)} = U_{0||}T_1\,{}_1'T_{||}e^{-j\phi_1}e^{-jk(S_1'x_1'+C_1'z_1')}, \tag{21}$$

$$_{||}U^T_{\perp(1B)} = U_{0||}T_1\,{}_1'T_{\perp}e^{-j\phi_1}e^{-jk(S_1'x_1'+C_1'z_1')},$$

$$_{||}U^T_{||(2B)} = U_{0||}T_2\,{}_2'T_{||}e^{-j\phi_2}e^{-jk(S_2'x_2'+C_2'z_2')},$$

and

$$_{||}U^T_{\perp(2B)} = U_{0||}T_2\,{}_2'T_{\perp}e^{-j\phi_2}e^{-jk(S_2'x_2'+C_2'z_2')},$$

where

$$\phi_1 = kn_1r_1\cos(\theta_{p_1}-\theta_{g_1});\quad \phi_2 = kn_2r_2\cos(\theta_{p_2}-\theta_{g_2})$$

and all phases are referenced to the original origin of coordinates at point O, i.e., $x=y=z=0$.

The fields of the ray reflected back to the gyrotropic medium to the point O_1'' and O_2'' are

$$U^R_{(1B)} = U_{0||}T_1'\,{}_1'R_3'e^{-j(\phi_1+\phi_1')} \tag{22}$$

and

$$U^R_{(2B)} = U_{0||}T_2\,{}_2'R_4'e^{-j(\phi_2+\phi_2')},$$

where

$$\phi_1' = kn_3'r_3'\cos(\theta_{p_3}'-\theta_{g_3}') \text{ and } \phi_2' = kn_4'r_4'\cos(\theta_{p_4}'-\theta_{g_4}'),$$

and all phases are referenced to the original origin O.

The use of ray optics for the gyrotropic body has considered a ray incident upon the body from the external medium and followed it completely through one internal reflection. All of the fields associated with this case have been given. Any additional internal reflections may be treated simply by repeating these same steps. In addition, any coupling terms may also be readily included in any case where coupling becomes significant.

Any changes in amplitude introduced by diverging ray systems, i.e., spatial attenuation, have been neglected. However, this problem can be handled by the modified geometrical optics method.

3. Conclusions

A modified geometrical optics method has been described for determining the scattered fields of gyrotropic bodies. This method makes use of a ray optics technique developed for gyrotropic media. The general methods given are applicable to any anisotropic body. Other known techniques cannot be applied to find these scattered fields.

4. Appendix. Reflection and Transmission Coefficients at Planar "Free-Space Gyrotropic" Interfaces

The coordinate system used in this appendix is chosen so that $S_1=S$ and $S_2=0$. There are four possible waves in the gyrotropic medium; paired as upgoing waves, designated by subscripts (1) and (2), and downgoing waves, designated by subscripts (3) and (4). Thus a wave (1) or (2) in the gyrotropic medium may be reflected as wave (3) or (4), the reflection coefficient of which is designated ${}_1R_3$, ${}_1R_4$, ${}_2R_3$, or ${}_2R_4$, etc.

4.1. Wave Incident From Free Space Up to Gyrotropic Medium

For parallel polarized plane wave incidence ($E_y^I=0$)

$$
\begin{aligned}
{}_{\|}T_1 &= \frac{E_{z_1}^T}{\eta H_y^I} = C\frac{E_{z_1}^T}{E_x^I} = \frac{2C}{D_1}\left(\pi_{y_2} - \frac{1}{C}\eta_{x_2}\right), \\
{}_{\|}T_2 &= \frac{E_{z_2}^I}{\eta H_y^I} = C\frac{E_{z_2}^T}{E_x^I} = -\frac{2C}{D_1}\left(\pi_{y_1} - \frac{1}{C}\eta_{x_1}\right),
\end{aligned}
\tag{A-1}
$$

$$
{}_{\|}R_{\|} = \frac{H_y^R}{H_y^I} = \frac{E_x^R}{E_x^I} = \frac{1}{D_1}\Big[(\eta_{x_1}\eta_{y_2} - \eta_{y_1}\eta_{x_2}) + \frac{1}{C}(\pi_{x_1}\eta_{x_2} - \eta_{x_1}\pi_{x_2})
$$

$$
-C(\pi_{y_1}\eta_{y_2} - \eta_{y_1}\pi_{y_2}) - (\pi_{x_1}\pi_{y_2} - \pi_{y_1}\pi_{x_2})\Big],
$$

