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VLF and LF Fields Propagating Near and Into a Rough Sea

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A heuristic theory for VLF and LF fields near and in a rough sea surface is obtained by first finding the electromagnetic field configuration in air and then calculating the undersea fields by means of the Helmholtz-Kirchoff Integral Theorem. The fields above the water are found by a succession of quasi-static approximations which depend on the observation that the scale of irregularities on the sea surface is very small compared with an EM wavelength.

The theory predicts that the configuration of the $\vec{H}$-field above the water depends on the direction of EM propagation relative to the wave crests. It also predicts that for underwater measurements made a few tens of feet below the troughs of the waves, the field variations due to one- or two-foot sea waves are averaged out; but for storm waves the phase and attenuation of the field observed underwater varies with instantaneous water height. These theoretical predictions have been confirmed experimentally.

1. Introduction

The well-known stability of LF and VLF propagation at great distances has produced an increasing number of applications to international comparison of frequency standards and to navigational aids. In this latter connection, it is important to have some estimate of local perturbations in field caused by a rough or stormy sea surface. Since at these frequencies, the attenuation of radio waves is only of the order of a few decibels per foot in sea water, it is also relevant to try to estimate the fields measured by an antenna located a few feet under the water surface.

Both the above-the-surface and subsurface aspects of this problem have received some attention [Rice, 1958; Herman, 1958; Wait, 1959] in the last few years. Only one of the works referred to, that of Wait, treats the problem of subsurface fields. Wait uses the "Leontovitch" [1948] assumptions concerning the relative magnitudes of the radius of curvature of the sea surface, the inverse of the propagation constant in sea water and the distance under the sea surface at which one wishes to calculate the fields.

There are, however, naturally occurring situations in which the Leontovitch assumptions do not hold, and in which we still may wish to calculate the fields. Our approach is to find a solution to the field problem above the sea surface on the assumption that the sea surface is perfectly conducting (to the extent that the E-field is everywhere normal to the surface) and that the water waves make the sea surface a trochoidal cylinder; then (dropping the assumption that sea water is a perfect conductor) we use the integral theorem of Helmholtz and Kirchoff to find the fields under the sea surface.

The general behavior of the fields above the sea surface can be readily ascertained from the equations which are developed for these fields. The E-field in the air is locally enhanced over the crests of the waves and diminished over the troughs, in accordance with the local curvature of the surface. Whether this effect is passed on to the H-field and (through continuity of H at the boundary) to the subsurface fields, depends on the angle between the direction of propagation and the direction of wave crests.

The behavior of the fields at some distance beneath the sea surface is not so evident from the equations derived for them. The theoretical results obtained have been evaluated for typical sea conditions at VLF. The result of these calculations is to predict that an observation made at reasonable depths under storm seas should show variations in amplitude and phase proportional to the instantaneous height of water above the point of observation. On the other hand, in calm seas (waves up to 1 or 2 ft) the effect of the waves on the field observed at depth is averaged out.

2. Trochoidal Coordinates

The shape of a periodic gravity wave on an infinitely extensive sea tends to be a trochoid [Lamb, 1945], a curve which is a generalization of a cycloid. Such curves are described by a point on the radius of a circle as the circle is rolled on a straight line; they tend to have steeper peaks and broader troughs than would be the case for a sine wave.

[1] Operated with support from the U.S. Air Force.

Actual sea waves depart from the cycloidal curves assumed in this paper in two ways: First, if the waves have sufficient amplitude so that they are cusped and breaking, the cycloidal-type trochoid assumed is only an approximation to the actual surface [Lamb, 1945]. Second, in actuality the wind excites a random spectrum of waves, so that each "wave" is actually a superposition of wavelets and no two waves are really alike. Moreover, the sea is sometimes "crossed" by waves propagating in several directions. Further, although real sea wave "crests" do stretch for distances comparable to a wavelength, they do not stretch indefinitely to produce a cylindric sea.

The willingness to ignore these differences between the real sea and the model is based on the assumption that local sea-surface conditions affect the surface fields in that same local area only. For the fields above the water, this assumption can be justified on the grounds that local surface irregularities (the water waves) would produce nonlocalized effects only if they tend to scatter the propagating radio wave. These irregularities, however, have dimensions which are so small compared with a radio wavelength in air [2] that little such scattering can occur.

Beneath the sea, the assumption of local influence is justified by the high attenuation of radio waves propagating through the water. The field beneath the sea surface can be thought of as due to a continuous distribution of sources along the surface (the strength of the field to be calculated by a Green's Theorem approach or the Helmholtz-Kirchoff Integral Theorem). The effect of a given source area diminishes exponentially with the distance between the surface source and a given subsurface observation point. Since this attenuation is of the order of 1 dB per foot at 15 kc/s, the subsurface field is for all practical purposes determined by the surface field in the local area above the subsurface point.

Thus, if we find the infinite trochoidal cylinder which matches the *local* sea conditions we are concerned with, we may trust that the failure of the match in other areas of the sea surface will not produce local fields noticeably different from these which we calculate.

Since the boundary conditions we must satisfy are applied on the surface of a trochoidal cylinder, it is convenient to use trochoidal coordinates in solving the field problem. We generate these coordinates as follows:

$$\text{let } \hat{z}=x+jy \qquad \hat{w}=u+jv, \qquad j=\sqrt{-1} \tag{2.1}$$

$\hat{z}$ and $\hat{w}$ represent points in two Cartesian planes, the abscissa and ordinate being given, respectively, by x and y in the $\hat{z}$ plane and u and v in the $\hat{w}$ plane.

[2] At 15 kc/s the electromagnetic wavelength in air is 20 km. Ocean wavelengths seldom exceed 300 m. Ocean wave heights (peak to trough) seldom exceed 10 m.

Further, let

$$\hat{w}=\hat{z}+ja \ \exp(j\hat{z}). \tag{2.2}$$

Equation (2.2) defines a transformation from the $\hat{z}$ plane to the $\hat{w}$ plane. All straight lines parallel to the x axis in the $\hat{z}$ plane are transformed or mapped by (2.2) into trochoids in the $\hat{w}$ plane (whether or not $ae^{-y}<1$). The mapping of the $\hat{z}$ and $\hat{w}$ is also conformal for values of $\hat{z}$ such that ae^{-y} is less than unity. We now define trochoidal coordinates s and n, *in the w plane*, as follows:

$$s=x \qquad n=y. \tag{2.3}$$

The curves, $n=$const are trochoids, and furthermore if $y>\ln a$,[3] the curves $s=$constant intersect the curves $n=$constant at right angles (because the transformation (2.2) is conformal).

We may combine (2.1) and (2.2) to obtain

$$u=x-ae^{-y}\sin x$$

$$v=y+ae^{-y}\cos x \tag{2.4}$$

and using (2.3) we get

$$u=s-ae^{-n}\sin s$$

$$v=n+ae^{-n}\cos s. \tag{2.5}$$

The scale factor h (the same for n as for s because of conformality) may be found from

$$h=\left[\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial v}{\partial s}\right)^2\right]^{1/2} \tag{2.6}$$

so that

$$h=(1-2ae^{-n}\cos s+a^2e^{-2n})^{1/2}. \tag{2.7}$$

We need only add a third coordinate axis normal to the u, v plane to change trochoidal coordinates to trochoidal cylindrical coordinates. If the third coordinate is denoted by z, then u, v, and z are the Cartesian coordinates of the three-dimensional space. The scale factor associated with the third coordinate is, of course, unity. We will denote the unit vectors in the s, n, and z-directions by $\vec{e}_s$, $\vec{e}_n$, and $\vec{e}_z$.

3. First Solution in Air

A radio wave propagating over the surface of an unrippled sea at or below broadcast frequencies is for all practical purposes vertically polarized. To find the field outside the sea, we assume the sea to be a perfect conductor with the surface having the form of a trochoidal cylinder.

[3] We use ln to denote the natural logarithm.

The conductivity of the sea water is so large (4 mho-meter^{-1} at standard temperature and salinity) compared with the susceptance (10^{-5} mho-meter^{-1} at about 150 kc/s) of the air above it, that what little loss power that enters the sea propagates essentially straight down. If the sea surface is disturbed by waves, the EM field above it will be correspondingly perturbed. Since the conductivity of even a few inches depth of water is high compared with either the susceptance or the Y_0 of the air immediately above it, we can expect that the effect of the disturbed sea surface on the E-fields above it will be nearly the same as that of a perfect conductor—the external E-fields must remain perpendicular to the sea surface.

