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Effect of Electron Collisions on the Formulas of Magneto-Ionic Theory

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In the standard simple treatment of magneto-ionic theory the effect of collisions is allowed for by assuming that an electron experiences a retarding force proportional to its velocity, and the two complex refractive indices of an ionized medium are then given by the Appleton-Hartree formula. Experiments have shown, however, that the collision frequency is approximately proportional to the square of the electron's velocity, and proper allowance for this, using the Boltzmann equation, leads to a modification of the Appleton-Hartree formula which was given by Sen and Wyller. In this tutorial paper the same modified formula is derived, but by a different method which, it is hoped, can be followed by readers not intimately familiar with the previous literature. Some numerical results are presented in which the modified and unmodified formulas are compared. It is concluded that the standard Appleton-Hartree formula can be used without modification for nearly all radio propagation problems in the ionosphere, provided that the correct effective value of the collision frequency is used. The modifications may be important, however, in the theory of wave interaction and for waves of very low frequency whose wave normals are perpendicular to the earth's magnetic field.

1. Introduction

The two refractive indices for radio waves in an ionized medium are given by the Appleton-Hartree formula [Appleton, 1932; Ratcliffe, 1959]. They are, in general, complex numbers whose values are affected by the collisions of the electrons in the medium with other particles. In the simplest treatments, this is allowed for by the method of Lorentz [1952, sec. 120 and note 57; see also Ratcliffe, 1959, ch. 4, 5] in which it is assumed that an electron of mass m experiences a retarding force $-m\nu\mathbf{v}$ proportional to its velocity $\mathbf{v}$, where ν is the effective collision frequency and is assumed to be a constant. But experiments (Huxley, 1959; Phelps and Pack, 1959] show that the average number of collisions ν which an electron makes per unit time with other particles, depends on the electron velocity $\mathbf{v}$, and when this is allowed for, a more elaborate treatment of magneto-ionic theory is needed. The effect of collisions of electrons with neutral molecules and with charged ions is somewhat different [Ginzburg, 1964]. In the lowest part of the ionosphere the concentration of molecules is so great that collisions of electrons with molecules dominate, and collisions with charged ions can be ignored, and only this case is considered in the present paper. There is good evidence [Huxley, 1959; Phelps and Pack, 1959] that the average number of collisions of an electron with molecules per unit time is proportional to the square of the velocity.

This problem has already been discussed by Molmud [1959] and more fully discussed by Sen and Wyller [1960] whose results are the same as those given here. Thus there is nothing new in the formulas derived in the present paper, but the method of derivation is different. Sen and Wyller used a coordinate system with oblique axes, and they allowed for the harmonic time variation of the radio wave fields by including real factors $\cos \omega t$ and $\sin \omega t$. In the present paper we use orthogonal axes and a complex time factor $\exp(i\omega t)$ as was done, for example, by Allis [1956], and these lead to considerable simplification.

The results are important in studying the propagation of radio waves in the lowest part of the ionosphere. In one kind of experiment Gardner and Pawsey [1953] have observed the reflection of vertically incident radio waves from small discontinuities of electron density in the height range 65 km to 90 km. By measuring the ratio of the reflected amplitudes for the ordinary and extraordinary rays it was possible to deduce the effective collision frequency.

But Belrose and Burke [1964] have shown that the results are significantly different for the simplified Lorentz theory and for the more exact theory, which leads to the formulas of Sen and Wyller. The more exact theory may also be important when studying the reflection of waves of very low frequency from the ionosphere, for in some published work [Budden, 1955; Barron and Budden 1959] the simple Lorentz treatment was used. The modifications introduced by the more exact theory have been discussed by Johler and Harper [1962] and by Wait [1962, appendix B]. But recently Deeks (private communication) has calculated the reflection coefficients of the ionosphere for a range of very low frequencies, using both the modified and unmodified formulas, and finds that the differences are negligible (see sec. 9 below).

The treatment given both by Sen and Wyller and in the present paper is based on the Boltzmann equation, which is set out in section 3. It is simplified by assuming that the field of the radio wave is very small and gives only a small perturbation f_1 of the distribution function. This leads to a linear equation for f_1 given at the end of section 3. It contains a term which allows for collisions, and is discussed in section 4 where it is expressed as an integral—the collision integral. The equation for f_1 is then solved in section 5, and the solution is used in section 6 to derive the electric susceptibility matrix of the medium. From this the refractive indices can be derived, and this is done first for an isotropic medium in section 7. The application to the full magneto-ionic theory is discussed in section 8 where Sen and Wyller's modifications to the refractive indices and wave polarizations are derived. Some numerical results are presented and discussed in section 9. When the field of the radio wave is stronger, some of the terms which are neglected in the simple theory are no longer small, and the consequences of this are very briefly reviewed in section 10.

2. Notation

The formulas in this paper are valid in any system of self-consistent rationalized units. The principal symbols are as follows:

$\mathbf{B}$=magnetic induction of superimposed constant magnetic field.
c=velocity of electromagnetic waves in free space.
$\mathbf{E}$=electric field of radio wave, components E_x, E_y, E_z.
e=charge of electron.
f=Boltzmann distribution function for electrons.
$\mathbf{H}$=magnetic field of radio wave, components H_x, H_y, H_z.
$\mathbf{J}$=current density.
K=Boltzmann's constant.
$k=\omega/c$, 2π divided by wavelength in free space.
m=mass of electron.
N=number of electrons per unit volume.
N_m=number of molecules per unit volume.
n=refractive index.
$\mathbf{n}$=vector of length n in direction of wave normal.
$P=\mathbf{J}/i\omega$, electric polarization.
$\mathbf{r}$=position vector, components x, y, z.
$d^3\mathbf{r}=dxdydz$.
T=absolute temperature of electrons.
t=time.
$U=1-iZ$.
$U_{\text{eff}}=1-iZ_{\text{eff}}$.
$\mathbf{v}$=electron velocity vector, length v, components v_x, v_y, v_z.
$d^3\mathbf{v}=dv_xdv_ydv_z$.
$X=Ne^2/\epsilon_0 m\omega^2$.
$\mathbf{Y}=e\mathbf{B}/\omega m$, vector of length Y.
$Z=\nu(v)/\omega$.

$Z_m = \nu_m/\omega = 1/w$.

$Z_{\text{eff}} = \frac{5}{2}\nu_m/\omega$, effective value of Z used in standard magneto-ionic theory.

Z_0 = characteristic impedance of free space.

ϵ = scalar dielectric constant.

$\boldsymbol{\epsilon}$ = tensor dielectric constant, with elements ϵ_{ij}, $(i, j = x, y, z)$.

$\epsilon_1, \epsilon_2, \epsilon_3$ = principal axis values of dielectric constant; ϵ_3 refers to axis parallel to **B**.

θ = angle between vector **Y** and wave normal.

ν = collision frequency, a function of electron velocity.

$\nu_m = \frac{2}{3} \times$ average value of ν, when $\nu \propto v^2$.

ρ, ρ_0, ρ_x = wave polarizations in magneto-ionic theory.

$\omega = 2\pi \times$ frequency of wave.

$\text{grad}_{\mathbf{r}}$ = vector operator in ordinary space, with components $\partial/\partial x, \partial/\partial y, \partial/\partial z$.

$\text{grad}_{\mathbf{v}}$ = vector operator in velocity space, with components $\partial/\partial v_x, \partial/\partial v_y, \partial/\partial v_z$.

3. Linearized Boltzmann Equation

We consider a neutral, ionized, homogeneous medium containing N electrons and N positive ions per unit volume, and with neutral molecules whose concentration N_m is very large compared with N so that electron-electron and electron-ion encounters can be ignored. A molecule is very massive compared with an electron so that as a first approximation the molecules can be assumed to be infinitely massive and at rest. Thus the state of the medium can be described by the Boltzmann distribution function $f(\mathbf{r}, \mathbf{v})$ for the electrons. There is a superimposed constant magnetic field of induction **B**. The function f must satisfy the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\ \text{grad}_{\mathbf{r}}\ f + \mathbf{a}\ \text{grad}_{\mathbf{v}}\ f = \frac{\partial_e f}{\partial t} \tag{1}$$

[see, for example, Chapman and Cowling 1958, ch. 3] where **a** is the acceleration imparted to an electron by all forces except encounters. The term $\frac{\partial_e f}{\partial t}$ is assumed to arise only from electron-molecule encounters.

