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Scattering of Electromagnetic and Electroacoustic Waves by a Cylindrical Object in a Compressible Plasma

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The theory of wave propagation in an inhomogeneous compressible plasma is considered. Cylindrical configurations are chosen such that Maxwell's equations, when combined with a (single fluid) continuum theory of fluid dynamics, are separable. It is shown that general expressions for the fields are superpositions of TM (transverse magnetic), TE (transverse electric), and acoustic waves. For oblique incidence, these three wave types are mutually coupled except in special cases.

1. Introduction

The scattering of electromagnetic waves in free space from an infinite dielectric cylinder is a problem first solved by Lord Rayleigh in 1881 and again by Seitz and Ignatowski in 1905. Rayleigh [1918] in a later paper discussed the earlier work and gave some numerical results. The complete generalization of the solution to include the case of oblique incidence was attempted many years later by Wellmann [1937] whose results appear to be incorrect as pointed out by van de Hulst [1957]. The complete solution for oblique incidence on a dielectric cylinder was apparently first given only ten years ago [Wait, 1955].

An interesting aspect of the obliquity is that incident TE (transverse electric) waves will be partially converted to scattered TM (transverse magnetic) waves by the dielectric cylinder. The converse is also true in the case of incident TM waves. However, if the cylinder becomes perfectly conducting, this conversion process no longer takes place. In other words, incident TE waves scatter as TE waves and incident TM waves scatter as TM waves. The following question is now posed: What happens if the surrounding or ambient medium is a plasma instead of free space? Such a situation might occur in scattering of electromagnetic waves from metallic cylindrical objects which are immersed in the ionospheric plasma.

It should be self-evident to many that scattering of electromagnetic waves within a cold isotropic plasma is not really different from scattering in free space. At least, this is the situation when nonlinear terms may be ignored. This follows since the cold plasma, in the absence of a d-c magnetic field, may be characterized by a dielectric constant which differs only by a complex factor from that of free space. Consequently, an incident TE (or TM) wave should still give rise to a scattered TE (or TM) wave provided the cylindrical scatterer is effectively a perfect conductor.

An interesting situation arises when the surrounding medium is a warm plasma. As demonstrated below, it is then found that a perfectly conducting cylinder will convert TE to TM waves and TM to TE waves. Furthermore, electroacoustic-type modes enter into the picture. It is the purpose of the present paper to analyze this question employing rather an idealized cylindrical model. Some related questions are also discussed.

2. Basic Equations

The plasma medium is regarded to be a one-component electron fluid. That is, ions are neglected in the equations of motion. Also, it is assumed that signals and perturbations are sufficiently small for linearized equations to be valid [Oster, 1960; Ginzburg, 1964]. The average number density of electrons is n_0 and is taken to be a constant. The pressure deviation of the electrons from the mean is p and their velocity is $\vec{v}$. As usual, the electric and magnetic fields are denoted $\vec{E}$ and $\vec{H}$, respectively.

Neglecting any dissipative effects such as collisions, the linearized hydrodynamic equation of motion is written

$$mn_0 \frac{\partial \vec{v}}{\partial t} = n_0 e \vec{E} - \nabla p, \tag{1}$$

where e and m are the charge and mass of the electron, respectively. The linearized equation of continuity, when combined with the equation of state, is

$$u^2 m n_0 \nabla \cdot \vec{v} = -\frac{\partial p}{\partial t} \tag{2}$$

where u is the velocity of sound in the electron gas.

Maxwell's equations for the electromagnetic fields in the sourceless plasma are given by

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \tag{3}$$

and

$$\nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + n_0 e \vec{v}, \tag{4}$$

where ϵ_0 and μ_0 are the dielectric constant and magnetic permeability in free space. In what follows, all field quantities are taken to vary according to exp $(i\omega t)$. Furthermore, it is assumed that the plasma region under consideration is homogeneous.

After a certain amount of manipulation of (1) through (4), it is found that the pressure p satisfies

$$(\nabla^2 + k_p^2) p = 0, \tag{5}$$

where

$$k_p = \frac{\omega}{u} \left(1 - \frac{\omega_0^2}{\omega^2}\right)^{1/2} \text{ and } \omega_0^2 = \frac{n_0 e^2}{\epsilon_0 m}.$$

On the other hand, by making use only of (1) and (4), it is found that the magnetic field is obtained from

$$\vec{H} = -\nabla \times \vec{A}, \tag{6}$$

where $\vec{A}$ satisfies

$$(\nabla^2 + k_e^2) \vec{A} = 0, \tag{7}$$

and where $k_e^2 = \epsilon \mu_0 \omega^2$ and $(\epsilon/\epsilon_0) = 1 - (\omega_0/\omega)^2$. As seen by its definition, ϵ is the dielectric constant of the plasma. In the case of a cold plasma, the electron acoustic velocity u tends to zero so that the acoustic wave number k_p becomes infinitely large.