In the case of a static E-field in cylindric coordinates, the problem is one of finding a scalar potential φ which satisfies Laplace's equation

$$\nabla^2\varphi=0$$

and for which φ is constant on the boundary.

In coordinates derived from a conformal transformation, the two-dimensional Laplacian ∇^2 is [Morse and Feshbach, 1953]

$$\nabla^2\varphi=\frac{1}{h^2}\left(\frac{\partial^2\varphi}{\partial s^2}+\frac{\partial^2\varphi}{\partial n^2}\right). \tag{3.1}$$

Obviously, then, if $\varphi=E_0 n$, φ satisfies Laplace's equation and

$$\vec{E}(s,n)=-\nabla\varphi=-\frac{E_0}{h}\vec{e}_n \tag{3.2}$$

is the corresponding electric field.

The equation we are really interested in solving is the electromagnetic wave equation. When the time variation is separated out of this partial differential equation, the result is called the Helmholtz equation. If $\vec{F}$ is a vector function of the spacial coordinates, the Helmholtz equation has the form

$$\nabla^2\vec{F}+k^2\vec{F}=0 \tag{3.3}$$

where for good dielectrics $k^2=\omega^2\mu\epsilon$, and for good conductors $k^2=-j\omega\mu\sigma$. Here ω is the radian frequency of the signal; μ, ϵ and σ are the (magnetic) permeability, permittivity (electric), and conductivity of the relevant medium, respectively.

It is a well-known property [Ramo and Whinnery, 1953] of cylindrical systems that if $\vec{E}(\xi_1, \xi_2)$ is a static two-dimensional field solution (i.e., one derivable as the gradient of a potential satisfying Laplace's equation) in the plane perpendicular to the cylinder axis (ξ_1 and ξ_2 being the coordinates of this plane), then

$$\vec{E}(\xi_1, \xi_2)e^{-jkz}$$

satisfies the Helmholtz equation (3.3). One can now state that

$$\vec{E}=\frac{-E_0}{h}e^{-jkz}\vec{e}_n \tag{3.4}$$

satisfies (3.3) from (3.1), (3.2) and the discussion between them. This field expression represents a TEM wave traveling along the cylinder axis. In the terms of the original physical problem, it represents a TEM wave traveling *parallel* to the sea wave crests. If we examine the form of h [see (2.7)], we note that the field becomes uniform as n becomes infinite. This is what we would expect from physical reasoning, as water wave action at the surface of the sea should not affect the fields at arbitrarily large distances from it. We also note that the rate of decay of this "surface effect" has nothing to do with the electromagnetic wave number, k. The way in which (3.4) has derived from a *static* field solution makes it obvious why this should be so.

We also note that

$$-j\omega\mu\vec{H}=\text{curl }\vec{E}$$

hence

$$\vec{H}=\sqrt{\frac{\epsilon}{\mu}}\frac{E_0}{h}e^{-jkz}\vec{e}_s. \tag{3.5}$$

4. Second Solution in Air

The results of section 3 apply only to TEM waves, those propagating parallel to the wave crests. We wish now to examine cases of EM propagation at an angle with respect to the sea wave crests; in particular, propagation perpendicular to the crests. This problem is inherently more difficult than the previous one, in that it requires manipulation of solutions of the *vector* Helmholtz equation in curvilinear coordinates. Helmholtz' equation for a vector $\vec{E}$ and a scalar φ are symbolically the same.

$$\nabla^2\varphi+k^2\varphi=0 \tag{4.1}$$

$$\nabla^2\vec{E}+k^2\vec{E}=0 \tag{4.2}$$

but the interpretations that must be given to the operators ∇^2 are (in curvilinear coordinates) different;[4] for this reason (4.2) may be solved by separation techniques in only six coordinate systems, whereas (4.1) can be so solved in eleven. With respect to other coordinate systems (including those appropriate to the present problem), the difficulty is not so much in finding solutions to the equations, as in finding solutions that will also satisfy the boundary conditions.

It is much easier to find solutions to (4.1) than to (4.2), whether or not the equations separate. If φ is a solution to the scalar wave equation (4.1) in a cylindrical coordinate system, then a set of orthog-

[4] $\nabla^2\varphi$ is div grad φ for φ a scalar; $\nabla^2\vec{E}$ is grad div $\vec{E}$$-$curl curl $\vec{E}$ for $\vec{E}$ a vector [Moon and Spencer, 1961].

onal solutions to (4.2), $\vec{L}$, $\vec{M}$, and $\vec{N}$, may be constructed from φ as follows:

$$\vec{L}=\text{grad}\,\varphi \tag{4.3}$$

$$\vec{M}=\text{curl}\,(\varphi\vec{e}_z) \tag{4.4}$$

$$\vec{N}=\text{curl curl}\,(\varphi\vec{e}_z) \tag{4.5}$$

in which $\vec{e}_z$, the axial unit vector, is parallel to the surface of the water, $\vec{L}$ is a "longitudinal" wave, and $\vec{M}$ and $\vec{N}$ are called "transverse" waves. In particular, the tangential component of $\vec{M}$ can be made to vanish along a boundary parallel to $\vec{e}_z$; it is therefore the appropriate wave by which to describe the $\vec{E}$-field outside the water. Thus, the problem of finding the vector field $\vec{E}$ reduces to that of finding an appropriate scalar φ in the trochoidal curvilinear coordinate system.

In trochoidal cylinder coordinates, the scalar Helmholtz equation is

$$\frac{1}{h^2}\left(\frac{\partial^2\varphi}{\partial s^2}+\frac{\partial^2\varphi}{\partial n^2}\right)+\frac{\partial^2\varphi}{\partial z^2}+k^2\varphi=0. \tag{4.6}$$

If the propagation is parallel to the wave crests, then for plane waves

$$\frac{\partial^2\varphi}{\partial z^2}=-k^2\varphi$$

and (4.6) reduces to

$$\frac{\partial^2\varphi}{\partial s^2}+\frac{\partial^2\varphi}{\partial n^2}=0 \tag{4.7}$$

which is precisely the problem treated in section 3. Let the propagation be at some angle θ with respect to the z-direction; because the boundary has cylindric symmetry, because the electrical wavelengths are very long, and because we would like a solution closely related to the plane wave solution of section 3, we assume that the variations of φ with z are exponential only:

$$\varphi(x, y, z)=(x, y)e^{-jkz\cos\theta}. \tag{4.8}$$

In this manner (4.6) becomes the two-dimensional scalar wave equation:

$$\frac{1}{h^2}\left(\frac{\partial^2\varphi}{\partial s^2}+\frac{\partial^2\varphi}{\partial n^2}\right)+k^2\varphi\sin^2\theta=0. \tag{4.9}$$

In discussing the solution of this equation, nothing is lost by setting $\sin\theta=1$, i.e., considering propagation perpendicular to the wave crests.

Unfortunately (4.9) does not separate in the trochoidal coordinate system.[5] Even though a suitable exact solution cannot be found for (4.9) in closed form, it may be possible to build up a series solution in ascending powers of k, if k is small; we can terminate the series at whatever term yields a satisfactory approximation to the desired solution. The method of obtaining the series solution is described in appendix A, together with the details of the calculation for the case of interest here. The result is

$$\psi=\frac{E_0}{jk}\,e^{-jks}\left[1+\frac{ka^2e^{-kn}}{2}-k^2a(n+1)e^{-n}\cos s-\frac{k^2a^2}{4}\,e^{-2n}\right]. \tag{4.10}$$

The principal part of ψ in (4.10) is the quasi-static term $(E_0/jk)\,e^{-jks}$. All the other terms are smaller by at least a factor of k and die out with increasing curvilinear height, n, above the water surface. We have assumed a wavelength of 2π for the water wavelengths, whose actual wavelengths L range between 40 and 300 m. The actual wavelengths λ of EM waves in air at VLF and LF range from 3000 to 30,000 m; the corresponding value of k to be used in these equations is found by expressing λ in units of $L/2\pi$:

$$k=\frac{2\pi}{\lambda}\div\frac{2\pi}{L}=\frac{L}{\lambda}; \tag{4.11}$$

thus k ranges from less than 0.001 for short L at 10 kc/s to 0.05 for long L at 100 kc/s.