The acceleration is given by

$$\mathbf{a} = \frac{e}{m}\{\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\} \tag{2}$$

where **E** is the electric field of the radio wave, which is assumed to be so weak that the force exerted on an electron by the magnetic field **H** of the wave can be neglected. It is now further assumed that when **E** is zero, and when the electrons have reached a state of equilibrium, the function f is given by the Maxwellian distribution function

$$f_0(v) = N\left(\frac{m}{2\pi KT}\right)^{3/2} \exp\ (-mv^2/2KT). \tag{3}$$

This is independent of time so that both $\frac{\partial f_0}{\partial t}$ and $\frac{\partial_e f_0}{\partial t}$ are zero. In the ionosphere the electron concentration N in (3) is a function of the height z and might give a contribution to the term $\mathbf{v} \cdot \text{grad}_{\mathbf{r}} f$ in (1). But it can be shown that the order of magnitude of this term is small enough to be neglected for frequencies greater than about 1 kc/s, though it could possibly become important in the ELF range (less than about 1 kc/s). Thus we shall assume that N is independent of **r** so that the medium is homogeneous.

It follows from (3) that $\text{grad}_{\mathbf{v}} f_0$ is a vector parallel to **v**. Thus its scalar product with the term $\mathbf{v} \wedge \mathbf{B}$ in (2) is zero. The field **E** is assumed to be very small, and when it is present

the function f undergoes a small change so that we now have

$$f(\mathbf{v}, \mathbf{r}, t)=f_0+f_1(\mathbf{v}, \mathbf{r}, t) \tag{4}$$

where the perturbation f_1 is so small that squares and products of f_1 and $\mathbf{E}$ can be neglected. If these results are used in (1) we obtain

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\operatorname{grad}_{\mathbf{r}} f_1+\frac{e}{m}\,\mathbf{E}\operatorname{grad}_{\mathbf{v}} f_0+\frac{e}{m}\,(\mathbf{v}\wedge\mathbf{B})\operatorname{grad}_{\mathbf{v}} f_1=\frac{\partial_e f_1}{\partial t}. \tag{5}$$

This equation is linear in $\mathbf{E}$ and f_1. The electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ of the radio wave must satisfy Maxwell's equations which are also linear. We seek a solution of the equations which represents a monochromatic wave of angular frequency ω. It follows from the linearity of the equations that a solution exists in which f_1, $\mathbf{E}$, and $\mathbf{H}$ all vary harmonically in time with angular frequency ω. This may be allowed for by assuming that each of them includes a complex time factor $\exp(i\omega t)$ which is customarily omitted from the equations, and when interpreting the complex f_1, $\mathbf{E}$ and $\mathbf{H}$ it is understood that the real parts are to be taken. Thus in (5) we may set

$$\frac{\partial f_1}{\partial t}=i\omega f_1. \tag{6}$$

The term $\dfrac{\partial_e f_1}{\partial t}$ is considered in the next section.

In magneto-ionic theory we seek a solution which represents a plane monochromatic wave. Thus f_1, $\mathbf{E}$, and $\mathbf{H}$ all contain a factor

$$\exp i(\omega t-k\mathbf{n}\cdot\mathbf{r}) \tag{7}$$

where the vector $\mathbf{n}$ has the direction of the wave normal and (complex) magnitude equal to the refractive index. If this is used in the second term of (5) it gives

$$\mathbf{v}\cdot\operatorname{grad}_{\mathbf{r}} f_1=-ik\mathbf{n}\cdot\mathbf{v}f_1. \tag{8}$$

This term depends on how f_1 varies in space, and its effect on the results is sometimes called "spatial dispersion" [Ginzburg, 1964]. It is very small compared with (6) provided that

$$n<<c/v. \tag{9}$$

Now in magneto-ionic theory we are usually concerned with refractive indices of the order of unity or less, whereas the number of electrons whose velocity v is comparable with c is very small. Thus (9) is satisfied and we may neglect the spatial dispersion term (8). By retaining the term (8) it is possible to study plasma waves in the medium, but this is not part of classical magneto-ionic theory. The term (8) also leads to Landau damping which is negligibly small for radio waves when n is of order unity. This term has a very small effect on the polarizations and refractive indices of magneto-ionic theory [see Ginzburg 1964, sec. 12]. In the rest of this paper it is neglected, except for a brief mention at the end of section 5.

The third term in (5) can be evaluated from (3) and gives

$$\frac{e}{m}\,\mathbf{E}\operatorname{grad}_{\mathbf{v}} f_0=-\frac{e}{KT}\,\mathbf{E}\cdot\mathbf{v}f_0. \tag{10}$$

With these simplifications (5) gives

$$i\omega f_1-\frac{e}{KT}\,\mathbf{E}\cdot\mathbf{v}f_0+\frac{e}{m}\,(\mathbf{v}\wedge\mathbf{B})\cdot\operatorname{grad}_{\mathbf{v}} f_1=\frac{\partial_e f_1}{\partial t}. \tag{11}$$

This is the form of the linearized Boltzmann equation which will be used in the rest of this paper.

4. Collision Integral

The Boltzmann distribution function $f(\mathbf{r}, \mathbf{v})$ is defined so that

$$f(\mathbf{r}, \mathbf{v}) d^3\mathbf{r} d^3\mathbf{v} \tag{12}$$

is the number of electrons in the small volume $d^3\mathbf{r}$ of ordinary space, having velocities within the small cell $d^3\mathbf{v}$ of velocity space. Thus the electrons of this small group all have a velocity very close to a given value $\mathbf{v}$. When one electron of the group makes a collision it is removed from the cell $d^3\mathbf{v}$. Other electrons in different cells make collisions which sometimes result in an electron moving into the cell $d^3\mathbf{v}$. It is these two processes which contribute to the term $\frac{\partial_e f}{\partial t}$ and an expression for this must now be derived.

It is assumed that a collision is practically instantaneous so that $\mathbf{E}$ and f_1 do not change appreciably while it is occurring. Thus the duration of a collision must be very small compared with $1/\omega$. This is believed to be true for electron-molecule encounters when ω refers to a radio wave of low or medium frequency. It may not be true for electron-ion encounters or for very high frequencies and microwaves. The problem is then more difficult and it is not always possible to use the Boltzmann equation in these cases.

The molecules are assumed to be infinitely massive and at rest. Thus an electron which approaches a molecule is deflected by the encounter but the magnitude of its velocity is unchanged. It jumps from the cell $d^3\mathbf{v}$ to another cell $d^3\mathbf{v}'$ on the same sphere, of radius v, in velocity space. Suppose that there is a molecule at the point M in ordinary space (fig. 1), and that an electron is approaching it along the line AB parallel to the velocity $\mathbf{v}$. The volume of the cell $d^3\mathbf{r}$ in ordinary space, though small, is very large compared with the region represented by figure 1. This cell thus contains a large number $N_m d^3\mathbf{r}$ of molecules, where N_m is the number of molecules per unit volume. An electron which makes an encounter with one of these molecules remains in the cell $d^3\mathbf{r}$ in ordinary space, though it jumps to a different cell in velocity space. Now imagine a plane drawn through M normal to AB, and take polar coordinates b, α in this plane with origin at M, radius b and angle α measured from some reference line MC. The radius b is called the impact parameter. Then PQRS is a small area $bdbd\alpha$ of this plane. The probability that an encounter occurs in time dt, with an electron approaching the area PQRS, is equal to the probability that there is an electron in the volume $bdbd\alpha vdt$ of ordinary space, so that the probability is

$$f(\mathbf{v}) d^3\mathbf{v} b db d\alpha v dt.$$

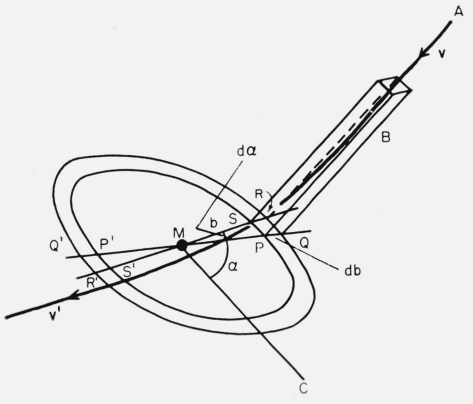

FIGURE 1. *Direct encounter of an electron with velocity* $\mathbf{v}$ *approaching an infinitely massive molecule at rest at* M.

The thick line represents the path of the electron.