and

$$
{}_{\|}\mathrm{R}_{\perp} = \frac{E_y^R}{\eta_0 H_y^I} = -\frac{2}{D_1}\left[\pi_{y_1}\eta_{x_2} - \eta_{x_1}\pi_{y_2}\right].
$$

where

$$
\pi_x = \frac{-(Sq+M_{xz})(C^2-q^2+M_{yy})+M_{xy}M_{yz}}{(1-q^2+M_{xx})(C^2-q^2+M_{yy})-M_{xy}M_{yx}},
$$

$$
\pi_y = \frac{-(1-q^2+M_{xx})M_{yz}+(S_q+M_{xz})M_{yz}}{(1-q^2+M_{xx})(C^2-q^2+M_{yy})-M_{xy}M_{yx}},
$$

$$
\eta_x = S_2 - q\pi_y,
$$

$$
\eta_y = -S_1 + q\pi_y,
$$

$$
\eta_z = -S_2\pi_x + S_1\pi_z,
$$

and the numerical subscript is the subscript for the particular q involved.

For perpendicular polarized plane wave incidence ($E_x^I=0$),

$$
{}_{\perp}T_1 = \frac{E_{z_1}^T}{E_y^I} = -\frac{2}{D_1}(\pi_{x_2} - C\eta_{y_2}).
$$

$$
{}_{\perp}T_2 = \frac{E_{z_2}^T}{E_y^I} = \frac{2}{D_1}(\pi_{x_1} + C\eta_{y_1}), \tag{A-2}
$$

$$
{}_{\perp}R_{\perp} = \frac{E_y^R}{E_y^I} = \frac{1}{D_1}\Big[-(\eta_{x_1}\eta_{y_2} - \eta_{y_1}\eta_{x_2}) + \frac{1}{C}(\pi_{x_1}\eta_{x_2} - \eta_{x_1}\pi_{x_2})
$$

$$
- C(\pi_{y_1}\eta_{y_2} - \eta_{y_1}\pi_{y_2}) + (\pi_{x_1}\pi_{y_2} - \pi_{y_1}\pi_{x_2})\Big],
$$

and

$$
{}_{\perp}R_{||} = \frac{\eta_0 H_y^R}{E_y^I} = \frac{2}{D_1}(\pi_{x_1}\eta_{y_2} - \eta_{y_1}\pi_{x_2}),
$$

where

$$
D_1 = (\eta_{x_1}\eta_{y_2} - \eta_{y_1}\eta_{x_2}) - \frac{1}{C}(\pi_{x_1}\eta_{x_2} - \eta_{x_1}\pi_{x_2}) - C(\pi_{y_1}\eta_{y_2} - \eta_{y_1}\pi_{y_2}) + (\pi_{x_1}\pi_{y_2} - \pi_{y_1}\pi_{x_2}).
$$

Transmission into the gyrotropic medium yields coefficients of the form ${}_{||}T_1$, ${}_{||}T_2$, where the subscript $_{||}$ designates the polarization of the incident wave and (1) or (2) designates one of the upgoing waves. Conventional definitions of polarization are used so that subscript $_{||}$ or $_{\perp}$ means that the E vector is "parallel to" or is "perpendicular to" the plane of incidence, respectively.

4.2. Wave Incident From Gyrotropic Medium Up to Free Space

For wave (1) ($E_{z_2}^I = 0$)

$$
{}_1T_{||} = \frac{\eta H_y^T}{E_{z_1}^I} = \frac{1}{C}\frac{E_x^T}{E_{z_1}^I} = \frac{\frac{1}{C}}{D_2}\{\pi_{x_1}(\eta_{x_3}\eta_{y_4} - \eta_{y_3}\eta_{x_4}) - \eta_{x_1}(\pi_{x_3}\eta_{y_4} - \eta_{y_3}\pi_{x_4})
$$

$$
+ \eta_{y_1}(\pi_{x_3}\eta_{x_4} - \eta_{x_3}\pi_{x_4}) + C\pi_{x_1}(\pi_{y_3}\eta_{y_4} - \eta_{y_3}\pi_{y_4})
$$

$$
- C\pi_{y_1}(\pi_{x_3}\eta_{y_4} - \eta_{y_3}\pi_{x_4}) + C\eta_{y_1}(\pi_{x_3}\pi_{y_4} - \pi_{y_3}\pi_{x_4})\},
$$