The $\vec{E}$-field follows directly from (4.10).

$$\vec{E}=\nabla\times(\psi\vec{e}_z)=\frac{E_0}{h}\,e^{-jks}\left[1+jk\{a(n+1)e^{-n}\sin s\}-k^2\left\{\frac{a^2}{4}\,e^{-2n}-\frac{a^2}{2}\,e^{-kn}+a(n+1)e^{-n}\cos s\right\}\right]\vec{e}_n$$
$$-\frac{jkE_0}{h}\,e^{-jks}\left[\frac{a^2}{2}\,(e^{-2n}-e^{-kn})+ane^{-n}\cos s\right]\vec{e}_s. \tag{4.12}$$

The main features of this expression are first, that the E-field is essentially a propagating static field; second, that the tangential component of $\vec{E}$, the terms in $\vec{e}_s$, does in fact vanish at the water's surface, $n=0$; third, that the correction terms are small.

The expression for the magnetic intensity $\vec{H}$ is somewhat simpler in that we have from Maxwell's equations

$$\vec{H}=\frac{j}{\omega\mu}\,\nabla\times\vec{E}. \tag{4.13}$$

[5] The condition for separation is that h^2 be the sum of a function of s only and a function of n only [see Morse and Feshbach, *op. cit.* pp. 498–500].

But since div $(\psi\vec{e}_z)$ is zero, we have from Maxwell's equations [6]

$$\nabla\times\vec{E}=\nabla\times\nabla\times(\psi\vec{e}_z)=k^2(\psi\vec{e}_z).$$

Hence

$$\vec{H}=\frac{jk^2}{\omega\mu}\psi\vec{e}_z=jkY_0\psi\vec{e}_z$$

in which Y_0, the characteristic admittance of free space has been written for $k/\omega\mu$.[7] What is noteworthy about this expression for $\vec{H}$ is that it does not contain the scale factor h. Thus, the $\vec{E}$ fields are essentially the same and essentially given by the static field configuration for propagation both parallel and perpendicular to the wave crests, provided k is small. The $\vec{H}$ fields, however, are not the same; in the former case, the magnitude of $\vec{H}$ varies with the scale factor h from point to point near the waves; in the latter case, the magnitude of $\vec{H}$ is substantially unvarying with respect to position above the waves.

Finally, we return to the question of propagation at an arbitrary angle, θ, with respect to the wave crests. With respect to the entire argument in appendix A, we can replace k by $k \sin \theta$ without changing either the argument or the result. In accordance with (4.8), we can multiply $\psi(k \sin \theta)$ by exp $(-jkz \cos \theta)$ to obtain a function φ which satisfies the three-dimensional scalar Helmholtz equation

$$\varphi=\psi(k\sin\theta)e^{-jkz\cos\theta}. \tag{4.14}$$

The $\vec{E}$-field is then given by

$$\vec{E}=\nabla\times[\psi(k\sin\theta)e^{-jkz\cos\theta}\vec{e}_z], \tag{4.15}$$

but the term exp $(-jkz \cos \theta)$ is constant with respect to the curl operation on a vector in the z-direction; hence

$$\vec{E}=e^{-jkz\cos\theta}\nabla\times[\psi(k\sin\theta)\vec{e}_z]. \tag{4.16}$$

Thus the $\vec{E}$-field is exactly the same as that obtained with $\sin \theta=1$, except that k is replaced by $k \sin \theta$ and the whole field is multiplied by exp $(-jkz \cos \theta)$. We find $\vec{H}$ by taking the curl of $\vec{E}$ to obtain

$$\vec{H}=\frac{j}{\omega\mu}\nabla\times\vec{E}$$

$$=\frac{j}{\omega\mu}\nabla\times\{e^{-jkz\cos\theta}\nabla\times[\psi(k\sin\theta)\vec{e}_z]\}. \tag{4.17}$$

By directly performing the curl operations [Morse and Feshbach, 1953, p. 115)] and by noting (4.9), we obtain after some manipulations

$$\frac{\vec{H}}{E_0}=-Y_0\frac{\cos\theta}{h}e^{-jkz\cos\theta}\left[\frac{\partial\psi(k\sin\theta)}{\partial s}\vec{e}_s+\frac{\partial\psi(k\sin\theta)}{\partial n}\vec{e}_n\right]$$
$$+Y_0jk\sin^2\theta e^{-jkz\cos\theta}\psi(k\sin\theta)\vec{e}_z \tag{4.18}$$

in which we have once more written Y_0 for $k/\omega\mu$. If we discard all terms having higher order in k, the expressions for ψ, $\vec{E}$, and $\vec{H}$ reduce, respectively, to

$$\psi=\frac{E_0e^{-jk(s\sin\theta+z\cos\theta)}}{jk\sin\theta} \tag{4.19}$$

$$\vec{E}=\frac{E_0}{h}e^{-jk(s\sin\theta+z\cos\theta)}\vec{e}_n \tag{4.20}$$

$$\vec{H}=Y_0E_0\left(\frac{\cos\theta}{h}\vec{e}_s+\sin\theta\vec{e}_z\right)e^{-jk(s\sin\theta+z\cos\theta)}. \tag{4.21}$$

5. Modified Helmholtz-Kirchoff Integral Theorem

The Helmholtz-Kirchoff Integral Theorem [Born and Wolf, 1959] provides a method of calculating the fields inside a source free volume when the fields and their normal derivatives are known on the surfaces bounding it. If a scalar field, U, satisfies the Helmholtz equation

$$(\nabla^2+k^2)U=0 \tag{5.1}$$

everywhere within a volume V whose bounding surface is S, and U is continuous and has continuous first and second derivatives in V and on S, then the field at a point P within the volume is given by [Born and Wolf, 1959]

$$U(P)=\frac{1}{4\pi}\iint\left[U\frac{\partial}{\partial n}\left(\frac{e^{jk\xi}}{\xi}\right)-\frac{e^{jk\xi}}{\xi}\frac{\partial U}{\partial n}\right]dS \tag{5.2}$$

where ξ is the distance from the point P within the volume to the element of area dS on the surface. The normal derivatives $\frac{\partial}{\partial n}\left(\frac{e^{jk\xi}}{\xi}\right)$ and $\frac{\partial U}{\partial n}$ are taken along the inward normal at the location of the surface element dS. Although (5.2) is usually derived for a scalar field, in Cartesian coordinates, u, v, z, the Helmholtz equation for a vector field $\vec{H}$ in a source free isotropic medium is simply

$$\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\vec{H}=0. \tag{5.3}$$

Hence, (5.1) holds individually for each of the Cartesian components of $\vec{H}$ and (5.2) may be applied to each of these in turn.

[6] Morse and Feshbach, *op. cit.*, p. 1766, eq 13.1.6.

[7] While k or λ is measured in units of $L/2\pi$ as in (3.41), Y_0 is invariant to a change of scale in length and is measured in whatever units are consistent with those assigned to E and H, e.g., 1/377 mho for E in volts per meter and H in amperes per meter. The units of μ are then dependent on those chosen for E, H, and length.

Now, the geometry of our problem is cylindric so we may adapt (5.2) to simplify computation in this case. We let S be a cylindric surface with the cylinder axis in the z-direction. The Cartesian coordinates of the plane perpendicular to the cylinder axis at $z=0$ will be denoted by u (abscissa) and v (ordinate). Furthermore, let us suppose that $U(u, v, z)$ can be expressed as follows:

$$U(u, v, z)=U(u, v)e^{-jbz} \tag{5.4}$$

where b is a constant. Then we may rewrite the integral (5.2) as

$$U(P)=\iint_S\left[U(u, v)e^{-jbz}\frac{\partial}{\partial n}\left(\frac{e^{jk\xi}}{\xi}\right)\right.$$
$$\left.-\frac{e^{jk\xi}}{\xi}\frac{\partial U(u, v)}{\partial n}e^{-jbz}\right]dS. \tag{5.5}$$

Now, a plane, z=constant, intersects S in a curve C, which is the same for all such constants because of the cylindric geometry. In appendix B this surface integral is reduced to a line integral along C by integration with respect to z. The result obtained is

$$U(P)=\frac{j}{4}\int_C\left[kUH_1^{(1)}(kr)\frac{\vec{r}\cdot\vec{n}}{r}-H_0^{(1)}(kr)\frac{\partial U}{\partial n}\right]dS \tag{5.6}$$

in which $\vec{r}$ is the vector distance from a point p on C to P, C being taken so as to lie in the same plane as P. The vector $\vec{n}$ in (5.6) is a unit inward normal from C at point P; the functions $H_0^{(1)}$ and $H_1^{(1)}$ are Hankel functions of the first kind, zero and first order. This result is the Modified Helmholtz-Kirchoff Integral Theorem (MHKIT) for cylindric geometry.