After the encounter this electron has a new velocity $\mathbf{v}'$ where $v'=v$ but $\mathbf{v}'\neq\mathbf{v}$. The probability of this kind of encounter for all the molecules in the volume d^3r is

$$N_m v f(\mathbf{v}) d^3\mathbf{v} d^3\mathbf{r} b db d\alpha dt. \tag{13}$$

Electrons also enter the cell $d^3\mathbf{v}$ through collisions which are called "inverse encounters." There will clearly be some electrons which, after an encounter, are traveling with velocity $\mathbf{v}$ so that they are then in the cell $d^3\mathbf{v}$. Some examples are illustrated in figure 2b, in which the electrons initially have a velocity $\mathbf{v}'$ or $\mathbf{v}''$ or $\mathbf{v}'''$ all with the same magnitude v but with directions which may be anywhere within the solid angle 4π. Of these we consider only those electrons whose final velocity vector $\mathbf{v}$, when projected backwards, intersects the area P′Q′R′S′ of figure 1, which lies near the point with polar coordinates b, $\alpha+\pi$. Thus for these collisions, the impact parameter b is the same as for the direct encounter considered in (13), and the angle through which the electron is deflected is the same for both cases. Hence the initial velocity vector $\mathbf{v}'$ of the inverse encounter must be the same as the final velocity vector $\mathbf{v}'$ for the direct encounter, and the electron orbits for the two encounters are identical in shape, though differently oriented in space. The process of counting these inverse encounters is exactly the same as counting the direct encounters, which led to (13). If a small change is made in $\mathbf{v}$ so that it remains within the cell $d^3\mathbf{v}$ of velocity space, then there is a resulting change in $\mathbf{v}'$ which remains within a corresponding cell $d^3\mathbf{v}'$. Now the volumes $d^3\mathbf{v}$ and $d^3\mathbf{v}'$ of the two cells are equal. This can be seen by expressing $\mathbf{v}$ and $\mathbf{v}'$ in spherical polar coordinates, v,ϑ,φ and v',ϑ',φ' respectively, using as polar axis the bisector of the angle between $\mathbf{v}$ and $\mathbf{v}'$. Then $v=v'$, $\vartheta=\vartheta'$, $\varphi=\varphi'+\pi$ whence $d^3\mathbf{v}=v^2\sin\vartheta dv d\vartheta d\varphi=d^3\mathbf{v}'$. This is a special case of the principle of detailed balancing and a formal proof, for a more general type of binary collision, is given by Chapman and Cowling [1958, sec. 3.52].

The number of electrons leaving the cell $d^3\mathbf{r}d^3\mathbf{v}$ in time dt is given by (13). The number entering the cell from the inverse encounters is the same expression with $f(\mathbf{v})d^3\mathbf{v}$ replaced by $f(\mathbf{v}')d^3\mathbf{v}'=f(\mathbf{v}')d^3\mathbf{v}$. The resultant number entering $d^3\mathbf{v}d^3\mathbf{r}$ in time dt is thus

$$N_m\{f(\mathbf{v}')-f(\mathbf{v})\}v d^3\mathbf{v} d^3\mathbf{r} b db d\alpha dt. \tag{14}$$

To find $\dfrac{\partial_e f}{\partial t}$, equation (14) is divided by $d^3\mathbf{v}d^3\mathbf{r}dt$ and then integrated with respect to b and α. Thus

$$\frac{\partial_e f}{\partial t}=N_m v\int_0^{2\pi}\int_0^{\infty}\{f(\mathbf{v}')-f(\mathbf{v})\}b db d\alpha. \tag{15}$$

This is the collision integral. In the integrand the vector $\mathbf{v}$ is a constant, and $\mathbf{v}'$ and $\mathbf{v}$ have equal magnitudes. But the direction of $\mathbf{v}'$ is a function of b and α.

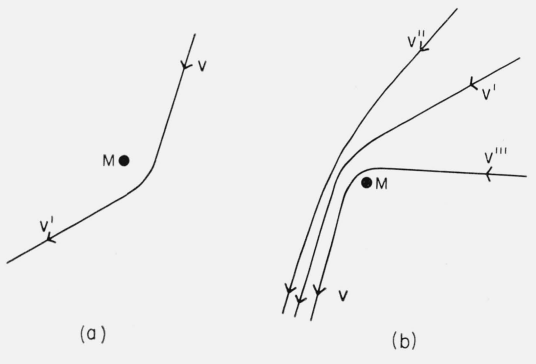

FIGURE 2. *In (a) a direct encounter is shown, in which an electron initially with velocity* $\mathbf{v}$ *leaves the cell* $d^3\mathbf{v}$ *centered on the point* $\mathbf{v}$ *in velocity space.*

Some inverse encounters are shown in (b), in which electrons with various initial velocities $\mathbf{v}'$, $\mathbf{v}''$, $\mathbf{v}'''$ all enter the same cell $d^3\mathbf{v}$ after the encounter.

When the impact parameter b is large, it usually happens that the electron is almost undeflected by the encounter, so that $\mathbf{v}' \approx \mathbf{v}$ and the integrand is very small. This occurs for electron-molecule encounters, where the forces are of short range, and the integral (15) converges as b increases. Thus the upper limit for b is taken as infinity. But for electron-ion encounters the inverse square law of force is of longer range and the integral does not converge. Then the upper limit for b must be assessed by considering the screening effect of the neighboring plasma. In the present paper, however, these considerations need not concern us since, following Sen and Wyller [1960], we shall use experimental results to give the final form of the term $\frac{\partial_e f}{\partial t}$.

When $f(\mathbf{v})$ is independent of the direction of $\mathbf{v}$, the integrand of (15) is zero. Thus $f_0(\mathbf{v})$ contributes nothing to the collision integral, and the f in (15) may be replaced by f_1. Equations (11) and (15) then give

$$i\omega f_1 - \frac{e}{KT}\mathbf{E}\cdot\mathbf{v}f_0 + \frac{e}{m}(\mathbf{v}\wedge\mathbf{B})\cdot\mathrm{grad}_{\mathbf{v}}\, f_1 = N_m v\int_0^{2\pi}\int_0^{\infty}\{f_1(\mathbf{v}') - f_1(\mathbf{v})\}b\,db\,d\alpha \tag{16}$$

which applies for each value of the velocity vector $\mathbf{v}$, and is to be solved to find the function $f_1(\mathbf{v})$.

5. Solution of the Equation

Equation (16) must apply at each point $\mathbf{v}$ in velocity space and we shall study it for those points which lie on a sphere of fixed radius v. Then f_1 is a function of the direction of $\mathbf{v}$, and may be expanded in a series of spherical harmonics thus

$$f_1(\mathbf{v}) = f_{10}(v) + f_{11}(\mathbf{v}) + f_{12}(\mathbf{v}) + \ \ldots \tag{17}$$

where $f_{1r}(\mathbf{v})$ is a spherical harmonic of order r. This is substituted into (16) and the terms representing harmonics of the various orders are equated. The second term in (16) is a spherical harmonic of the first order, which could be written $-\frac{e}{KT}f_0 Ev\cos\vartheta$ where ϑ is the angle between $\mathbf{E}$ and $\mathbf{v}$. The polar axis is then the direction of $\mathbf{E}$ and may be a complex direction since $\mathbf{E}$ in general represents an elliptically polarized wave [Budden, 1964b].

Now the term $f_{10}(v)$ is independent of the direction of $\mathbf{v}$, so that $\mathrm{grad}_{\mathbf{v}}\, f_{10}(v)$ is a vector parallel to $\mathbf{v}$ and thus contributes nothing to the third term of (16). Similarly $f_{10}(v)$ contributes nothing to the collision integral because $f_{10}(v') = f_{10}(v)$. The only spherical harmonic of zero order in (16) is in the term $i\omega f_1$, so that

$$f_{10}(v) = 0. \tag{18}$$

Next consider the spherical harmonic of first order $f_{11}(\mathbf{v})$ which proves to be the most important term. It may be written

$$f_{11}(\mathbf{v}) = \mathbf{v}\cdot\mathbf{V}(v) \tag{19}$$

where the vector $\mathbf{V}$ depends only on the magnitude of $\mathbf{v}$ and not on its direction. Now

$$\mathrm{grad}_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{V}) = \mathbf{V} + \left(\mathbf{v}\cdot\frac{\partial\mathbf{V}}{\partial v}\right)\mathbf{v}/v. \tag{20}$$

The last term of (20) is a vector parallel to $\mathbf{v}$ and contributes nothing to the third term of (16), which therefore gives

$$\frac{e}{m}(\mathbf{v}\wedge\mathbf{B})\cdot\mathrm{grad}_{\mathbf{v}} f_{11} = \frac{e}{m}(\mathbf{v}\wedge\mathbf{B})\cdot\mathbf{V} = \frac{e}{m}(\mathbf{B}\wedge\mathbf{V})\cdot\mathbf{v} \tag{21}$$

which is a spherical harmonic of the first order. The contribution of (19) to the collision

integral is

$$N_m v\mathbf{V}\cdot\int_0^{2\pi}\int_0^{\infty}(\mathbf{v}'-\mathbf{v})bdbd\alpha. \tag{22}$$

To evaluate this, choose Cartesian axes v_x, v_y, v_z in velocity space so that v_x is in the reference direction MC in figure 1, and v_z is in the direction of $\mathbf{v}$. Suppose that in the encounter the electron is deflected through an angle $\chi(b)$. Then the Cartesian components of $\mathbf{v}'-\mathbf{v}$ are

$$-v\sin\chi\cos\alpha,-v\sin\chi\sin\alpha,\ v(\cos\chi-1). \tag{23}$$

Integration of the first two with respect to α gives zero so that only the v_z component remains and this is parallel to $\mathbf{v}$. Hence (22) gives

$$-2\pi N_m v\mathbf{V}\cdot\mathbf{v}\int_0^{\infty}(1-\cos\chi)bdb. \tag{24}$$

Now let

$$A(v)=2\pi\int_0^{\infty}(1-\cos\chi)bdb. \tag{25}$$

This is called the "transport area of cross section" for the encounter. It may depend on v, but is independent of the direction of $\mathbf{v}$. It is determined by the mechanics of an electron-molecule encounter. For example, it is easy to show that if the molecules are rigid spheres of radius a, then $A=\pi a^2$.