It is of interest to note that the pressure deviation p is related to the static pressure p_0 and density perturbation n by the adiabatic relation

$$p = \gamma p_0 n / n_0, \tag{8}$$

where γ, the classical ratio of specific heats, takes the value 3 for 1 degree of freedom in the gas. Thus, in terms of the kinetic temperature T, it is found that

$$p=3\kappa Tn, \tag{9}$$

where κ is Boltzmann's constant. In terms of the root-mean square electron velocity u,

$$p=mu^2n. \tag{10}$$

For the remainder of this paper, we shall refer only to the pressure deviation p but one should keep in mind that this is related directly to the density perturbation of the plasma [Cohen, 1961].

On combining (1) and (4) for the harmonic time dependence it is easy to see that

$$\vec{E}=\frac{1}{i\epsilon\omega}(\nabla\times\vec{H})-\frac{\omega_0^2}{n_0e(\omega^2-\omega_0^2)}\nabla p \tag{11}$$

and

$$\vec{v}=-\frac{e}{\omega^2m\epsilon}(\nabla\times\vec{H})-\frac{\epsilon_0}{i\omega mn_0e}\nabla p. \tag{12}$$

3. General Field Representations

In the present paper, we are particularly interested in obtaining representations which are applicable to cylindrical coordinates (ρ, ϕ, z). Thus, $\vec{A}$ and p are first expressed in a Fourier representation in the manner [Wait, 1959]

$$\vec{A}=\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\vec{\alpha}_n e^{-ihz}dhe^{-in\phi} \tag{13}$$

and

$$p=\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\beta_n e^{-ihz}dhe^{-in\phi}, \tag{14}$$

where $\vec{\alpha}_n$ and β_n are the respective transforms.

In operational notation, equations (13) and (14) may be written

$$\vec{A}=\Gamma\vec{\alpha}_n \tag{13'}$$

and

$$p=\Gamma\beta_n \tag{14'}$$

where Γ signifies multiplication by exp $[-ihz-in\phi]$ and subsequent integration over h, and summation over integer values of n.

In view of (5) and (7), it is seen that $\vec{\alpha}_n$ and β_n satisfy

$$(\nabla_t^2+u_e^2)\vec{\alpha}_n=0 \tag{15}$$

and

$$(\nabla_t^2+u_p^2)\beta_n=0, \tag{16}$$

where ∇_t^2 is the two-dimensional (i.e., transverse) Laplacian operator, while

$$u_e^2=k_e^2-h^2,$$

and

$$u_p^2=k_p^2-h^2.$$

In cylindrical coordinates (ρ, ϕ, z), it is well known that

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{n^2}{\rho^2}.$$

Therefore, solutions of (15) are cylindrical Bessel functions $Z_n(u_e\rho)$ of order n and argument $u_e\rho$. Similarly, solutions of (16) are cylindrical Bessel functions $Z_n(u_p\rho)$. In the subsequent development, it proves to be convenient to use the Bessel functions J_n and the Hankel function $H_n^{(2)}$ as the two independent solutions. Furthermore, rather than working with the vector potential $\vec{A}$, it is more convenient to employ the two scalar quantities E_z and H_z in a manner similar to that used for dielectric media [Wait, 1959].

With some consideration, we now find that

$$E_z=\Gamma[u_e^2a_nH_n+u_e^2A_nJ_n]+\frac{ie}{m\omega^2\epsilon}\Gamma[hc_n\hat{H}_n+hC_n\hat{J}_n], \tag{17}$$

$$H_z=\Gamma[u_e^2b_nH_n+u_e^2B_nJ_n], \tag{18}$$

and

$$p=\Gamma[c_n\hat{H}_n+C_n\hat{J}_n], \tag{19}$$

where

$$H_n=H_n^{(2)}(u_e\rho),\quad \hat{H}_n=H_n^{(2)}(u_p\rho),$$

$$J_n=J_n(u_e\rho),\quad \hat{J}_n=J_n(u_p\rho),$$

and a_n, A_n, b_n, B_n, c_n, C_n are coefficients which do not depend on the coordinates.