Both (5.6) and the usual form of the Theorem (5.5) determine the field U at any point P inside the curve C from the value of U and its normal derivative on the curve, given k for the material within this boundary. While this is indeed a correct result, it seems to require too much information in that either the field *or* its normal derivative at the surface should suffice to specify the function inside the sea [Stratton, 1941]. It is beyond the scope of this paper to enter an extended discussion of the mathematical subtleties of this apparent overspecification of the boundary conditions. The following observations are nevertheless helpful:

(a) The Helmholtz-Kirchoff Theorem is essentially an application of Green's functions. If we could find the Green's function for the bounded region, V, we could calculate the fields in the interior from their *values* on the surface alone [Morse and Feshbach, 1953, pp. 803–807]. In our problem, we know of no way of determining the requisite Green's function. The Helmholtz-Kirchoff Integral Theorem is a means of using the Green's function appropriate to a known solvable problem in a problem which has an arbitrary boundary. The usual HKIT makes use of the Green's function for a spherical boundary; the modified theorem uses the Green's function for a circular cylinder. The price that one pays for using the simple Green's function is that one must know both the field and its normal derivative at the boundary. So, if we are able to provide extra information, we can use the MHKIT to avoid mathematical difficulties.

(b) Once the field U is specified at the boundary, the $\partial U/\partial n$ required in the equation is also determined. It must be calculated or estimated by other means before the Integral Theorems can be used. The required $\frac{\partial U}{\partial n}$ generally cannot be found from the theorem itself, even by successive approximations [Franz, 1949; Shelkunoff, 1951].

We have already calculated the electromagnetic field propagating above a trochoidal sea.

What remains is the question of determining the normal derivative of the tangential $\vec{H}$ on the sea surface. The tangential component of the magnetic field is always continuous across a boundary between two media, but its normal derivative is not. For this normal derivative to be continuous across a boundary, it is necessary that both the conductivity and permittivity of the two media be the same. This condition is not satisfied here. Our method of attack is outlined in the next section.

6. Estimating the Normal Derivative

Suppose we have a circular cylindrical (air filled) cavity in an infinite conducting medium. The conducting medium has wave number $\hat{k}=(-1+j)\sqrt{\pi f\mu\sigma}$. Now suppose there is a cylindrical scalar wave propagating outward from the cylinder axis which is given in the *conducting medium only* by

$$U(r)=AH_0^{(1)}(\hat{k}r) \tag{6.1}$$

where r and θ are the usual cylindrical coordinates. Now, if the radius of the circular cross section of the cavity is ρ, the field on the surface of the conducting medium may be denoted by U_s and we have

$$U(r)=\frac{H_0^{(1)}(\hat{k}r)}{H_0^{(1)}(\hat{k}\rho)}U_s \tag{6.2}$$

which reduces to $U(r)=U_s$ at the cylinder surface $r=\rho$. For the normal derivative we then have

$$\left.\frac{dU}{dr}\right|_\rho=-k\frac{H_1^{(1)}(\hat{k}\rho)}{H_0^{(1)}(\hat{k}\rho)}U_s. \tag{6.3}$$

What we shall attempt to do is to estimate the value of the normal derivative on the sea surface by representing the tangential component of the field at each point on the surface (or just beneath it) by a cylindrical wave of the form given in (6.2), where ρ will be chosen to be the radius of curvature of the cross section of the sea surface normal to the wave crests, at the point in question. Hence, (5.6) becomes

$$U(P)=\frac{\hat{k}j}{4}\int U\left[H_1^{(1)}(kr)\frac{\vec{r}\cdot\vec{n}}{r}+H_0^{(1)}(kr)\frac{H_1^{(1)}(\hat{k}\rho)}{H_0^{(1)}(\hat{k}\rho)}\right]ds. \tag{6.4}$$

This procedure suggests that the sea surface is made to conform locally, with the boundary of the cylindrical cavity referred to above. Now, the center of the circle (of radius equal to the radius of curvature of the curve at point ρ) that best conforms to the curve at ρ is on the concave side of the curve. Throughout most of its length, the cross section of the sea surface is concave upward, so the idea of a cylindrical wave propagating into the sea from an axis above it makes some sort of physical sense. On the portions of sea surface cross section which are concave downward, we expect the cylinder axis in question to be beneath the sea surface. The wave appropriate to the physical conditions of the problem is then an $H_0^{(2)}$ wave. $H_0^{(2)}$ is a Hankel function of zero order and *second* kind, and represents a wave propagating inward toward the cylinder axis.

Now, if

$$U(r)=AH_0^{(2)}(\hat{k}r)\vec{e}_z \tag{6.5}$$

$$\frac{dU}{dn}=AkH_1^{(2)}(\hat{k}r)\vec{e}_z. \tag{6.6}$$

The expression has a positive sign because the inward normal makes an angle of π radians with the direction of increasing r. If again we are given that the surface value of U is U_s,

$$A=\frac{U_s}{H_0^{(2)}(\hat{k}\rho)} \tag{6.7}$$

and

$$\left.\frac{dU}{dn}\right|_\rho=k\frac{H_1^{(2)}(\hat{k}\rho)}{H_0^{(2)}(\hat{k}\rho)}U_s. \tag{6.8}$$

If the rectangular coordinates (u, v) of a curve are given parametrically in terms of an auxiliary variable x, then the radius of curvature, ρ, is given by

$$\rho=\frac{\pm\left[\left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dx}\right)^2\right]^{3/2}}{\left(\frac{du}{dx}\frac{d^2v}{dx^2}-\frac{dv}{dx}\frac{d^2u}{dx^2}\right)}, \tag{6.9}$$

where the sign is chosen to achieve a positive result. In particular, for a trochoid, we can place $n=0$ in (2.3) and (2.5) to obtain from (6.9) the radius of curvature

$$\rho=\pm\frac{(1-2a\cos x+a^2)^{3/2}}{a(a-\cos x)}. \tag{6.10}$$

Any point p on the curve, at which the curve is concave away from the sea, corresponds to a value of x for which $(a-\cos x)>0$. For such values of p, we use (6.3) to estimate the normal derivative $\partial U/\partial n$. Infinite radius of curvature occurs when $(a-\cos x)=0$; and any point p at which the curve is concave into the sea corresponds to a value of x for which $(a-\cos x)<0$, for which (6.8) is used to estimate $\partial U/\partial n$.

7. Calculating the Subsurface Fields

We must now assemble the results derived in previous sections to obtain an expression for the subsurface magnetic field.

At VLF and LF frequencies, it suffices to use the quasi-static approximation to $\vec{H}$ above the water, as given by (4.21)

$$\vec{H}=Y_0E_0\left(\frac{\cos\theta}{h}\vec{e}_s+\sin\theta\vec{e}_z\right)e^{-jk(s\sin\theta+z\cos\theta)}. \tag{4.21}$$

In order to use the MHKIT on this vector $\vec{H}$, we need to express it in rectangular coordinates. For a trochoid expressed parametrically as in (2.5) with $n=0$, the unit tangent $\vec{e}_s$ to the water surface is given by

$$\vec{e}_s=\frac{1-a\cos x}{h}\vec{e}_u-\frac{a\sin x}{h}\vec{e}_v. \tag{7.1}$$

Thus, the magnetic field becomes in rectangular coordinates

$$\vec{H}=Y_0E_0\left(\frac{(1-a\cos x)\cos\theta}{h^2}\vec{e}_u-\frac{a\sin x\cos\theta}{h^2}\vec{e}_v+\sin\theta\vec{e}_z\right)e^{-jk(s\sin\theta+z\cos\theta)}. \tag{7.2}$$

Moreover, s can be expressed parametrically in terms of x, so that

$$ds=hdx. \tag{7.3}$$

Since the MHKIT applies individually to each of the rectangular components of $\vec{H}$, it applies to the vector $\vec{H}$ when it is expressed in rectangular coordinates as in (7.2). By placing (7.2) and (7.3), (6.10), and either (6.3) or (6.8), as required, in (5.6), we could write down a general integral for the subsurface field in which all variables are expressed in terms of the parameter x. Such a general expression is complicated. Either it must simplify by way of approximations, or the entire process must be reduced to a computer calculation. Unfortunately, at VLF and LF frequencies, skin depths in sea water are of the order of a few meters (1 to 3 dB of attenuation per foot in the 10 to 100 kc/s region). wave heights are of the order of a few meters, and the depth at which one can make observations is limited by attenuation from a few meters to a few tens of meters. Thus, many of the variables that appear in the MHKIT must be integrated over ranges in which the Hankel functions do not have the sort of simple asymptotic form which would permit analytic simplification of our expressions.