$$
{}_1T_{\perp} = \frac{E_y^T}{E_{z_1}^I} = \frac{1}{D_2}\{\pi_{y_1}(\eta_{x_3}\eta_{y_4} - \eta_{y_3}\eta_{x_4}) - \eta_{x_1}(\pi_{y_3}\eta_{y_4} - \eta_{y_3}\pi_{y_4})
$$

$$
+ \eta_{y_1}(\pi_{y_3}\eta_{x_4} - \eta_{x_3}\pi_{y_4}) - \frac{1}{C}\pi_{x_1}(\pi_{y_3}\eta_{x_4} - \eta_{x_3}\pi_{y_4})
$$

$$
+ \frac{1}{C}\pi_{y_1}(\pi_{x_3}\eta_{x_4} - \eta_{x_3}\pi_{x_4}) - \frac{1}{C}\eta_{x_1}(\pi_{x_3}\pi_{y_4} - \pi_{y_3}\pi_{x_4})\},
$$

$$
{}_1R_3 = \frac{E_{z_3}^R}{E_{z_1}^I} = -\frac{1}{D_2}\Big[(\eta_{x_1}\eta_{y_4} - \eta_{y_1}\eta_{x_4}) + \frac{1}{C}(\pi_{x_1}\eta_{x_4} - \eta_{x_1}\pi_{x_4})
$$

$$
+ C(\pi_{y_1}\eta_{y_4} - \eta_{y_1}\pi_{y_4}) + (\pi_{x_1}\pi_{y_4} - \pi_{y_1}\pi_{x_4})\Big], \quad \text{(A–3)}
$$

and

$$
{}_1R_4 = \frac{E_{z_4}^R}{E_{z_1}^I} = \frac{1}{D_2}\Big[(\eta_{x_1}\eta_{y_3} - \eta_{y_1}\eta_{x_3}) + \frac{1}{C}(\pi_{x_1}\eta_{x_3} - \eta_{x_1}\pi_{x_3})
$$

$$
+ C(\pi_{y_1}\eta_{y_3} - \eta_{y_1}\pi_{y_3}) + (\pi_{x_1}\pi_{y_3} - \pi_{y_1}\pi_{x_3})\Big].
$$

For wave (2) ($E^I_{z_1}=0$)

$$
\begin{aligned}
{}_2T_{\|}=\frac{\eta H_y^T}{E^I_{z_2}}=\frac{1}{C}\frac{E_x^T}{E^I_{z_2}}=\frac{\frac{1}{C}}{D_2}\{&\pi_{x_2}(\eta_{x_3}\eta_{y_4}-\eta_{y_3}\eta_{x_4})-\eta_{x_2}(\pi_{x_3}\eta_{y_4}-\eta_{y_3}\pi_{x_4})\\
&+\eta_{y_2}(\pi_{x_3}\eta_{x_4}-\eta_{x_3}\pi_{x_4})+C\pi_{x_2}(\pi_{y_3}\eta_{y_4}-\eta_{y_3}\pi_{y_4})\\
&-C\pi_{y_2}(\pi_{x_3}\eta_{y_4}-\eta_{y_3}\pi_{x_4})+C\eta_{y_2}(\pi_{x_3}\pi_{y_4}-\pi_{y_3}\pi_{x_4})\},
\end{aligned}
$$

$$
\begin{aligned}
{}_2T_{\|}=\frac{E_y^T}{E^I_{z_2}}=\frac{1}{D_2}\{&\pi_{x_2}(\eta_{x_3}\eta_{y_4}-\eta_{y_3}\eta_{x_4})-\eta_{x_2}(\pi_{y_3}\eta_{y_4}-\eta_{y_3}\pi_{y_4})\\
&+\eta_{y_2}(\pi_{y_3}\eta_{x_4}-\eta_{x_3}\pi_{y_4})-\frac{1}{C}\pi_{x_2}(\pi_{y_3}\eta_{x_4}-\eta_{x_3}\pi_{y_4})\\
&+\frac{1}{C}\pi_{y_2}(\pi_{x_3}\eta_{x_4}-\eta_{x_3}\pi_{x_4})-\frac{1}{C}\eta_{x_2}(\pi_{x_3}\pi_{y_4}-\pi_{y_3}\pi_{x_4})\},
\end{aligned}
$$