Using the basic equations (1) to (4), it is a straightforward matter to obtain, from (17), (18), and (19), general expressions for the transverse components of the fields. Thus, we find

$$E_\phi=\Gamma\left[i\mu_0\omega b_n\frac{\partial H_n}{\partial\rho}-\frac{nh}{\rho}a_nH_n+i\mu_0\omega B_n\frac{\partial J_n}{\partial\rho}-\frac{nh}{\rho}A_nJ_n\right]+\frac{ie}{m\omega^2\epsilon\rho}\Gamma[nc_n\hat{H}_n+nC_n\hat{J}_n], \tag{20}$$

$$E_\rho=\Gamma\left[-\frac{n\mu_0\omega}{\rho}b_nH_n-iha_n\frac{\partial H_n}{\partial\rho}-\frac{n\mu_0\omega}{\rho}B_nJ_n-ihA_n\frac{\partial J_n}{\partial\rho}\right]-\frac{e}{m\omega^2\epsilon}\Gamma\left[c_n\frac{\partial\hat{H}_n}{\partial\rho}+C_n\frac{\partial\hat{J}_n}{\partial\rho}\right], \tag{21}$$

$$H_\phi=\Gamma\left[-\frac{nh}{\rho}b_nH_n-\frac{ik_e^2}{\mu_0\omega}a_n\frac{\partial H_n}{\partial\rho}-\frac{nh}{\rho}B_nJ_n-\frac{ik_e^2}{\mu_0\omega}A_n\frac{\partial J_n}{\partial\rho}\right], \tag{22}$$

$$H_\rho=\Gamma\left[-ihb_n\frac{\partial H_n}{\partial\rho}+\frac{nk_e^2}{\mu_0\omega\rho}a_nH_n-ihB_n\frac{\partial J_n}{\partial\rho}+\frac{nk_e^2}{\mu_0\omega\rho}A_nJ_n\right]. \tag{23}$$

The components of the velocity $\vec{v}$ are now found easily by using (12). Thus, for example, the radial component is given by

$$v_\rho=-\frac{e}{\omega^2m\epsilon}\left(\frac{1}{\rho}\frac{\partial H_z}{\partial\phi}-\frac{\partial H_\phi}{\partial z}\right)-\frac{\epsilon_0}{i\omega mn_0\epsilon}\frac{\partial p}{\partial\rho} \tag{24}$$

or, explicitly,

$$v_\rho=-\frac{e}{\omega^2m\epsilon}\Gamma\left[-\frac{in}{\rho}k_e^2b_nH_n-\frac{in}{\rho}k_e^2B_nJ_n+\frac{k_e^2h}{\mu_0\omega}a_n\frac{\partial H_n}{\partial\rho}+\frac{k_e^2h}{\mu_0\omega}A_n\frac{\partial J_n}{\partial\rho}\right]$$
$$-\frac{\epsilon_0}{i\omega mn_0\epsilon}\Gamma\left[c_n\frac{\partial\hat{H}_n}{\partial\rho}+C_n\frac{\partial\hat{J}_n}{\partial\rho}\right]. \tag{25}$$

4. Treatment of Cylindrically Stratified Media

The representations developed above are particularly suitable when dealing with cylindrically stratified media. For example, we could deal with a column of a compressible plasma whose properties varied only in the ρ direction. The idea would be to represent the region by a finite number of concentric cylindrical homogeneous regions. In a typical region bounded in ρ by ρ_i and ρ_{i+1}, the electroacoustic velocity is u_i and the electron plasma frequency is $(\omega_0)_i$. In the region ρ_{i+1} to ρ_{i+2}, the corresponding plasma properties are then designated u_{i+1} and $(\omega_0)_{i+1}$. In each region, the field representations would have the form given by (17) to (25). Thus, in the ith region, the six unknown coefficients are a_{in}, A_{in}, b_{in}, B_{in}, c_{in}, and C_{in}. In the $(i+1)$th region, there is a corresponding set of six coefficients. In principle, the boundary value problem is solved if six linear equations can be obtained from the boundary conditions at $\rho=\rho_i$.