If each molecule presents an effective area $A(v)$ to the oncoming electrons, then the probability of an electron encountering a molecule within a path dx is $N_m A dx$. Thus the mean free path is

$$\lambda(v)=1/\{N_m A(v)\} \tag{26}$$

and the average number of collisions per unit time is

$$\nu(v)=v/\lambda=N_m A v. \tag{27}$$

Equation (27) is adopted as the definition of the collision frequency $\nu(v)$. Huxley [1959] has described experiments which show that A is proportional to v, so that $\nu(v)$ in (27) is proportional to v^2, and this result is used later. If (27) is combined with (25), the contribution (24) to the collision integral is

$$-\mathbf{V}\cdot\mathbf{v}\nu(v) \tag{28}$$

which is a spherical harmonic of the first order in velocity space. It is shown below that the spherical harmonics of second and higher order in (17) do not contribute any terms of first order in (16). Hence the spherical harmonics of first order give

$$i\omega\mathbf{V}\cdot\mathbf{v}-\frac{e}{KT}f_0\mathbf{E}\cdot\mathbf{v}+\frac{e}{m}(\mathbf{B}\wedge\mathbf{V})\mathbf{v}+\nu\mathbf{V}\cdot\mathbf{v}=0 \tag{29}$$

which can now be solved to find $\mathbf{V}$. This is done in the following section.

The spherical harmonic of the second order in (17) may conveniently be written in tensor notation thus

$$f_{12}(\mathbf{v})=W_{ij}(v)v_iv_j \qquad (i,j=x,\ y,\ z) \tag{30}$$

where W_{ij} is a symmetric tensor whose trace is zero, and the summation convention for repeated suffixes is used. The contribution of (30) to the collision integral in (16) is

$$N_m v W_{ij}\iint(v_i'v_j'-v_iv_j)\ bdbd\alpha \tag{31}$$

and here the integral can be evaluated by using the same Cartesian coordinate system as was

used above to evaluate (22). The elements of the integrand of (31) can then be written in terms of the polar coordinates v, χ, α of $\mathbf{v}'$ (compare (23)) and integrated with respect to α. In this way it can be shown that (31) gives

$$-N_m v W_{ij} v_i v_j \int_0^\infty \frac{3}{2} \sin^2 \chi b db \tag{32}$$

which is a spherical harmonic of the second order and contributes nothing to the first order terms in (29).

The contribution of (30) to the third term in (16) may similarly be found, since on converting to tensor notation

$$a_k = \{\mathrm{grad}_{\mathbf{v}} f_{12}(\mathbf{v})\}_k = \frac{\partial}{\partial v_k}\{W_{ij}(v) v_i v_j\} = 2W_{ik} v_i + \frac{v_i v_j v_k}{v} \frac{\partial W_{ij}}{\partial v} \tag{33}$$

and

$$(\mathbf{v} \wedge \mathbf{B}) \cdot \mathbf{a} = \epsilon_{ijk} B_j a_k v_i \tag{34}$$

where ϵ_{ijk} is the third order isotropic tensor [Jeffreys and Jeffreys, 1956, ch. 3]. Substitution of (34) into (33) gives for the third term of (16)

$$\frac{e}{m}\left\{ 2\epsilon_{ijk} B_j W_{lk} v_i v_l + \epsilon_{ijk} B_j \frac{\partial W_{lm}}{\partial v} v_i v_k v_l v_m / v \right\} \tag{35}$$

in which the first term is a spherical harmonic of order two, and the second term is of order four, and there is no term of the first order.

More generally it can be shown that a spherical harmonic of order r in (17) does not contribute terms of smaller order in the third term of (16). By expressing it as Legendre functions and associated Legendre functions it can further be shown [Allis, 1956, p. 409] that it contributes only a spherical harmonic of tbe same order in the collision integral. Thus the spherical harmonic terms of order $r=2$ or greater, in (17), when inserted into (16) give a set of linear homogeneous equations which do not contain the field **E**. Since we are only interested in the disturbances produced by the radio wave field **E**, we may clearly take the terms $f_{1r}(\mathbf{v})$ to be zero when $r \geq 2$.

It may be noted that this simplification does not occur if the spatial dispersion term (8) is included, for then the spherical harmonic of first order, $f_{11}(\mathbf{v})$, when inserted in (8), gives a spherical harmonic of order two. Since $f_{11}(\mathbf{v}) = \mathbf{V} \cdot \mathbf{v}$ depends on **E**, as is shown by (29), this introduces a term depending on **E** into the second order spherical harmonic component of (16). Thus it is no longer permissible to take $W_{ij}=0$, which leads to some complication, but it is beyond the scope of this paper.

6. Susceptibility Matrix of the Medium

In the last section it was shown that only the second term of (17) is to be retained and it is given by (19). Thus the distribution function (4) for electrons becomes

$$f = f_0 + \mathbf{V} \cdot \mathbf{v} \tag{36}$$

where the vector $\mathbf{V}(v)$ is given by (29). Every term of (29) is the scalar product of **v** with some other vector, and since the equation must be true for all **v**, we may omit the factor $\cdot\,\mathbf{v}$ and obtain

$$\{i\omega + \nu(v)\}\mathbf{V} + \frac{e}{m}\mathbf{B} \wedge \mathbf{V} = \frac{e}{KT} f_0 \mathbf{E}. \tag{37}$$

Now it is convenient to use the standard notation (U.R.S.I. 1956) of magneto-ionic theory, in which the vector $\frac{e}{m\omega}\mathbf{B}$ is denoted by **Y** and is antiparallel to **B** because e is a negative number.

Similarly ν/ω is denoted by Z. This is slightly different from the usual Z because here $\nu(v)$ and $Z(v)$ may depend on the electron velocity. Finally we use $U(v)=1-iZ$. Then (37) gives

$$U\mathbf{V}-i\mathbf{Y}\wedge\mathbf{V}=\frac{e}{i\omega KT}f_0\mathbf{E}. \tag{38}$$

This is very similar to an equation used in standard magneto-ionic theory [Budden, 1961, eq. (3.15)] and may be solved in the same way. Choose Cartesian axes with the z-axis parallel to $\mathbf{Y}$ and write (38) in matrix form thus

$$\begin{pmatrix} U & iY & 0 \\ -iY & U & 0 \\ 0 & 0 & U \end{pmatrix}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}=\frac{e}{i\omega KT}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \tag{39}$$

By inversion of the 3×3 matrix this gives

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}=\frac{1}{U(U^2-Y^2)}\begin{pmatrix} U^2 & -iUY & 0 \\ iUY & U^2 & 0 \\ 0 & 0 & U^2-Y^2 \end{pmatrix}\frac{ef_0}{i\omega KT}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \tag{40}$$

This is more concisely written by using the matrix notation

$$\mathbf{Q}=\frac{1}{U(U^2-Y^2)}\begin{pmatrix} U^2 & -iUY & 0 \\ iUY & U^2 & 0 \\ 0 & 0 & U^2-Y^2 \end{pmatrix} \tag{41}$$

so that (40) gives

$$\mathbf{V}=\frac{ef_0}{i\omega KT}\mathbf{QE}. \tag{42}$$

The current density vector $\mathbf{J}$ in the ionized medium is given in terms of the Boltzmann distribution function thus

$$\mathbf{J}=e\iiint f(\mathbf{v})\mathbf{v}d^3\mathbf{v} \tag{43}$$

the integral being taken through the whole of velocity space. To this integral the symmetric function $f_0(\mathbf{v})$ contributes nothing so that insertion of (36) gives

$$\mathbf{J}=e\iiint(\mathbf{v}\cdot\mathbf{V})\mathbf{v}\cdot d^3\mathbf{v}. \tag{44}$$

Now use spherical polar coordinates v, ϑ, φ in velocity space and choose the direction of $\mathbf{V}$ as the polar axis. Then the components of (44) perpendicular to $\mathbf{V}$ are easily shown to be zero so that (44) gives

$$\mathbf{J}=e\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\mathbf{V}(v)v^4\cos^2\vartheta\sin\vartheta dvd\vartheta d\varphi=\frac{4}{3}\pi e\int_0^{\infty}\mathbf{V}(v)\cdot v^4dv. \tag{45}$$

Instead of $\mathbf{J}$ it is convenient to define an electric polarization vector

$$\mathbf{P}=\mathbf{J}/i\omega \tag{46}$$

as is done in standard magneto-ionic theory [Ratcliffe, 1959; Budden, 1961]. Then a combination of (46), (45), and (42) gives

$$\mathbf{P}=\epsilon_0\mathbf{M}\cdot\mathbf{E}, \tag{47}$$

where

$$\mathbf{M}=-\frac{4\pi e^2}{3\epsilon_0\omega^2 KT}\int_0^\infty f_0\mathbf{Q}v^4dv \tag{48}$$

is called the susceptibility matrix of the medium.