In such a case, the recourse is computation. The expressions for $\vec{H}$ were programmed onto a 7090 computer. Typical results are shown for 2-ft waves at 20 kc/s in a "normal" sea [8] in figure 2 and for 25-ft waves in figure 3. The values of a were taken to be $\pi/20$, values typical of actual sea conditions. The figures give the attenuation (solid lines) and phase shift (dashed lines) of the magnetic field at a receiver located in a fixed horizontal plane, which plane is located 25 ft below the troughs of the waves. The data are plotted as functions of the height of the water *directly above* the position of the receiver in the horizontal plane. The portions of the curves drawn with thin solid lines represent points interpolated from computed (solid and dashed) data.

The reason for the existence of the vertical component of $\vec{H}$, for propagation parallel to the wave crests, can be surmised by substituting H_r from (7.2) for U in (6.4). This component is zero at any point directly under the crest or trough of a wave because of symmetry. It has a maximum value (for a *fixed* vertical coordinate) somewhere between the wave crest and the wave trough.

[8] Conductivity assumed 4 mho meter $^{-1}$.

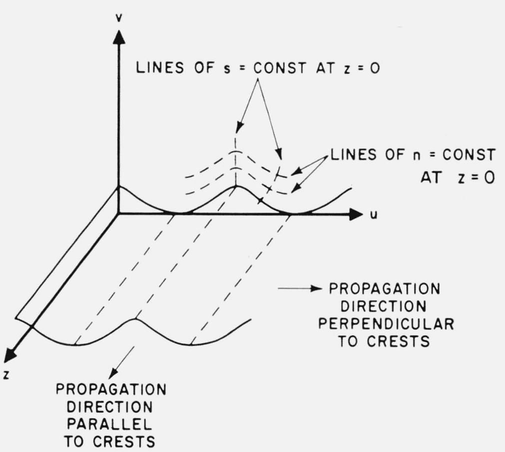

FIGURE 1. *Coordinates for EM propagation over trochoidal cylinder.*

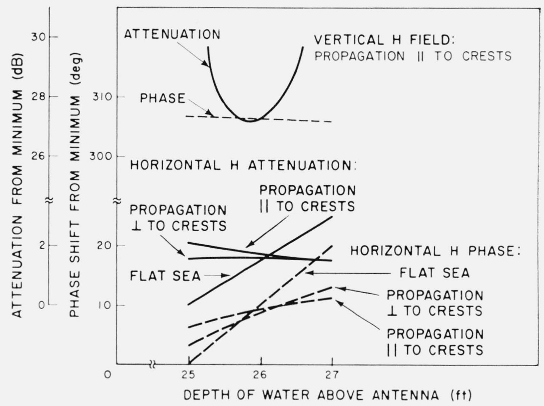

FIGURE 2. *Fields at a horizontal plane 25 ft under troughs of 2-ft waves.*

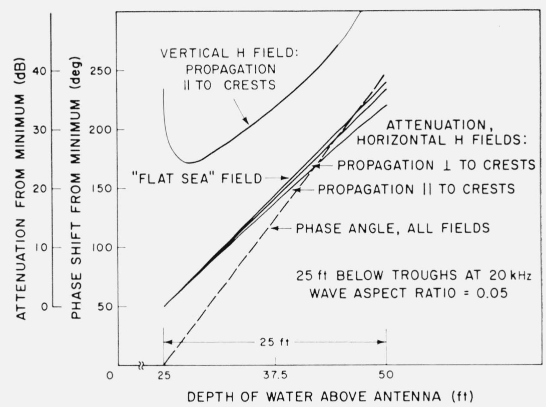

FIGURE 3. *Fields at a horizontal plane 25 ft under troughs of 25-ft storm waves.*

8. Discussion of Results

There are three points worth noting about the foregoing results:

(a) With respect to the theory, we have assumed a simple trochoidal cylinder as the curve for the surface of the sea. However, the methods developed can be applied directly to any cylindric surface whose generating curve can be put in the form

$$\begin{cases} u=s-a_1 \sin b_1s-a_2 \sin b_2s-a_3 \sin b_3s- \dots \\ v=a_1 \cos b_1s+a_2 \cos b_2s+ \dots \end{cases} \tag{8.1}$$

In particular, a wave which is the sum of individual trochoidal wavelets has this form (provided the sea is not crossed). The static fields above such a surface can be obtained by a conformal transformation which is an extension of (2.2):

$$\hat{w}=\hat{z}+ja_1 \exp jb_1\hat{z}+ja_2 \exp jb_2\hat{z}+ \dots \tag{8.2}$$

and so on, through all the formulas of section 2. In particular, the scale factor h is given by

$$h=\sqrt{1-2\sum a_k e^{-b_k n}\cos b_k s+\sum_{m,k} a_m a_k e^{-(b_k+b_m)n}\cos (b_k-b_m)s}.$$

If all the quantities a_k are much less than one (as is usually the case for waves in the ocean), then this scale factor is approximately

$$h=1-\sum_k a_k e^{-b_k n}\cos b_k s \tag{8.3}$$

so that, roughly speaking, the superposition of waves results in a superposition of effects on the fields outside the water. With respect to fields inside the water, the effects of superposing several waves are not clear *a priori*, since the boundary surface itself is altered.

It is also worthwhile to remark that the MHKIT for the underwater fields was obtained under the assumption that the external fields do not vary with the z-coordinate; clearly, except for $\theta=90°$ (propagation perpendicular to the wave crests), $\vec{H}$ does have a z-variation. However, the underwater propagation constant $\hat{k}$ is so large as compared with $\hat{k}\sin\theta$ above the water that we are justified in ignoring the effects of a z-variation in the integrand in arriving at (5.6) from (5.5).[9]

(b) With respect to the calculations, figures 2 and 3 represent two extreme situations. In the former, the skin depth is large compared to the wave height and the depths of observation are comparable with the radius of curvature of the waves. Accordingly, the fields at depth show substantial averaging-out of the variations. In the case of the 25-ft waves, the skin depth is less than the wave height and the depths of observation are small as compared with the radius of curvature of the waves. The fields in this case seem to behave substantially as if a flat sea were moving up and down over the point of observation.

(c) With respect to possible experimental confirmation of the theory, there are three possibilities. One can attempt to verify the predicted difference in the dependence of $\vec{H}$ on the configuration of the waves, either above or below the water surface. In figure 2, the phases of vertical and horizontal components of $\vec{H}$ depend differently on the amount of water above the observation point, and therefore vary differently with time; one could attempt to verify the elliptization thus introduced by the boundary. Finally, there is the difference in the phase behavior of the fields predicted at modest depths under relatively calm and strong sea conditions, as in figures 2 and 3. All three experiments are made uncertain by the fact that the real sea is not a simple trochoidal cylinder. The last two are also difficult in that the effect to be measured is small . . . only a few degrees of elliptization or of phase change or a decibel or two difference in field strength.

Experimental observations of VLF fields do, however, confirm both the gross dependence of attenuation and phase on instantaneous wave height as predicted by the theory and also confirm the dependence of the local scale factor effect on the direction of atmospheric propagation with respect to the running of the sea.[10] The details of these experiments are described in the next section.

9. Experimental Results

The two conclusions of the theory that we put to experimental test are as follows:

(a) For sea waves running from 5 ft swell to 20 to 30 ft storm waves, the subsurface field is the same as if the sea were flat and the antenna were moving up and down, to within a few degrees in phase and a decibel or two in attenuation.

(b) The E-field in the air is locally enhanced over the crests of the waves and diminished over the troughs, in accordance with the local curvature of the surface. Whether this effect is passed on to the H-field and (through continuity of H at the boundary) to the subsurface fields depends on the angle between the direction of propagation and the direction of the wave crests.

To test the validity of conclusion (a) we made arrangements to tow a submerged float about 30 ft below the troughs of 20 to 30 ft storm seas in the North Atlantic, and to record the instantaneous variations in the magnitude (in decibels) and phase of VLF transmissions received on antennas in the float.

Some of the results are shown in the strip chart recording of figure 4. The upper two traces are the received field strengths of VLF station NSS at 22.3 kc/s on each of two underwater loops having orthogonal directivity patterns. The next two traces are the corresponding variations in the received phase measured with a recording phasemeter. The bottom trace is a record of the variation in height of water above the towed float, as measured by a pressure transducer. Except that the pressure trace varies in the opposite sense, there is a clear excellent correlation between these curves, instant by instant in time.

To properly observe the scale factor effect, conclusion (b), requires better control over experimental conditions than is available in the open sea. To this end we made use of a field site at an old quarry wharf at Folly Cove in Rockport, Mass.