Under the rather idealized assumption that the waves do not deform the boundary, kinematic considerations require the continuity of the normal fluid velocity. Thus,

$$(v_\rho)_i=(v_\rho)_{i+1} \text{ at } \rho=\rho_i. \tag{26}$$

It is clear that, if this condition were not satisfied, the fluids would separate at the boundary as pointed out by Field [1956].

A further hydrodynamic-type boundary condition is obtained requiring that the pressure is continuous. Thus, as indicated by Field [1956],

$$p_i=p_{i+1} \text{ at } \rho=\rho_i, \tag{27}$$

which is a requirement imposed by force equilibrium at the boundary. The idealized kinematic and hydrodynamic boundary conditions have also been used by Yildiz [1963] and Johler [1964] who considered scattering from a compressible plasma sphere.

Four additional boundary conditions are readily obtained by imposing the usual electrodynamic property that tangential electric and magnetic fields are continuous. Thus,

$$\left.\begin{aligned}(E_z)_i&=(E_z)_{i+1}\\(E_\phi)_i&=(E_\phi)_{i+1}\\(H_z)_i&=(H_z)_{i+1}\\(H_\phi)_i&=(H_\phi)_{i+1}\end{aligned}\right] \text{ at } \rho=\rho_i. \tag{28}$$

For $P+1$ homogeneous regions, there are only P interfaces. Thus, it would appear, in general, that insufficient boundary equations are available. Fortunately, however, additional physical considerations allow us to dispense with some of the coefficients in the outermost and the innermost cylindrical regions. For example, if the source is in the outermost (0th) homogeneous region, the coefficients $(A_n)_0$, $(B_n)_0$, and $(C_n)_0$ would be known since they are specified by the form of the incident field. The scattered field in the outermost region would be described entirely by the coefficients $(a_n)_0$, $(b_n)_0$, and $(c_n)_0$. On the other hand, in the core or innermost $(P+1\text{th})$ homogeneous region, the coefficients $(A_n)_{P+1}$, $(B_n)_{P+1}$, and $(C_n)_{P+1}$ are zero since the fields must be finite at $\rho=0$.

5. Application to Rigid Perfectly Conducting Cylinder

As an illustration of this method for handling cylindrical boundary value problems in compressible plasma, we shall consider a highly idealized situation. The cylindrical boundary at $\rho=a$ is regarded as perfectly conducting and rigid. The exterior region $\rho>a$ is taken to be a homogeneous compressible plasma with properties u and ω_0. With these simplifying and rather restrictive assumptions, it follows that the required boundary conditions are

$$\left.\begin{aligned} v_\rho&=0\\ E_z&=0\\ E_\phi&=0\end{aligned}\right]\text{at } \rho=a. \tag{29}$$

The same conditions have been employed by Seshadri et al. [1964] in the study of scattering of normally incident plane waves from a perfectly conducting cylinder immersed in the compressible plasma. Using (17), (20), and (25), it now readily follows from (29) that, in matrix form,

$$\begin{bmatrix} u_e^2 H_n & 0 & i\mathcal{E} h\hat{H}_n \\ -\frac{nh}{a} H_n & i\mu_0\omega H_n' & in\mathcal{E}\hat{H}_n \\ -\frac{k_e^2 h u_e}{\mu_0\omega} H_n' & \frac{in\mathcal{E} k_e^2}{a} H_n & -\frac{\epsilon_0 u_p}{i\omega m n_0 \epsilon}\hat{H}_n' \end{bmatrix}\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$$

$$=\begin{bmatrix} -u_e^2 J_n A_n + 0 - i\mathcal{E} h\hat{J}_n C_n \\ \frac{nh}{a} J_n A_n - i\mu_0\omega J_n' B_n - im\mathcal{E}\hat{J}_n C_n \\ \frac{k_e^2 h u_e}{\mu_0\omega} J_n' A_n - \frac{in\mathcal{E} k_e^2}{a} J_n B_n + \frac{\epsilon_0 u_p}{i\omega m n_0\epsilon}\hat{J}_n' C_n \end{bmatrix}, \tag{30}$$

where the arguments of the Bessel functions are evaluated at $\rho=a$ and where $\mathcal{E}=\frac{e}{m\omega^2\epsilon}\cdot$ The column matrix on the right of the previous equation contains the driving terms which are specified by the form of the incident field. Thus, in principle, the boundary-value problem is solved since a_n, b_n, and c_n are now expressed in terms of known quantities.