$$
{}_2R_3=\frac{-1}{D_2}\left[(\eta_{x_2}\eta_{y_4}-\eta_{y_2}\eta_{x_4})+\frac{1}{C}(\pi_{x_2}\eta_{x_4}-\eta_{x_2}\pi_{x_4})+C(\pi_{y_2}\eta_{y_4}-\eta_{y_2}\pi_{y_4})+(\pi_{x_2}\pi_{y_4}-\pi_{y_2}\pi_{x_4})\right], \quad \text{(A-4)}
$$

and

$$
{}_2R_4=\frac{E^R_{z_4}}{E^I_{z_2}}=\frac{1}{D_2}\left[(\eta_{x_2}\eta_{y_3}-\eta_{y_2}\eta_{x_3})+\frac{1}{C}(\pi_{x_2}\eta_{x_3}-\eta_{x_2}\pi_{x_3})+C(\pi_{y_2}\eta_{y_3}-\eta_{y_2}\pi_{y_3})+(\pi_{x_2}\pi_{y_3}-\pi_{y_2}\pi_{x_3})\right],
$$

where

$$
D_2=(\eta_{x_3}\eta_{y_4}-\eta_{y_3}\eta_{x_4})+\frac{1}{C}(\pi_{x_3}\eta_{x_4}-\eta_{x_3}\pi_{x_4})+C(\pi_{y_3}\eta_{y_4}-\eta_{y_3}\pi_{y_4})+(\pi_{x_3}\pi_{y_4}-\pi_{y_3}\pi_{x_4}).
$$

The dominant reflection coefficients are ${}_1R_3$ and ${}_2R_4$. Coefficients ${}_1R_4$ and ${}_2R_3$ are usually negligible.

4.3. Wave Incident From Gyrotropic Medium Down to Free Space

For wave (3) ($E^I_{z_4}=0$)

$$
\begin{aligned}
{}_3T_{\|}=\frac{\eta H_y^T}{E^I_{z_3}}=-C\frac{E_x^T}{E^I_{z_3}}=\frac{-C}{D_3}\{&\pi_{x_3}(\eta_{x_1}\eta_{y_2}-\eta_{y_1}\eta_{x_2})-\eta_{x_3}(\pi_{x_1}\eta_{y_2}-\eta_{y_1}\pi_{x_2})\\
&+\eta_{y_3}(\pi_{x_1}\eta_{x_2}-\eta_{x_1}\pi_{x_2})-C\pi_{x_3}(\pi_{y_1}\eta_{y_2}-\eta_{y_1}\pi_{y_2})+C\pi_{y_3}(\pi_{x_1}\eta_{y_2}-\eta_{y_1}\pi_{x_2})-C\eta_{y_3}(\pi_{x_1}\pi_{y_2}-\pi_{y_1}\pi_{x_2})\},
\end{aligned}
$$

$$
\begin{aligned}
{}_3T_{\perp}=\frac{E_y^T}{E^I_{z_3}}=\frac{1}{D_3}\Big\{&\pi_{y_3}(\eta_{x_1}\eta_{y_2}-\eta_{y_1}\eta_{x_2})-\eta_{x_3}(\pi_{y_1}\eta_{y_2}-\eta_{y_1}\pi_{y_2})\\
&+\eta_{y_3}(\pi_{y_1}\eta_{x_2}-\eta_{x_1}\pi_{y_2})+\frac{1}{C}\pi_{x_3}(\pi_{y_1}\eta_{x_2}-\eta_{x_1}\pi_{y_2})\\
&-\frac{1}{C}\pi_{y_3}(\pi_{x_1}\eta_{x_2}-\eta_{x_1}\pi_{x_2})+\frac{1}{C}\eta_{x_3}(\pi_{x_1}\pi_{y_2}-\pi_{y_1}\pi_{x_2})\Big\},
\end{aligned}
$$

$$
\begin{aligned}
{}_3R_1=\frac{E^R_{z_1}}{E^I_{z_3}}=\frac{1}{D_3}\Big\{&(\eta_{x_2}\eta_{y_3}-\eta_{y_2}\eta_{x_3})-\frac{1}{C}(\pi_{x_2}\eta_{x_3}-\eta_{x_2}\pi_{x_3})\\
&-C(\pi_{y_2}\eta_{y_3}-\eta_{y_2}\pi_{y_3})+(\pi_{x_2}\pi_{y_3}-\pi_{y_2}\pi_{x_3})\Big\}, \quad \text{(A-5)}
\end{aligned}
$$

and

$$
{}_3R_2=\frac{E^R_{z_2}}{E^I_{z_3}}=\frac{1}{D_3}\left\{-(\eta_{x_1}\eta_{y_3}-\eta_{y_1}\eta_{x_3})+\frac{1}{C}(\pi_{x_1}\eta_{x_3}-\eta_{x_1}\pi_{x_3})+C(\pi_{y_1}\eta_{y_3}-\eta_{y_1}\pi_{y_3})-(\pi_{x_1}\pi_{y_3}-\pi_{y_1}\pi_{x_3})\right\}.
$$