[9] The effect of assuming an e^{ikz} dependence of the fields amounts to replacing $\hat{k}$ by $\sqrt{(\hat{k})^2+(k)^2}$ in (5.6). Since $k<<\hat{k}$, this change is small indeed.

[10] The effect to be measured is the difference between h and unity. From (2.7) the ratio of the maximum to minimum value of h is $(1+a)/(1-a)$. In a sea driven by strong winds a can rise from its normal value of 0.05 to 0.15 to the nearly breaking-waves value of 0.3 to 0.4.

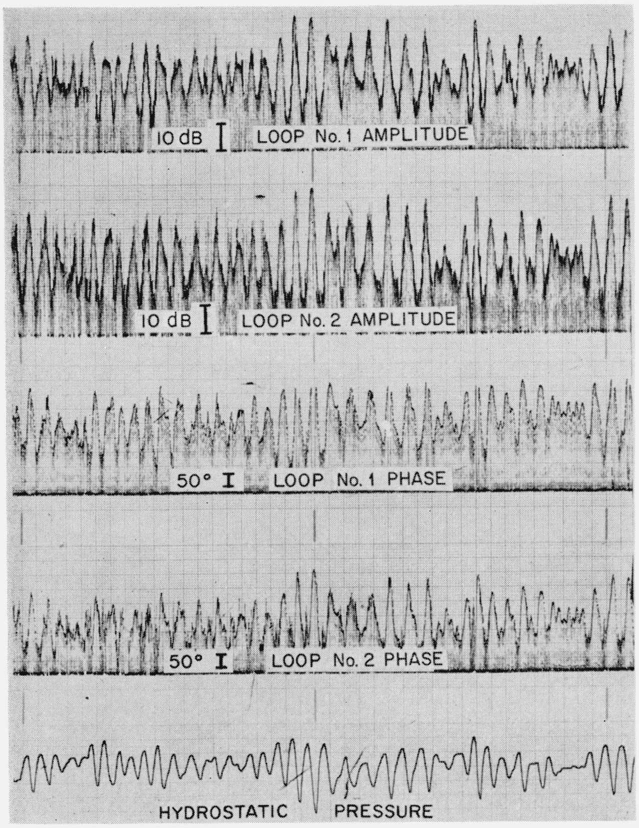

FIGURE 4. *Magnetic field and pressure measurements—open sea.*

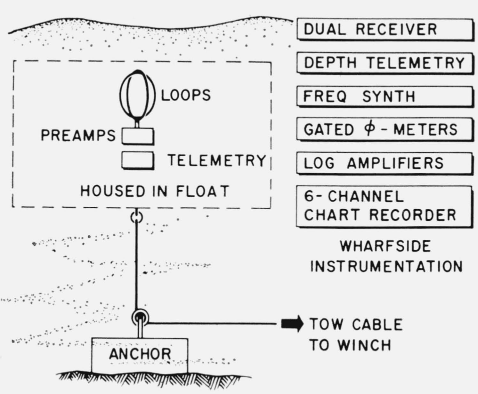

FIGURE 5. *Folly Cove instrumentation.*

Part of the test equipment is in an underwater float in 70 ft of water 100 yards off shore. This includes a pair of loop antennas, preamplifiers, pressure transducers to measure depth, and telemetry as shown in figure 5. An anchor cable passes through a pulley to a winch on shore, so that the assembly can be raised and lowered. Other cables and ropes orient the loops and carry signals. On shore there are instruments for simultaneously receiving and recording two stations at once; they measure signal amplitudes in decibels and signal phase. A six-channel chart recorder on which to record and compare these measurements along with the corresponding variations in head of water above the antenna loops is also used. The phasemeters operate by comparing the received phase of standard transmissions such as those of NBA with that of a local precision frequency synthesizer. Since the transmissions of NBA and GBR are often keyed, we key the averaging circuits of the phase meter, so as not to measure the phase of the intervening noise. It has been found that besides those stations set up for standard frequency transmissions, many other VLF stations exhibit sufficient carrier stability to permit us to measure the phase of the received signal with our equipment. Thus, we have had greater flexibility in choosing signals than we might have originally expected.

Since we measure depth with pressure transducers, we must take into account the fact that the pressure variation p due to the overhead passage of a given wave diminishes exponentially with the average depth d of the transducer. This and the hydrodynamic relationship between the wavelength λ and period T of gravity waves in open ocean are given by

$$p=p_0e^{-2\pi d/\lambda}$$

$$\lambda=1.56T^2 \text{ meters, seconds}$$

$$p=p_0e^{-4d/T^2}$$

in which d is depth of observation and p_0 is the magnitude of the pressure variation as it would be at the surface. The last formula is a combination of the first two. The waves we see at Folly Cove do obey these laws. Apparently the presence of an ocean bottom at depth D (150 to 250 ft) for many miles simply eliminates by friction at the bottom those waves for which λ is greater than the depth. The waves that remain are true gravity waves with sufficiently short wavelength as to have little interaction with the bottom.

What the theory predicts is this: if the atmospheric propagation is *across* the sea wave crests, the underwater field will vary in attenuation according to the flat sea model as the waves go up and down. But if the atmospheric propagation is *parallel* to the sea wave crests, in addition to the flat sea effect there will be an enhancement of the field at the crests and diminution at the troughs of the waves. This effect operates in the opposite sense to the attenuation with depth, so as to reduce the total variation in attenuation. The phase shift, however, is not at all affected. If the direction of atmospheric propagation makes some general angle θ with the direction of the wave crests, the theory predicts total underwater H field can be written as the sum of two orthogonal components, $H_\perp$ and $H_\parallel$ one of which varies with surface shape and one of which does not.

In order to make any observation of this effect, it must be of the order of 4 to 6 dB, which in turn requires almost-breaking waves (wave height to wavelength ratios of the order of 0.1 or greater) of a kind that exist only in a heavy storm.

There was one storm at Folly Cove that produced almost ideal conditions for observing the effect, on a day on which a NW gale blew for 18 hours. The results are illustrated in figure 6. This is a strip chart recording in which the upper pair of traces are the received amplitudes (decibel scale) of signals transmitted from distant stations operating at 18.6 kc/s and 22.3 kc/s. The third and fourth traces are the corresponding received phases. The bottom trace is pressure (this time recorded in the same sense as the signal variations). By good luck, the 22.3 kc/s signal was propagating at about 45° with respect to the waves; moreover, the antenna loop directivity patterns were such that loop No. 1 saw mainly the component of H that is unaffected by surface shape and loop No. 2 saw mainly that component of H which is affected by surface shape. One would expect that the variation of amplitude introduced in the 22.3 kc/s signal in loop No. 2 by the surface shape would almost completely cancel out the 6–9 dB variation of amplitude due to the varying height of water above the antenna (4 to 6 ft waves). One would also expect the 22.3 kc/s signal into loop No. 1 to show the full range of amplitude variation.

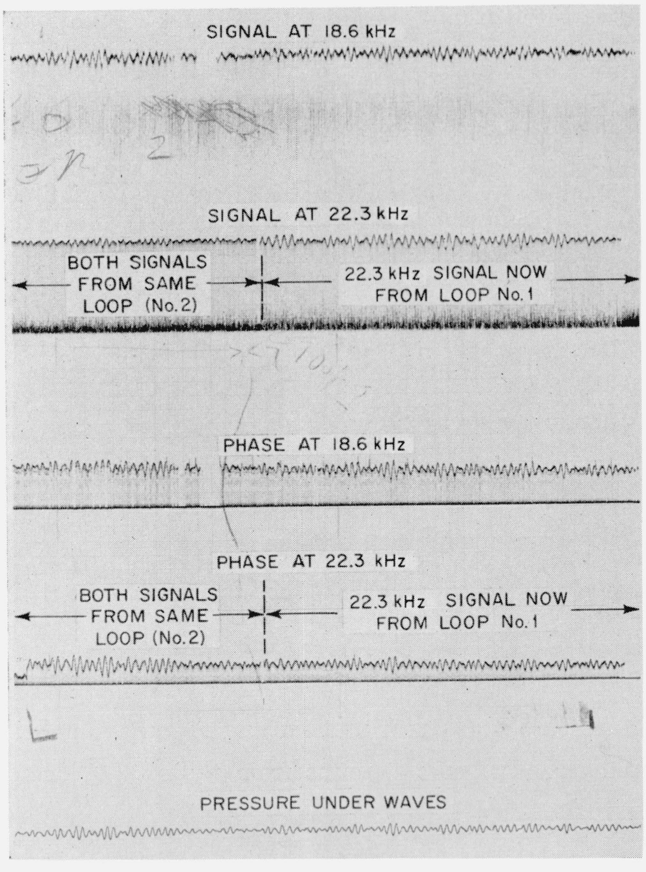

FIGURE 6. *Magnetic field and pressure measurements—Folly Cove.*

In the left hand part of the records in figure 6, both receivers were operating from loop No. 2; in accordance with theoretical expectation, the amplitude of the 22.3 kc/s signal shows little or no variation. Midway through the record, the 22.3 kc/s receiver was switched to loop No. 1; as predicted, the full 6 to 9 dB range of signal variation is apparent. Note that the phase variations are the same in both cases.