A special case of some interest is the cold plasma. The transition to this case is obtained by letting the velocity $u\to 0$, whence $k_p\to\infty$. From (30), we readily find that

$$a_n=-\frac{J_n}{H_n} A_n=-\frac{J_n(u_e a)}{H_n^{(2)}(u_e a)} A_n \tag{31}$$

and

$$b_n=-\frac{J_n'}{H_n'} B_n=-\frac{J_n'(u_e a)}{H_n^{(2)\prime}(u_e a)} B_n. \tag{32}$$

The first of these equations indicates that if the incident wave is purely TM and characterized by

$$E_z^{\text{inc}}=\Gamma[A_n J_n],\ H_z^{\text{inc}}=0, \tag{33}$$

then the secondary field is also purely TM and given by

$$E_z^s=\Gamma[a_n H_n],\ H_z^s=0, \tag{34}$$

$$E_\phi^s=\Gamma\left[-\frac{nh}{\rho} a_n H_n\right],\ H_\phi^s=\Gamma\left[-\frac{ik_e^2}{\mu_0\omega} a_n \frac{\partial H_n}{\partial\rho}\right], \tag{35}$$

$$E_\rho^s=\Gamma\left[-iha_n\frac{\partial H_n}{\partial\rho}\right],\ H_\rho^s=\Gamma\left[\frac{nk_e^2}{\mu_0\omega\rho} a_n H_n\right] \tag{36}$$

(where the arguments of the Bessel functions are ua).

Similarly, (32) indicates that if the incident wave is purely TE and characterized by

$$H_z^{\mathrm{inc}}=\Gamma[B_nJ_n],\ E_z^{\mathrm{inc}}=0, \tag{37}$$

then the secondary fields are also purely TE and given by

$$H_z^s=\Gamma[b_nH_n],\ E_z^s=0, \tag{38}$$

$$H_\phi^s=\Gamma\left[-\frac{nh}{\rho}\,b_nH_n\right],\ E_\phi^s=\Gamma\left[i\mu_0\omega b_n\,\frac{\partial H_n}{\partial\rho}\right], \tag{39}$$

$$H_\rho^s=\Gamma\left[-ihb_n\,\frac{\partial H_n}{\partial\rho}\right],\ E_\rho^s=\Gamma\left[-\frac{n\mu_0\omega}{\rho}\,b_nH_n\right]. \tag{40}$$

As seen above, scattering of electromagnetic waves from a perfectly conducting cylinder in a cold plasma is identical in form to scattering of electromagnetic waves from a perfectly conducting cylinder in free space (e.g., as discussed in Wait, 1959). The only change required is the replacement of the dielectric constant ϵ of the plasma with that for free space ϵ_0.

Another interesting special case occurs when we let the charge density in the medium approach zero. Thus, setting $e=0$, it is easily found from (30) that

$$c_n=-\frac{\hat{J}_n'}{\hat{H}_n'}\,C_n=-\frac{J_n'(u_pa)}{H_n^{(2)\prime}(u_pa)}\,C_n, \tag{41}$$

where $u_p=(k_p^2-h^2)^{1/2}$ and $k_p=\omega/u$ in terms of the acoustic velocity u for an uncharged medium of n_0 particles, of mass m, per unit volume. For this case, an incident acoustic wave, whose pressure perturbation is given by

$$p^{\mathrm{inc}}=\Gamma[C_nJ_n(u_p\rho)], \tag{42}$$

leads to a scattered acoustic wave of the form

$$p^{\mathrm{sec}}=\Gamma[c_nH_n^{(2)}(u_p\rho)]. \tag{43}$$

This is the well-known result for pure acoustic scattering from a rigid cylindrical obstacle [Morse and Feshbach, 1953].

We see from the above that our general solution contains both pure electromagnetic and pure acoustic scattering as special cases. In the general case, it is easy to see from (30) that a pure incident TM electromagnetic wave (i.e., $A_n\neq 0$, $B_n=C_n=0$) will generate scattered TM and TE electromagnetic waves in addition to a scattered acoustic wave. In a similar fashion, a pure incident TE electromagnetic wave (i.e., $B_n\neq 0$, $A_n=C_n=0$) will scatter TM and TE electromagnetic waves and the acoustic waves. Of particular interest is the fact that a pure incident acoustic wave (i.e., $C_n\neq 0$, $A_n=B_n=0$) will generate both the TM and TE electromagnetic waves in addition to the acoustic waves.