In standard magneto-ionic theory it is assumed that ν and Z are independent of the electron velocity v, and then $\mathbf{Q}$ is independent of v and may be taken outside the integral in (48). Now f_0 is given by (3) so that the integral may be evaluated and the result is

$$\mathbf{M}=-X\mathbf{Q} \tag{49}$$

which is the same as in standard magneto-ionic theory [Budden, 1961, eq (3.24)] and leads to the Appleton-Hartree formula.

7. Isotropic Ionosphere

For an isotropic ionosphere the earth's magnetic induction $\mathbf{B}$ is neglected so that $\mathbf{Y}$ is zero and $\mathbf{Q}$ in (41) becomes a scalar $1/U$. Then the susceptibility $\mathbf{M}$ also becomes a scalar M which is related to the complex dielectric constant ϵ, and the complex refractive index n thus:

$$n^2=\epsilon=1+M=1-\frac{4\pi e^2}{3\epsilon_0\omega^2 KT}\int_0^\infty \frac{f_0(v)v^4}{U}\,dv. \tag{50}$$

Again if U is independent of v this leads to the standard result

$$n^2=1-X/U. \tag{51}$$

Now it might be expected that when an electron moves in a gas of infinitely heavy molecules which are at rest, the probability of its encountering a molecule would depend simply on the position and direction of its path, and not on its speed. Thus the mean free path would be independent of velocity, but the average number of collisions made in unit time would be proportional to the velocity. This case has been considered by Margenau [1946] and by Ginzburg [1964, sec. 6] who gives curves showing how the real and imaginary parts of ϵ depend upon frequency ω.

The true position is less simple because fast moving electrons can more easily excite the rotational states of molecules, so that significant collisions occur more easily for fast electrons [Huxley, 1959]. Careful experiments [Crompton, Huxley, and Sutton, 1953; Huxley, 1959; Phelps and Pack, 1959] have shown that the collision frequency $\nu(v)$ is more closely proportional to the square of the velocity. Hence following Sen and Wyller [1960] it will be assumed that

$$\nu(v)=\frac{m\nu_m}{2KT}v^2. \tag{52}$$

By using (3) it can be shown that the average value of ν is then $\frac{3}{2}\nu_m$. Now let

$$u^2=\nu/\nu_m=mv^2/2KT. \tag{53}$$

By using this with (3) and (50):

$$n^2=\epsilon=1-\frac{4X}{3\sqrt{\pi}}\int_0^\infty \frac{u^{3/2}e^{-u}du}{1-iu\nu_m/\omega}. \tag{54}$$

This integral may be expressed in terms of standard integrals defined and tabulated by Dingle, Arndt, and Roy [1956–1957], namely

$$\mathcal{C}_p(w)=\frac{1}{p!}\int_0^\infty \frac{u^p e^{-u}du}{u^2+w^2}. \tag{55}$$

Now (54) is separated into its real and imaginary parts, and with

$$w=\omega/\nu_m \tag{56}$$

we obtain

$$n^2=\epsilon=1+M=1-X\{w^2\mathcal{C}_{3/2}(w)+\tfrac{5}{2}iw\mathcal{C}_{5/2}(w)\} \tag{57}$$

which replaces the standard formula (51).

When ν_m/ω is very small the integral in (54) may be evaluated approximately by using a binomial expansion for $(1-iu\nu_m/\omega)^{-1}$ and integrating term by term. This gives

$$n^2=\epsilon\approx 1-X\left\{1-\frac{35}{4}\left(\frac{\nu_m}{\omega}\right)^2+\cdots\right\}-iX\left(\frac{5}{2}\frac{\nu_m}{\omega}+\cdots\right). \tag{58}$$

When the formula (51) of standard theory is used, it is assumed that Z can be given an effective value Z_{eff} which is independent of electron velocity. Then (51) gives

$$n^2=\epsilon=1-X/(1+Z_{\text{eff}}^2)-iXZ_{\text{eff}}/(1+Z_{\text{eff}}^2). \tag{59}$$

If both ν_m/ω and Z_{eff} are so small that squares and higher powers can be neglected, (59) agrees with (58) provided that

$$\nu_{\text{eff}}/\omega=Z_{\text{eff}}=\frac{5}{2}\nu_m/\omega=5/(2w),\ \nu_{\text{eff}}=\frac{5}{2}\nu_m. \tag{60}$$

Thus, for very small values of Z the standard formula (59) is approximately correct provided that Z and ν have the effective values given by (60). Note that ν_{eff} is not the average value $\frac{3}{2}\nu_m$ of ν, but is $\frac{5}{3}$ of this average value. For greater values of Z, (59) is in error and (57) must be used. To obtain (57) from (59) the real and imaginary parts of $\epsilon-1$ must be multiplied by correcting factors K_r and K_i, respectively, so that, with (60)

$$\epsilon=1-K_rX/\{1+25/(4w^2)\}-5iK_i/\{2w+25/(2w)\} \tag{61}$$

and by equating this to (57)

$$K_r=\left(w^2+\frac{25}{4}\right)\mathcal{C}_{3/2}(w);\ K_i=\left(w^2+\frac{25}{4}\right)\mathcal{C}_{5/2}(w). \tag{62}$$

These two functions are shown in figure 3 plotted with $\log_{10}(1/Z_{\text{eff}})=\log_{10}(\frac{2}{5}w)$ as abscissa. Similar curves were given by Ginzburg [1964, sec. 6] for the case where $\nu(v)$ is proportional to v.

When $\nu_m/\omega=\frac{2}{5}Z_{\text{eff}}$ is very large (w very small) the integral in (54) may again be evaluated approximately by using a binomial expansion for the factor $\{1+i\omega/(u\nu_m)\}^{-1}$ and integrating term by term. This gives

$$n^2=\epsilon\sim 1-\frac{4}{3}Xw^2-i\frac{2}{3}Xw \tag{63}$$

and by comparison with (61), for small w

$$K_r\sim\frac{25}{3};\ K_i\sim\frac{5}{3}. \tag{64}$$

The curves of figure 3 approach these limiting values when Z_{eff} is large.

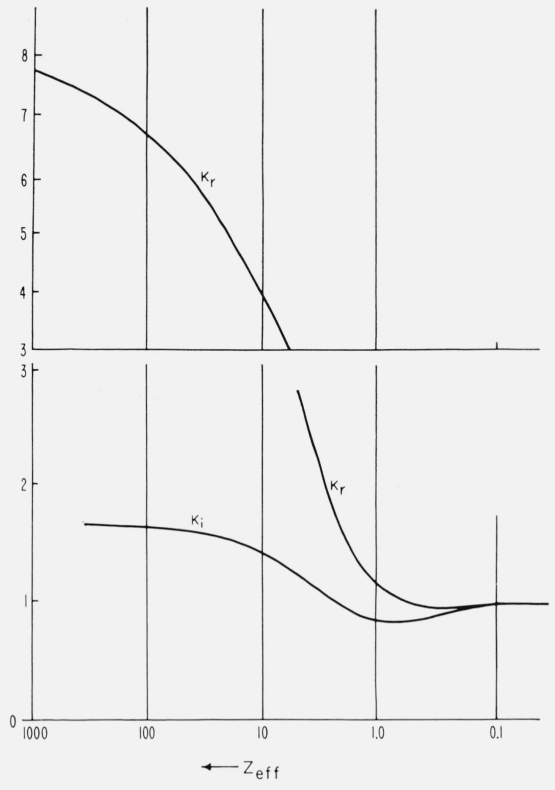

FIGURE 3. *The correction factors* K_r *and* K_i *used for converting the real and imaginary parts respectively of* $\epsilon-1$, *as computed with the simple formula (59), to the more accurate values given by (57).*