As a matter of fact, the influence of the scale factor effect on the field observed in a directive underwater antenna (e.g., a loop) varies in a complicated way with the angle between the direction of atmospheric propagation and the wave crests, the orientation of the antenna, and its pattern. It is significant that by taking these factors into account it was possible to calculate the approximate direction of the two stations being received in figure 6 from the known orientation of the loops and the changes in average signal strength and phase as the receivers were switched between loops. These calculated directions agreed within ±5° of the directions derived from the known locations of the transmitters. It is even more significant that taking account, in addition, the observed changes in the range of amplitude variations as the signals are switched between loops, it is possible to calculate the direction of the running of the sea. This calculation agrees (within 30°) with what was observed as closely as it was possible to estimate that direction by eye and hand compass.

10. Appendix A

We wish, here, to find a solution of the scalar Helmholtz equation in two dimensions

$$\nabla^2\varphi(s,n)+k^2\varphi(s,n)=0 \tag{A1}$$

in order to obtain a scalar function φ from which to derive an $\vec{E}$-field

$$\vec{E}=\text{curl}\ \varphi\vec{e}_z \tag{A2}$$

where $\vec{E}$ is the electric field associated with an electromagnetic wave which propagates in a direction normal to the sea wave crests. The tangential component of $\vec{E}$ must vanish at the surface $n=0$.

Since, in the limit $k=0$, the fields must approach the static field pattern, we begin by taking

$$\varphi=\psi_0=\frac{E_0}{jk}\,e^{-jks}. \tag{A3}$$

If we place this ψ_0 in the Helmholtz equation (4.1) or (4.9), we obtain

$$\nabla^2\psi_0+k^2\psi_0=\frac{E_0}{jk}\,k^2\left(1-\frac{1}{h^2}\right)e^{-jks} \tag{A4}$$

where since we consider propagation perpendicular to the wave crests, $\sin\theta=1$. Thus in the limit $k\to 0$,

$$\nabla^2\psi_0=0$$

satisfying Laplace's equation, and

$$\vec{M}=\vec{E}=\text{curl}\,\psi_0\vec{e}_z=\frac{1}{h}E_0\vec{e}_n. \tag{A5}$$

This is precisely the static field that results from conformal transformation of the field between parallel plates. It is unusual only in expressing $\vec{E}$ (in a charge-free region) as the curl of a vector potential rather than as the gradient of a scalar potential. When $k\neq 0$, ψ_0 satisfies the scalar wave equation only if $h^2\equiv 1$; in this case, the coordinate system would be rectangular and ψ_0 would represent a plane wave traveling in the x-direction between parallel-plate bounding surfaces.

In the present problem, $h^2\not\equiv 1$, the right-hand side of (A4) is not identically zero, so that ψ_0 is not an exact solution to the wave equation in trochoidal coordinates. Suppose, now, that we can write a second approximation to the solution of the wave equation as

$$\psi_1=\psi_0+\frac{Q_1E_0}{jk}e^{-jks} \tag{A6}$$

in which Q_1 is a function of n and s for which

$$\nabla^2Q_1=-k^2(1-1/h^2) \tag{A7}$$

and in which Q_1 is a function whose maximum magnitude is of the order of magnitude k^2. Substituting ψ_1 in the Helmholtz equation, we obtain a result which has the form

$$\nabla^2\psi_1+k^2\psi_1=\frac{E_0}{jk}P_1(n,s)e^{-jks} \tag{A8}$$

where

$$P_1(n,s)=\left[2jk\frac{\partial Q_1}{\partial s}+k^2(1-1/h^2)Q_1\right]. \tag{A9}$$

P_1 is a function of the order of magnitude k^3 instead of one of the order of magnitude k^2 as in the case of ψ_0 and (A4). Once more, we can take a new trial solution ψ_2.

$$\psi_2=\psi_1+\frac{Q_2E_0}{jk}e^{-jks} \tag{A10}$$

in which Q_2 is a function of n and s for which

$$\nabla^2Q_2=-P_1(n,s) \tag{A11}$$

and in which Q_2 is a function whose maximum magnitude is of the order of magnitude k^3! This in turn leads to an equation which has the form

$$\nabla^2\psi_2+k^2\psi_2=\frac{E_0}{jk}P_2(n,s)e^{-jks} \tag{A12}$$

in which P_2 is a function of the order of magnitude k^4 instead of k^3 as for the case of ψ_2. Thus, we are led to a sequence of functions

$$\psi_0,\ \psi_1,\ \psi_2,\ \ldots,\ \psi_n$$

which satisfy the Helmholtz equation in the limit as n is taken sufficiently large,

$$\lim_{n\to\infty}\nabla^2\psi_n+k^2\psi_n=0 \qquad k<1$$

provided that k is less than one. At the same time, we have found a solution to the scalar Helmholtz equation

$$\psi=\lim_{n\to\infty}\psi_n=\psi_0+(Q_1+Q_2+\ldots)\frac{E_0}{jk}e^{-jks}$$

in which the maximum amplitudes of the Q_m is of the order of magnitude k^{m-1} times that of Q_1.

The original trial solution ψ_0 satisfied the boundary condition that $\vec{M}=\vec{E}$ be normal to the water surface, but did not satisfy the Helmholtz equation. At each step of the approximation, it is necessary to pick the Q's so that ψ_1, ψ_2, . . . continue to satisfy the boundary condition. It is beyond the scope of this paper to show that in the differential equations for the Q's there is sufficient flexibility to carry through the program just outlined.[11]

Now, let us carry out the program for the case at hand. Our ψ_0 is given by (A3). When substituted in (A2), ψ_0 leads to an $\vec{E}$-field normal to the surface $n=0$, but does not satisfy the Helmholtz equation (A1). Using the expression for ∇^2 in (3.1) and that for h in (2.7), we find that in trochoidal coordinates (A7) for Q_1 becomes

$$\frac{\partial^2Q_1}{\partial s^2}+\frac{\partial^2Q_1}{\partial n^2}=2k^2ae^{-n}\cos s^{-k^2a^2e^{-2n}}. \tag{A13}$$

A particular solution of (A13) is

$$\hat{Q}_1=-k^2ane^{-n}\cos s-\frac{k^2a^2}{4}e^{-2n}. \tag{A14}$$

To this solution, we can add any number of terms which are solutions of the homogeneous equation associated with (A13)

$$\overline{Q}_1=A_re^{-rn}\cos rs \tag{A15}$$

in which r is any real number and A_r any complex number. In general, terms from the set $\{\overline{Q}_1\}$ can

[11] Indeed, in order to satisfy the boundary conditions exactly at each step, one must generally add to Q_m a term which is an approximate solution of the Helmholtz equation rather than being a solution to a differential equation like (A7). Such terms have sufficiently small coefficients so that they do not upset the convergence of the sequence of functions, ψ_1, ψ_2, ... They amount to slight modifications of ψ_0, having relative magnitudes less than k^m, based on the requirements of the mth step.

be added to $\hat{Q}_1$ for the purpose of satisfying the boundary that $\vec{E}$ be normal to the surface $n=0$. We remark that both terms of $\hat{Q}_1$ lead to tangential components of $\vec{E}$ at $n=0$. In order to eliminate the tangential $\vec{E}$ at $n=0$ due to the first term on the right of (A14), we add to (A14) a term from the set $\{\overline{Q}_1\}$ to obtain for Q_1

$$Q_1=-k^2a(n+1)e^{-n}\cos s-\frac{k^2a^2}{4}e^{-2n}. \quad \text{(A16)}$$

The tangential components of $\vec{E}$ due to the second term of (A14) cannot be counteracted by terms from the set (A15) with $r=0$, because such terms do not meet the reasonable requirement that their effect tend to zero as n gets very large. To complete the satisfaction of the boundary condition, we add to ψ_1 a further term φ_1.

$$\varphi_1=\frac{ka^2}{2}e^{-kn}\frac{E_0}{jk}e^{-jks}. \quad \text{(A17)}$$

Thus, the modified second trial solution to the Helmholtz equation (A1) becomes

$$\begin{aligned}\psi&=\psi_0(1+Q_1)+\varphi_1,\\ &=\left[1-k^2a(n+1)e^{-n}\cos s-\frac{k^2a^2}{4}e^{-2n}\right.\\ &\qquad\left.+\frac{ka^2}{2}e^{-kn}\right]\frac{E_0}{jk}e^{-jks}. \quad \text{(A18)}\end{aligned}$$

This is the function ψ given in the text in (4.11). If we use the Helmholtz operator on ψ, we obtain

$$\begin{aligned}\frac{\nabla^2\psi+k^2\psi}{\psi}&=k^3\left[2ja(n+1)e^{-n}\sin s+\frac{a^2}{2}e^{-kn}\right]\\ &\qquad+\text{terms of higher order in } k. \quad \text{(A19)}\end{aligned}$$

Thus, ψ is a solution of the Helmholtz equation to the third order in small quantities, k.