Consistently, in the development given here, the field functions are preceded by the Fourier-summation operator Γ. Using this artifice allows us to consider quite general types of incident fields. For example, if an obliquely incident plane TE wave is considered,

$$E_z^{\mathrm{inc}}=E_0\sin\theta\exp\left[ik_e\rho\sin\theta\cos\phi-ik_ez\cos\theta\right], \tag{44}$$

where θ is the angle which the wave normal subtends with the z axis. Employing a well-known addition theorem [McLachlan, 1934], it is found that an alternate representation is

$$E_z^{\mathrm{inc}}=E_0\sin\theta\sum_{n=-\infty}^{+\infty}i^nJ_n(k_e\sin\theta)\exp\left[-in\phi-ik_ez\cos\theta\right], \tag{45}$$

which may be written in the form

$$E_z^{\text{inc}}=\Gamma[u_e^2 A_n J_n(u_e\rho)] \tag{46}$$

if $u_e^2 A_n=E_0\,(\sin\theta)i^n$, $h=k_e\cos\theta$, and $u_e=k_e\sin\theta$. The operator for this case of plane wave incidence implies multiplication by $\exp[-in\phi-ik_e z\cos\theta]$ and subsequent summation over integer values of n from $-\infty$ to $+\infty$.

The total fields in the compressible plasma exterior to the rigid perfectly conducting cylindrical object are thus given by (17) to (25) with the above meaning to the operator Γ. The coefficients a_n, b_n, and c_n are given by (30) where $B_n=C_n=0$ and A_n is defined as above.

The explicit form of the fields for obliquely incident plane TM electromagnetic waves or a plane acoustic wave is obtained in a manner almost identical to the above. The reader should have no difficulty satisfying himself that this is the case.

A case of some practical importance is when the rigid cylinder considered above is excited by an electric dipole located at (ρ_0, ϕ_0, z_0) where $\rho_0>a$. The electric dipole of length ds with current I is taken to be oriented in the z-direction (i.e., parallel to the cylinder axis). From earlier work, we know that [Wait, 1959]

$$E_z^{\text{inc}}=i\frac{Ids}{8\pi\epsilon\omega}\sum_{n=-\infty}^{+\infty} e^{-in(\phi-\phi_0)}\int_{-\infty}^{+\infty} H_n^{(2)}(u\rho_0)J_n(u\rho)\exp[-ih(z-z_0)]dh, \tag{47}$$

provided $\rho<\rho_0$. (For $\rho>\rho_0$, we merely interchange ρ and ρ_0 in the above representation.) Thus, we may write (for $\rho<\rho_0$)

$$E_z^{\text{inc}}=\Gamma[A_n J_n(u_e\rho)], \tag{48}$$

where

$$A_n=i\frac{Ids}{8\pi\epsilon\omega}H_n^{(2)}(u_e\rho_0)e^{in\phi_0}e^{ihz_0} \tag{49}$$

and Γ signifies multiplication by $\exp(-ihz-in\phi)$ and subsequent summation over n and integration over h.

The total fields are now again obtained from (17) to (25) with this general meaning attached to Γ. Thus, the fields are represented as a Fourier integral and an infinite sum. The formulas can be greatly simplified in the far field where both $k_e\rho$ and $k_p\rho>>1$. Then the scattered field integrals may be asymptotically approximated in the manner [Wait, 1959]

$$-\frac{i}{2}\int_{-\infty}^{+\infty} F(h)H_n^{(2)}(u\rho)\exp[-ih(z-z_0)]dh\simeq\exp(in\pi/2)F(k\sin\theta)\exp(-ikR)/R, \tag{50}$$

where $u=u_e$ or u_p and $k=k_e$ or k_p, which is valid when the function $F(h)$ is relatively slowly varying. Using this asymptotic relation, formulas for the radiation patterns of the dipole in the presence of the cylinder are readily obtained.

When the observer is near the cylinder, it is necessary to consider the influence of the pole singularities of the integrand. For example, $F(h)$ may have a finite number of complex poles. Some of these may lead to residue contributions which are identified with axial surface waves. This would be an important subject for further investigation. It will not be pursued here. The interested reader may consult a fundamental paper by Samaddar and Yildiz [1964] who consider the excitation of axially symmetric (i.e., $n=0$) surface waves in a compressible plasma cylinder surrounded by free space.