8. Refractive Indices of Anisotropic Ionosphere: Modified Appleton-Hartree Formula

For an anisotropic ionosphere Y is not zero and the elements of the matrix (48) must be found. Equation (41) shows that three different integrals are involved in (48) with factors U^2, UY, and U^2-Y^2 in the integrand. Further, the element M_{zz} is identical with M in (50), and is given by (57). The remaining nonzero elements are $M_{xx}=M_{yy}$ and $M_{xy}=-M_{yx}$, and (48) and (41) together show that

$$M_{xx}\mp iM_{xy}=-\frac{4\pi e^2}{3\epsilon_0\omega^2 KT}\int_0^\infty \frac{f_0 v^4}{U\pm Y}\,dv. \tag{65}$$

The two expressions in (65) together with M_{zz} comprise the three "principal axis" values of the susceptibility matrix (see Budden, 1961, sec. 3.11, and the references given there; 1964a, sec. 2.8), and the corresponding "principal axis" values of the dielectric constant matrix are

$$\epsilon_1=1+M_{xx}-iM_{xy},\ \epsilon_2=1+M_{xx}+iM_{xy},\ \epsilon_3=1+M_{zz}. \tag{66}$$

Now assume that $\nu(v)$ is given by (52) and use the variable u defined at (53). Then, with (3), equations (66) and (65) give

$$\left.\begin{matrix}\epsilon_1\\ \epsilon_2\end{matrix}\right\}=1-\frac{4X}{3\sqrt{\pi}}\int_0^\infty \frac{u^{3/2}e^{-u}du}{1\pm Y-iu\nu_m/\omega} \tag{67}$$

and ϵ_3 is given by (54). The real and imaginary parts of the integrals in (67) may be expressed

in terms of w, in (56), and of the $\mathcal{C}$ integrals given by (55). Thus it is easily shown that

$$\epsilon_1=1-X\left[(1+Y)w^2\,\mathcal{C}_{3/2}\{w(1+Y)\}+\frac{5}{2}\,iw\,\mathcal{C}_{5/2}\{w(1+Y)\}\right]$$

$$\epsilon_2=1-X\left[(1-Y)w^2\,\mathcal{C}_{3/2}\{w(1-Y)\}+\frac{5}{2}\,iw\,\mathcal{C}_{5/2}\{w(1-Y)\}\right]$$

$$\epsilon_3=1-X\left[w^2\,\mathcal{C}_{3/2}(w)+\frac{5}{2}\,iw\,\mathcal{C}_{5/2}(w)\right] \tag{68}$$

[note that these ϵ's are not the same as those used by Sen and Wyller, 1960]. These three complex numbers, involving six $\mathcal{C}$ integrals, are sufficient to determine the polarization and refractive index for a plane wave. The corresponding values according to standard magneto-ionic theory are

$$\left.\begin{aligned}\epsilon_1&=1-X/(U_{\text{eff}}+Y)\\ \epsilon_2&=1-X/(U_{\text{eff}}-Y)\\ \epsilon_3&=1-X/U_{\text{eff}}\end{aligned}\right\} \tag{69}$$

[Budden, 1964a, sec. 2.8] where $U_{\text{eff}}=1-iZ_{\text{eff}}$ is a constant independent of v.

The numbers in (68) determine the constitutive relation for the medium, which may be written

$$\epsilon_0\mathbf{E}+\mathbf{P}=\mathbf{D}=\epsilon_0\boldsymbol{\epsilon}\mathbf{E}=\epsilon_0(1+\mathbf{M})\mathbf{E}. \tag{70}$$

where $\boldsymbol{\epsilon}$ is the matrix dielectric constant. With the coordinate axes used in section 6, in which the z-axis was parallel to the vector $\mathbf{Y}$, the matrix $\boldsymbol{\epsilon}$ is given from (66), by

$$\boldsymbol{\epsilon}=\begin{pmatrix}\frac{1}{2}(\epsilon_1+\epsilon_2) & \frac{1}{2}i(\epsilon_1-\epsilon_2) & 0\\ -\frac{1}{2}i(\epsilon_1-\epsilon_2) & \frac{1}{2}(\epsilon_1+\epsilon_2) & 0\\ 0 & 0 & \epsilon_3\end{pmatrix}. \tag{71}$$

We now seek a solution of Maxwell's equations which represents a plane wave whose wave normal makes an angle θ with the vector $\mathbf{Y}$. There are several ways of doing this, used in standard works on magneto-ionic theory [Ratcliffe, 1959; Budden, 1961, 1964a; Sen and Wyller, 1960]. The following derivation is given only in outline, and uses the author's previously published method [Budden, 1961, 1964a].

Choose a Cartesian coordinate system in which the z-axis is the wave normal, and the x-axis is in the plane containing the wave normal and the vector $\mathbf{Y}$. In this system the dielectric constant $\boldsymbol{\epsilon}$ is given by

$$\boldsymbol{\epsilon}=\begin{pmatrix}\frac{1}{2}(\epsilon_1+\epsilon_2)\cos^2\theta+\epsilon_3\sin^2\theta, & \frac{1}{2}i(\epsilon_1-\epsilon_2)\cos\theta, & \frac{1}{2}(\epsilon_1+\epsilon_2)-\epsilon_3\sin\theta\cos\theta\\ -\frac{1}{2}i(\epsilon_1-\epsilon_2)\cos\theta, & \frac{1}{2}(\epsilon_1+\epsilon_2), & -\frac{1}{2}i(\epsilon_1-\epsilon_2)\sin\theta\\ \frac{1}{2}(\epsilon_1+\epsilon_2)-\epsilon_3\sin\theta\cos\theta, & \frac{1}{2}i(\epsilon_1-\epsilon_2)\sin\theta, & \epsilon_3\cos^2\theta+\frac{1}{2}(\epsilon_1+\epsilon_2)\sin^2\theta\end{pmatrix}. \tag{72}$$

For the plane wave, the differential operators with respect to the coordinates are

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial y}=0 \qquad \frac{\partial}{\partial z}=-ikn, \tag{73}$$

and when these are used in Maxwell's third and fourth equations, they give, respectively

$$nE_y=-Z_0H_x,\ nE_x=Z_0H_y,\ H_z=0 \tag{74}$$

and

$$\left.\begin{aligned} nZ_0H_y&=\epsilon_{xx}E_x+\epsilon_{xy}E_y+\epsilon_{xz}E_z \\ -nZ_0H_x&=\epsilon_{yx}E_x+\epsilon_{yy}E_y+\epsilon_{yz}E_z \\ \epsilon_{zx}E_x&+\epsilon_{zy}E_y+\epsilon_{zz}E_z=0 \end{aligned}\right\}. \tag{75}$$

The last equation (75) gives an expression for E_z which is substituted in the remaining equations. The ratio of the two equations in (74) and of the first two equations in (75) then gives

$$\rho=\frac{E_y}{E_x}=-\frac{H_x}{H_y}=\frac{\left(\epsilon_{yx}-\dfrac{\epsilon_{yz}\epsilon_{zx}}{\epsilon_{zz}}\right)E_x+\left(\epsilon_{yy}-\dfrac{\epsilon_{yz}\epsilon_{zy}}{\epsilon_{zz}}\right)E_y}{\left(\epsilon_{xx}-\dfrac{\epsilon_{xz}\epsilon_{zx}}{\epsilon_{zz}}\right)E_x+\left(\epsilon_{xy}-\dfrac{\epsilon_{xz}\epsilon_{zy}}{\epsilon_{zz}}\right)E_y} \tag{76}$$

so that

$$\rho=\frac{\epsilon_{zz}\epsilon_{yx}-\epsilon_{yz}\epsilon_{zx}+(\epsilon_{yy}\epsilon_{zz}-\epsilon_{yz}\epsilon_{zy})\rho}{\epsilon_{xx}\epsilon_{zz}-\epsilon_{xz}\epsilon_{zx}+(\epsilon_{xy}\epsilon_{zz}-\epsilon_{xz}\epsilon_{zy})\rho}. \tag{77}$$

The ratio ρ is a complex number which determines the polarization of the wave. Equation (77) is a quadratic equation for ρ and shows that the wave must have one of two possible polarizations. If the elements of ϵ from (72) are inserted into (77), then after rearrangement:

$$\rho^2+i\rho\,\frac{\sin^2\theta\{2\epsilon_1\epsilon_2-\epsilon_3(\epsilon_1+\epsilon_2)\}}{\cos\theta\epsilon_3(\epsilon_1-\epsilon_2)}+1=0. \tag{78}$$

This equation must hold both for standard magneto-ionic theory which uses (69), and for a velocity dependent collision frequency when (68) is used. If (69) is used, (78) may be simplified and leads to the well-known polarization equation of magneto-ionic theory. In both cases the product of the two roots is unity so that

$$\rho_o\rho_x=1 \tag{79}$$

where the subscripts o, x, denote "ordinary" and "extraordinary". This well-known result of magneto-ionic theory is therefore unaffected by a velocity dependent collision frequency.