To derive the $\vec{E}$-field from (A18), it is convenient to use the vector identity which states that if u is a scalar function and $\vec{F}$ is a vector function, respectively, of coordinates, then

$$\nabla\times(u\vec{F})=(\nabla u)\times\vec{F}+u\nabla\times\vec{F}. \quad \text{(A20)}$$

Thus, for (A2) we have

$$\begin{aligned}\vec{E}&=\text{curl }\psi\vec{e}_z=\nabla\times(\psi\vec{e}_z)\\ &=\nabla\psi\times\vec{e}_z\\ &=\frac{1}{h}\left(\frac{\partial\psi}{\partial n}\vec{e}_n+\frac{\partial\psi}{\partial s}\vec{e}_s\right)\times\vec{e}_z\\ &=\frac{1}{h}\left(\frac{\partial\psi}{\partial n}\vec{e}_s-\frac{\partial\psi}{\partial s}\vec{e}_n\right). \quad \text{(A21)}\end{aligned}$$

We can thus use (A21) to compute the $\vec{E}$-field due to the scalar solution determined in (A18). The operations are straightforward and lead to the result

$$\begin{aligned}\vec{E}&=\frac{E_0}{jkh}e^{-jks}\left[\frac{k^2a^2}{2}(e^{-2n}-e^{-kn})+k^2ane^{-n}\cos s\right]\vec{e}_s\\ &+\frac{E_0}{jkh}e^{-jks}\left[jk-k^2(a[n+1]e^{-n}\sin s)-jk^3\left(\frac{a^2}{4}e^{-2n}\right.\right.\\ &\qquad\left.\left.-\frac{a^2}{2}e^{-kn}+a[n+1]e^{-n}\cos s\right)\right]\vec{e}_n. \quad \text{(A22)}\end{aligned}$$

This is the same expression as that given in (4.13) of the text, except for the cancellation of the factor jk. The tangential component of $\vec{E}$ vanishes at $n=0$; the tangential terms in e^{-n} and e^{-2n} come from corresponding terms in Q_1; while the term in e^{-kn} comes from φ.

11. Appendix B. Reduction of the $H-K$ Integral

We wish to reduce the Helmholtz-Kirchoff integral which appears in (5.5) by integrating with respect to z.

$$\begin{aligned}U(P)=\iint_S&\left[U(u,v)e^{-jbz}\frac{\partial}{\partial n}\left(\frac{e^{jk\xi}}{\xi}\right)\right.\\ &\left.-\frac{e^{jk\xi}}{\xi}\frac{\partial U(u,v)}{\partial n}e^{jbz}\right]dS. \quad (5.5)\end{aligned}$$

We let a plane $z=z_0$ intersect the cylindric surface in a curve C, which has the same shape regardless of the value of the constant z_0. Let us denote the differential element of arc length on C by ds; then $dS=dsdz$ and we can rewrite (5.5) as

$$\begin{aligned}U(P)=&\frac{1}{4\pi}\int_C U(u,v)\left[\frac{\partial}{\partial n}\int_{-\infty}^{\infty}\frac{e^{jk\xi}}{\xi}e^{-jbz}dz\right]ds\\ &-\frac{1}{4\pi}\int_C\frac{\partial U(u,v)}{\partial n}\left[\int_{-\infty}^{\infty}\frac{e^{jk\xi}}{\xi}e^{-jbz}dz\right]ds. \quad \text{(B1)}\end{aligned}$$

Further, if the point P lies in the plane $z=z_0$, we can set $\xi=\sqrt{(z-z_0)^2+r^2}$, where r is the length of the projection on the plane $z=0$ of the line from P to a point on the curve C.

So we have

$$\int_{-\infty}^{\infty} \frac{e^{jk}}{\xi} e^{-jbz} dz = \int_{-\infty}^{\infty} \frac{e^{jk\sqrt{(z-z_0)^2+r^2}}}{\sqrt{(z-z_0)^2+r^2}} e^{-jbz} dz. \qquad \text{(B2)}$$

If we make a change of variables $\omega = z - z_0$, we get

$$\int_{-\infty}^{\infty} \frac{e^{jk\sqrt{(z-z_0)^2+r^2}}}{\sqrt{(z-z_0)^2+r^2}} e^{-jbz} dz = e^{-jbz_0} \int_{-\infty}^{\infty} \frac{e^{jk\sqrt{\omega^2+r^2}}}{\sqrt{\omega^2+r^2}} e^{-jb\omega} d\omega. \qquad \text{(B3)}$$

We note that

$$\frac{e^{jk\sqrt{\omega^2+r^2}}}{\sqrt{\omega^2+r^2}}$$

is an even function. Hence

$$\int_{-\infty}^{\infty} \frac{e^{jk\sqrt{\omega^2+r^2}}}{\sqrt{\omega^2+r^2}} e^{-jb\omega} d\omega = \int_{-\infty}^{\infty} \frac{e^{jk\sqrt{\omega^2+r^2}}}{\sqrt{\omega^2+r^2}} \cos b\omega d\omega. \qquad \text{(B4)}$$

From well-known tables,[12] we have

$$\int_{-\infty}^{\infty} \frac{e^{jk\sqrt{\omega^2+r^2}}}{\sqrt{\omega^2+r^2}} \cos b\omega d\omega = j\pi H_0^{(1)} [r(k^2-b^2)^{1/2}]. \qquad \text{(B5)}$$

Therefore, setting $(k^2-b^2)^{1/2} = k_1$, becomes

$$U(P) = j \frac{e^{-jbz_0}}{4} \int_C \left[U(u,v) \frac{\partial}{\partial n} H_0^{(1)}(k_1 r) - H_0^{(1)}(k_1 r) \frac{\partial U(u,v)}{\partial n} \right] ds.$$

We will note also that if we choose P and C in the same plane perpendicular to the cylinder axis, the factor e^{-jbz_0} disappears and we have

$$U(P) = \frac{j}{4} \int_C \left[U(u,v) \frac{\partial}{\partial n} H_0^{(1)}(kr) - H_0^{(1)}(kr) \frac{\partial U(u,v)}{\partial n} \right] ds. \qquad \text{(B6)}$$

Now if $\vec{r}$ is the radius vector from a point p on C to P and $\vec{n}$ is the inward unit normal to C at p, then

$$\frac{\partial H_0^{(1)}}{\partial n} (kr) = k H_1^{(1)}(kr) \frac{\vec{r} \cdot \vec{n}}{r} \qquad \text{(B7)}$$

$$U(P) = \frac{j}{4} \int_C \left[kU H_1^{(1)}(kr) \frac{\vec{r} \cdot \vec{n}}{r} - H_0^{(1)}(kr) \frac{\partial U}{\partial n} \right] ds. \qquad \text{(B8)}$$

This result is the Modified Helmholtz-Kirchoff Integral Theorem (MHKIT) for cylindric geometry.

[12] "Tables of Integral Transforms," Bateman Manuscript Project (McGraw-Hill Book Co., Inc., New York, N.Y., 1954), Formulas 1.7 (34) and 1.7 (30) on p. 26 and 1.5 (27) on p. 17. We need also $H_0^{(1)}(z) = J_0(z) + jY_0(z)$ and $K_0(z) = \frac{j\pi}{2} H_0^{(1)} \left(z e^{\frac{j\pi}{2}} \right)$ which may be found in "Higher Transcendental Functions," Bateman Manuscript Project (McGraw-Hill Book Co., Inc., New York, N.Y., 1953), Volume II, as Formulas 7.2.1 (5) and 7.2.2 (15) respectively (pp. 4 and 5).

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