6. Two-Dimensional Situations

Another important special case is when the fields do not depend on z [Wait, 1964a]. This occurs, for example, if the excitation is a normally incident plane wave or a cylindrical wave emanating from a uniform line source. In the general formulas listed from (17) to (25), this means that h is set equal to zero and the operator Γ now is defined by

$$p=\Gamma\beta_n=\sum_{n=-\infty}^{+\infty}\beta_n e^{-in\phi}. \tag{51}$$

It is evident that for z independent fields, (30) be partly decomposed to the set

$$H_n a_n=-J_n A_n \tag{52}$$

and

$$\begin{bmatrix} i\mu_0\omega k_e H_n' & in\mathcal{E}\hat{H}_n \\ \dfrac{in\mathcal{E}k_e}{a} & -\dfrac{\epsilon_0 k_p}{i\omega m n_0 e}\hat{H}_n' \end{bmatrix}\begin{bmatrix} b_n \\ c_n \end{bmatrix}=\begin{bmatrix} -i\mu_0\omega k_e J_n' B_n - in\mathcal{E}\hat{J}_n C_n \\ -\dfrac{in\mathcal{E}k_e^2}{a}J_n B_n+\dfrac{\epsilon_0 k_p}{i\omega m n_0 e}J_n' C_n \end{bmatrix}. \tag{53}$$

Here, the arguments of the Bessel functions J_n and H_n are $k_e a$ while the arguments of $\hat{J}_n$ and $\hat{H}_n$ are $k_p a$.

It is evident from the above that a normally incident TM wave will scatter only a TM wave even though the plasma is compressible. However, a normally incident TE wave will scatter both a TE wave and an acoustic-type wave. To illustrate this point more clearly, a specific case is considered. The incident TE wave is defined by

$$H_z^{\text{inc}}=H_0\exp(ik_e\rho\cos\theta)=H_0\sum_{n=-\infty}^{+\infty}i^n e^{-in\phi}J_n(k_e\rho),\ p^{\text{inc}}=0. \tag{54}$$

By utilizing the matrix equation (53), it is found that the secondary TE electromagnetic wave is obtained from

$$H_z^{\text{sc}}=H_0\sum_{n=-\infty}^{+\infty}i^n\Lambda_n H_n^{(2)}(k_e\rho)e^{-in\phi}, \tag{55}$$

where

$$\Lambda_n=-\frac{(k_e a)(k_p a)J_n'(k_e a)H_n^{(2)\prime}(k_p a)-n^2(\omega_0/\omega)^2 J_n(k_e a)H_n^{(2)}(k_p a)}{(k_e a)(k_p a)H_n^{(2)\prime}(k_e a)H_n^{(2)\prime}(k_p a)-n^2(\omega_0/\omega)^2 H_n^{(2)}(k_e a)H_n^{(2)}(k_p a)}. \tag{56}$$

It also follows from (53) that the scattered acoustic wave is to be obtained from

$$p=p^{\text{sc}}=H_0\sum_{n=-\infty}^{+\infty}i^n P_n H_n^{(2)}(k_p\rho)e^{-in\phi}, \tag{57}$$

where

$$P_n=\frac{(2in/\pi)(\omega_0/\omega)^2(m\omega/e)}{(k_e a)(k_p a)H_n^{(2)\prime}(k_e a)H_n^{(2)\prime}(k_p a)-n^2(\omega_0/\omega)^2 H_n^{(2)}(k_e a)H_n^{(2)}(k_p a)}. \tag{58}$$

In writing (58) in the above form, the well-known Wronskian relation for Bessel functions has been utilized. Equations (55) and (57) agree with the results derived directly by Seshadri et al. [1964]. Their analysis was restricted at the outset to normal (i.e., perpendicular) incidence.

It is easy to see that (56) may be rewritten in the form

$$\Lambda_n=\Lambda_n^c[1-\Delta_n], \tag{59a}$$

where

$$\Lambda_n^c=-\frac{J_n'(k_e a)}{H_n^{(2)\prime}(k_e a)}, \tag{59b}$$

and

$$1-\Delta_n=\frac{1-\dfrac{n^2(\omega_0/\omega)^2}{(k_p a)^2(k_e a)^2}\dfrac{J_n(k_e a)}{J_n'(k_e a)}\dfrac{H_n^{(2)}(k_p a)}{H_n^{(2)\prime}(k_p a)}}{1-\dfrac{n^2(\omega_0/\omega)^2}{(k_p a)^2(k_e a)^2}\dfrac{H_n^{(2)}(k_e a)}{H_n^{(2)\prime}(k_e a)}\dfrac{H_n^{(2)}(k_p a)}{H_n^{(2)\prime}(k_p a)}}. \tag{59c}$$