For wave (4) ($E^I_{z_3}=0$)

$$
{}_4T_{||}=\frac{\eta H^T_y}{E^I_{z_4}}=-C\frac{E^T_x}{E^I_{z_4}}=-\frac{C}{D_3}\{\pi_{x_4}(\eta_{x_1}\eta_{y_2}-\eta_{y_1}\eta_{x_2})-\eta_{x_4}(\pi_{x_1}\eta_{y_2}-\eta_{y_1}\pi_{x_2})+\eta_{y_4}(\pi_{x_1}\eta_{x_2}-\eta_{x_1}\pi_{x_2})
$$

$$
-C\pi_{x_4}(\pi_{y_1}\eta_{y_2}-\eta_{y_1}\pi_{y_2})+C\pi_{y_4}(\pi_{x_1}\eta_{y_2}-\eta_{y_1}\pi_{x_2})-C\eta_{y_4}(\pi_{x_1}\pi_{y_2}-\pi_{y_1}\pi_{x_2})\},
$$

$$
{}_4T_{\perp}=\frac{E^T_y}{E^I_{z_4}}=\frac{1}{D_3}\left\{\pi_{y_4}(\eta_{x_1}\eta_{y_2}-\eta_{y_1}\eta_{x_2})-\eta_{x_4}(\pi_{y_1}\eta_{y_2}-\eta_{y_1}\pi_{y_2})\right.
$$

$$
+\eta_{y_4}(\pi_{y_1}\eta_{x_2}-\eta_{x_1}\pi_{y_2})+\frac{1}{C}\pi_{x_4}(\pi_{y_1}\eta_{x_2}-\eta_{x_1}\pi_{y_2})
$$

$$
\left.-\frac{1}{C}\pi_{y_4}(\pi_{x_1}\eta_{x_2}-\eta_{x_1}\pi_{x_2})+\frac{1}{C}\eta_{x_4}(\pi_{x_1}\pi_{y_2}-\pi_{y_1}\pi_{x_2})\right\},
$$

$$
{}_4R_1=\frac{E^R_{z_1}}{E^I_{z_4}}=\frac{1}{D_3}\left\{(\eta_{x_2}\eta_{y_4}-\eta_{y_2}\eta_{x_4})-\frac{1}{C}(\pi_{x_2}\eta_{x_4}-\eta_{x_2}\pi_{x_4})\right.
$$

$$
\left.-C(\pi_{y_2}\eta_{y_4}-\eta_{y_2}\pi_{y_4})+(\pi_{x_2}\pi_{y_4}-\pi_{y_2}\pi_{x_4})\right\}, \qquad \text{(A–6)}
$$

and

$$
{}_4R_2=\frac{E^R_{z_2}}{E^I_{z_4}}=\frac{1}{D_3}\left\{-(\eta_{x_1}\eta_{y_4}-\eta_{y_1}\eta_{x_4})+\frac{1}{C}(\pi_{x_1}\eta_{x_4}-\eta_{x_1}\pi_{x_4})+C(\pi_{y_1}\eta_{y_4}-\eta_{y_1}\pi_{y_4})-(\pi_{x_1}\pi_{y_4}-\pi_{y_1}\pi_{x_4})\right\},
$$

where

$$
D_3=(\eta_{x_1}\eta_{y_2}-\eta_{y_1}\eta_{x_2})-\frac{1}{C}(\pi_{x_1}\eta_{x_2}-\eta_{x_1}\pi_{x_2})-C(\pi_{y_1}\eta_{y_2}-\eta_{y_1}\pi_{y_2})+(\pi_{x_1}\pi_{y_2}-\pi_{y_1}\pi_{x_2}).
$$

In this case the dominant reflection coefficients are ${}_3R_1$ and ${}_4R_2$. The coefficients ${}_3R_2$, ${}_4R_1$ are usually negligible.

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