Next H_y from the second equation (74) is substituted in the first equation (75), and E_z is eliminated by using the third equation (75). If the resulting equation is divided by E_x, we obtain

$$n^2=\frac{(\epsilon_{zz}\epsilon_{xx}-\epsilon_{xz}\epsilon_{zx})+\rho(\epsilon_{zz}\epsilon_{xy}-\epsilon_{xz}\epsilon_{zy})}{\epsilon_{zz}} \tag{80}$$

and on using the elements of ϵ from (72), this gives

$$n^2=\frac{\epsilon_3(\epsilon_1+\epsilon_2)+i\rho\cos\theta\epsilon_3(\epsilon_1-\epsilon_2)}{(\epsilon_1+\epsilon_2)\sin^2\theta+2\epsilon_3\cos^2\theta}. \tag{81}$$

Thus the refractive indices n are found by substituting the solutions of (78) for ρ into (81), and this leads to the formula used by Sen and Wyller, namely

$$n^2=\frac{\epsilon_1\epsilon_2\sin^2\theta+\frac{1}{2}\epsilon_3(\epsilon_1+\epsilon_2)(1+\cos^2\theta)\pm[\sin^4\theta\{\epsilon_1\epsilon_2-\frac{1}{2}\epsilon_3(\epsilon_1+\epsilon_2)\}^2+\cos^2\theta\epsilon_3^2(\epsilon_1-\epsilon_2)^2]^{1/2}}{(\epsilon_1+\epsilon_2)\sin^2\theta+2\epsilon_3\cos^2\theta}. \tag{82}$$

This gives the Appleton-Hartree formula if (69) is now used, but it gives the more accurate results of Sen and Wyller if (68) is used. Furtber, n^2 satisfies

$$n^4\{\tfrac{1}{2}(\epsilon_1+\epsilon_2)\sin^2\theta+\epsilon_3\cos^2\theta\}-n^2\{\epsilon_1\epsilon_2\sin^2\theta+\tfrac{1}{2}\epsilon_3(\epsilon_1+\epsilon_2)(1+\cos^2\theta)\}+\epsilon_1\epsilon_2\epsilon_3=0 \tag{83}$$

which shows that n is zero when any one of ϵ_1, ϵ_2, or ϵ_3 is zero. Thus in both the Appleton-Hartree and the Sen and Wyller case the zeros of n are independent of the direction θ of the wave normal. If ν_m is not zero, it is not possible for ϵ_1, ϵ_2, or ϵ_3 to be zero for any real value of X, but the zeros occur for complex values of X and are important when studying the contour integrals in the complex X-plane, as used in the phase integral method [Budden, 1961, ch. 20; 1964a, ch. 5].

9. Results

Many curves were given by Sen and Wyller showing how the real and imaginary parts of the refractive indices n depend upon X, which is proportional to the electron concentration N. They applied only to the cases $\theta=0$ and $\theta=\frac{1}{2}\pi$, that is to purely longitudinal and purely transverse propagation, respectively, and to the single value $Y=2.28$. For the ionosphere, in which the electron gyrofrequency is about 1.0 to 1.2 Mc/s, this means that their results applied only to a frequency of about 500 kc/s. In two of their curves they compared the results of their accurate formula using $Z_m=0.1$, with those of the Appleton-Hartree formula with $Z_{\text{eff}}=\frac{5}{2}Z_m=0.25$, as suggested by (60). This would correspond to a value $\nu_m\approx 3\times 10^5$ sec^{-1} which is about right for a height of 90 km in the ionosphere. In this case the differences between the two formulas were significant but small. In other curves they used larger values of Z_m namely 0.5 and 2, and here the differences between the formulas were more marked, although in these cases they used $Z_{\text{eff}}=Z_m$ or $\frac{3}{2}Z_m$ but not $\frac{5}{2}Z_m$.

The curves shown here were computed on E.D.S.A.C.2, the digital computer in the University Mathematical Laboratory, Cambridge. The two complex values of n and ρ were found for each set of values of X, Y, Z_m, and θ according to the Sen and Wyller formulas, and according to the formulas of standard magneto-ionic theory using $Z_{\text{eff}}=5Z_m/2$. The C integrals in (68) were computed from approximate formulas, given by Hara [1963], which express the integrals as rational functions with an error less than 0.7 percent.

The program was first checked by recomputing the results given by Sen and Wyller [1960]. For longitudinal propagation, $\theta=0$, the agreement was found to be as good as could be judged from the printed curves in their paper. For transverse propagation, $\theta=\pi/2$, the agreement was good for the extraordinary wave, but for the ordinary wave there was serious disagreement. For this case the refractive index should be the same as for an isotropic medium as given in section 7. The E.D.S.A.C. program was checked by hand computing a few values, and was found to be correct.

A series of calculations was then made using $\theta=30°$, so that the results would apply for radio waves traveling vertically upwards or downwards in the ionosphere in temperate latitudes. The first set of values used was $Y=1/2$, $Z_m=0.02$, $Z_{\text{eff}}=2.5\,Z_m=0.05$. They thus apply to a frequency of about 2.5 Mc/s and a value $\nu_{\text{eff}}=7.5\times 10^5$ sec^{-1}, that is to a height of about 90 km. The results showed that the differences between the two formulas were too small to show clearly on curves of the kind used by Sen and Wyller. A few typical values are given in table 1. These suggest that for waves of frequency greater than about 2 Mc/s and for heights greater than 90 km in the ionosphere, the standard Appleton-Hartree formula may safely be used with $\nu_{\text{eff}}=5\nu_m/2$.

TABLE 1. *Values of polarization* ρ *and refractive index* n *as calculated by the Sen and Wyller formula (SW) and by the Appleton-Hartree formula (AH), for* $Y=1/2$, $\theta=30°$, $Z_m=0.02$, $Z_{\text{eff}}=0.05$

		ρ (SW)		ρ (AH)		n (SW)		n (AH)	
X	Wave	R	I	R	I	R	I	R	I
0.3	Ord	−0.0065	−0.9030	−0.0066	−0.9027	0.8859	−0.0046	0.8858	−0.0046
	Ext	−.0080	1.1074	−.0081	1.1078	.6582	−.0439	.6568	−.0447
0.5	Ord	−.0121	−0.8676	−.0123	−0.8672	.7983	−.0091	.7982	−.0091
	Ext	−.0161	1.1524	−.0163	1.1530	.2548	−.2170	.2525	−.2224
0.95	Ord	−.2479	−0.3873	−.2505	−0.3838	.4634	−.1160	.4625	−.1175
	Ext	−1.1723	1.8316	−1.1928	1.8271	.9995	−.7499	1.0002	−.7380

Next the values $Y=1/2$, $Z_m=0.12$ were used. These give $\nu_{\text{eff}}\approx 4.5\times 10^6$ sec^{-1} which would apply at a height of about 70 km. The results are plotted in figure 4, and there is now a small but significant difference between the two formulas, which is greater for the extraordinary wave. This confirms Belrose and Burke's [1964] conclusion that for some purposes, at these low heights, the Appleton-Hartree formula may not be accurate enough.

In figures 5 and 6 the values used are $Y=6$, $Z_m=1$. Thus the curves refer to a frequency of about 200 kc/s and a height of about 70 km. In figures 7 and 8 the values are $Y=80$, $Z_m=12$ so that they refer to a frequency of about 16 kc/s and again to a height of about 70 km. Figures 5 and 7 give the refractive indices and show that in both cases the differences between the two formulas are significant though small. Figures 6 and 8 give the polarization ρ for the ordinary wave (sometimes called the "whistler mode"). The polarization for the other wave, according to (79), is simply the reciprocal of this. It is clear that the two formulas now show differences which are greater at the smaller values of X, that is below the level where these waves are reflected.

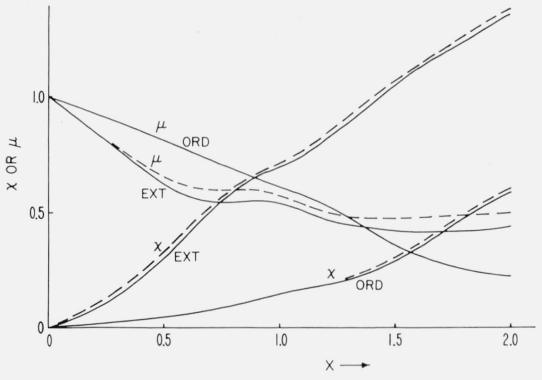

FIGURE 4. *A comparison of the values of the real parts* μ *and imaginary parts* $-\chi$ *of the refractive indices* n *according to the formula of Sen and Wyller (solid curves) and the Appleton-Hartree formula (broken curves).*

In this example $Y=\frac{1}{2}$, $\Theta=30°$. For the solid curves $Z_m=0.12$ and for the broken curves $Z_{\text{eff}}=0.3=2.5\ Z_m$. These values would apply for a frequency of about 2.5 Mc/s and a height of about 70 km. The difference between the μ values for the ordinary wave is too small to show clearly on the scale of this diagram.