This shows that for a cold plasma (i.e., $k_p a \rightarrow \infty$), $\Lambda_n = \Lambda_n^c$, which is the appropriate value for scattering (at normal incidence) of a plane wave from the perfectly conducting cylinder immersed in a cold plasma whose wave number is k_e. If either u/c or ω_0/ω is sufficiently small, the correction factor Δ_n itself may be very small. To within this approximation, it is not difficult to show that

$$\Delta_n \cong -\frac{(2n^2/\pi)(\omega_0/\omega)^2(u/c)}{(k_e a)^2 J_n'(k_e a) H_n^{(2)\prime}(k_e a)}, \tag{60}$$

where, as usual, $k_e = k(1-\omega_0^2/\omega^2)^{\frac{1}{2}}$ in terms of the free space wave number k. In most situations, the factor Δ_n is exceptionally small. For example, in the case of ionospheric plasma u/c is of the order of 10^{-3}.

7. Concluding Remarks

On the basis of the analysis in this paper, it is evident that TE, TM, and acoustic waves are coupled in a cylindrically stratified compressible plasma. However, when the fields are independent of the axial coordinate, the TM waves become decoupled, although the TE and the acoustic waves are still coupled. Also, in the limiting case of a cold (i.e., incompressible) plasma, the acoustic wave is not present but there may still be coupling between the TE and TM electromagnetic waves.

In the analysis in the present paper, the presence of a superimposed d-c magnetic field is ignored. As indicated by Field [1956] and much more recently by the author [Wait, 1964b], this will cause additional coupling. Furthermore, the influence of nonlinearities in the medium will greatly complicate matters [Ginzburg, 1964].

8. References

Cohen, M. H. (1961), Radiation in a plasma, I, Cerenkov effect, Phys. Rev. **123,** 711–721.
Field, G. B. (July–Nov. 1956), Radiation by plasma oscillations, Astrophys. J. **124,** 555–570.
Ginzburg, V. L. (1964), The propagation of electromagnetic waves in plasma (Pergamon Press, Oxford).
Johler, J. R. (1964), private communication.
McLachlan, N. W. (1934), Bessel functions for engineers (Oxford Univ. Press.).
Morse, P. M., and H. Feshbach (1953), Methods of Theoretical Physics, **II**, p. 1387 (McGraw-Hill Book Co. Inc., New York.).
Oster, L. (Jan. 1960), Linearized theory of plasma oscillations, Rev. Mod. Phys. **32,** 141–168.
Rayleigh, Lord (1918), On the dispersal of light by a dielectric cylinder, Phil. Mag. **36,** 365–376.
Samaddar, S. N., and M. Yildiz (Apr. 1964), Excitation of coupled electro-acoustical waves by a ring source in a compressible plasma cylinder, Can. J. Phys. **42,** 638–656.
Seshadri, S. R., I. L. Morris, and R. J. Mailloux (Mar. 1964), Scattering by perfectly conducting cylinder in a compressible plasma, Can. J. Phys. **42,** 465–476.
van de Hulst, H. C. (1957), Light Scattering by Small Particles p. 327 (John Wiley and Sons, Inc., New York).
Wait, J. R. (Feb. 1955), Scattering of a plane wave from a circular dielectric cylinder at oblique incidence, Can. J. Phys. **33,** 189–195.
Wait, J. R. (1959), Electromagnetic radiation from cylindrical structures (Pergamon Press, Oxford, and Soviet Radio Press, Moscow, 1963).
Wait, J. R. (Sept. 1964a), Radiation from sources immersed in compressible plasma media, Can. J. Phys. **42.**
Wait, J. R. (Nov. 1964b), On the theory of electromagnetic waves from the interface between a compressible magnetoplasma and a dielectric, Radio Sci. J. Res. NBS/USNC–URSI **68D,** No. 11, 1187–1191.
Wellmann, P. (1937), Die Absorption und Streuung durch kleine zylindrische Korper im intersetllaren Raum, Zeit. fur Astrophys. **14,** 195–207.
Yildiz, A. (Dec. 1963), Scattering of a plane plasma wave from a plasma sphere, Il Nuovo Cimento, Series X, **30,** 1182–1207.

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