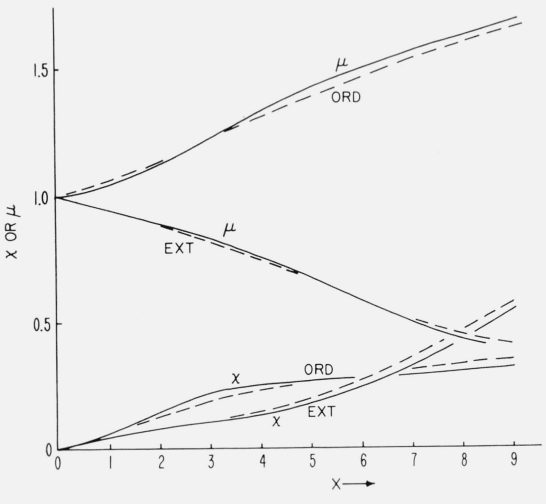

FIGURE 5. *Similar curves to figure 4 but for the values* $Y=6$, $\Theta=30°$, $Z_m=1$, $Z_{\text{eff}}=2.5$.

These would apply for a frequency of about 200 kc/s and a height of about 70 km.

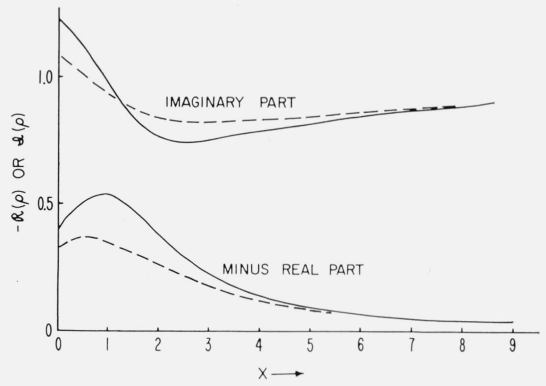

FIGURE 6. *A comparison of the values of the real parts and imaginary parts of the polarization* ρ *for the ordinary wave according to the formula of Sen and Wyller (solid curves) and the formula of standard magneto-ionic theory (broken curves).*

The values used are as in figure 5.

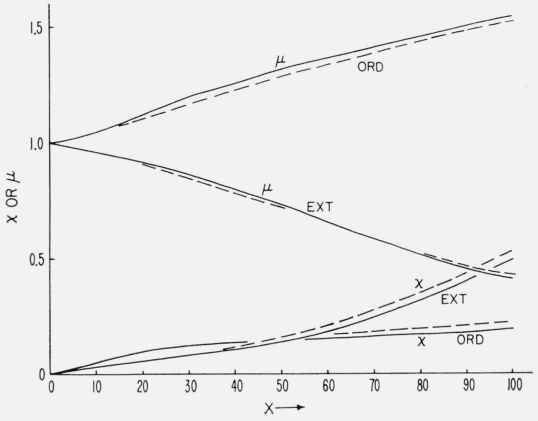

FIGURE 7. *Similar curves to figures 4 and 5 but for the values* $Y=80$, $\Theta=30°$, $Z_m=12$, $Z_{eff}=30$, *which would apply for a frequency of about 16 kc/s and a height of about 70 km.*

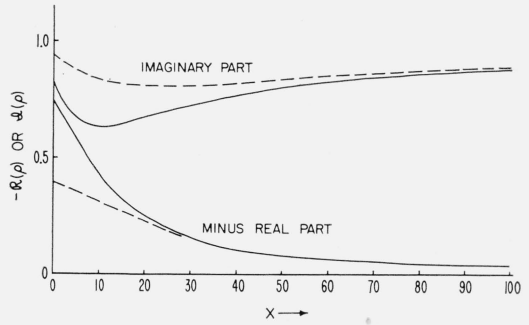

FIGURE 8. *Similar curves to figure 6 but with the values used in figure 7.*

For these very low frequencies we have $Y>>1$. Inspection of (68) shows that the arguments of the C functions in ϵ_1 and ϵ_2 are then approximately $\pm wY=\pm 2\pi f_H/\nu_m$. This is independent of frequency and even in the lowest part of the ionosphere (70 km) it is not less than 3 or 4. If we therefore assume that $|(1\pm Y)w|>>1$, the integrand in (67) may be expanded by the binomial theorem, and integrated term by term (compare (54) and (58)). If terms of the series beyond the second are neglected, the result leads to the values of standard magneto-ionic theory for ϵ_1 and ϵ_2, given by (69). Thus only in ϵ_3 should we expect a significant difference between the standard and modified theories. Figures 6 and 8 show that this affects the wave polarizations at low levels.

At these low frequencies a change of polarization with height gives coupling between the two characteristic waves [Budden, 1961, ch. 19], and since the wavelength is large there can be a significant change of ρ within one wavelength. The reflecting properties of the ionosphere must usually be computed by a "full wave" solution of the differential equations, so that it is difficult to assess immediately the effect of a change of ρ on the reflection coefficient, but figures 6 and 8 suggest that there might be a difference in the magnitude of the coupling process, according to the two formulas. The work of Deeks (private communication) has shown, however, that the resulting effect on the ionospheric reflection coefficients is negligible for steeply incident waves in temperate latitudes.

For waves whose wave normal is perpendicular to the earth's magnetic field, the refractive index for the ordinary wave is $\epsilon_3^{1/2}$ and is independent of ϵ_1 and ϵ_2. In this case the curves of figure 3 could be used to find n, and the difference between the standard and modified formulas is more serious. This suggests that significant differences between reflection coefficients calculated by the two methods might possibly be found for east-west propagation of VLF waves at grazing incidence, or at all angles of incidence near the magnetic equator.

10. Effect of Stronger Wave Fields

The foregoing results were derived with the assumption that the wave field E and the perturbation f_1 of the Boltzmann function f were so small that squares and products could be neglected. This may be called the first order approximation and an important consequence of it was that the isotropic part of f, that is f_0, the spherical harmonic component of zero order in velocity space, was unaltered when the field was switched on, and was given by the Max-

wellian distribution (3). The average electron energy associated with the perturbation f_1 is zero because f_1 varies harmonically in time. Thus to the first order, the total energy of the electrons is unaltered. Between encounters the electrons are accelerated by the field **E** and acquire energy from it, but the generation of this energy is governed by a nonlinear term which was neglected when deriving (5). The energy acquired by the electrons in a very weak field **E** is too small to affect the current density **J** in (45), so that it can be ignored when calculating refractive indices, but it does cause a change of the second order in the isotropic part f_{00} of the distribution function.

To get the second order approximation to the Boltzmann function f, we must include squares and products of the small quantities f_1 and **E**, but we may continue to neglect higher powers. Since the extra terms are small, it is accurate enough to use the first order approximation in them. The results of doing this will be indicated here only in brief outline.

Equations (4) and (17) are combined to give

$$f(\mathbf{v}, t)=f_{00}(v, t)+f_{11}(\mathbf{v}, t)+f_{12}(\mathbf{v}, t)+ \tag{84}$$

where the terms on the right are spherical harmonics of order 0, 1, 2 . . . in velocity space. This is substituted in the Boltzmann equation (1). For the reasons given in section 3, we may neglect the $\mathrm{grad}_{\mathbf{r}}$ term which would allow for spatial dispersion and for inhomogeneities in the medium. Then

$$\frac{\partial f}{\partial t}+\frac{e}{m}\,\mathbf{E}\,\mathrm{grad}_{\mathbf{v}}\,f_{11}+\frac{e}{m}\,(\mathbf{v}\wedge\mathbf{B})\,\mathrm{grad}_{\mathbf{v}}\,f=\frac{\partial_e f}{\partial t} \tag{85}$$

where the first order approximation f_{11} is used in the small second term. From this equation we now select the terms which are isotropic in velocity space (spherical harmonics of order zero), and equate them to zero. The third term makes no contribution, as was shown in section 5, so that

$$\frac{\partial f_{00}}{\partial t}+\frac{e}{m}\,\mathbf{E}\cdot\mathrm{grad}_{\mathbf{v}}\,f_{11}=\frac{\partial_e f_{00}}{\partial t}. \tag{86}$$

This equation is not linear, so that the complex number convention for harmonically varying quantities cannot immediately be used, although it can be reintroduced as follows.

When using the first order approximation f_{11}, (20) must be replaced by

$$\mathrm{grad}_{\mathbf{v}}\,f_{11}=\mathcal{R}\left[\left\{\mathbf{V}+\left(\mathbf{v}\cdot\frac{\partial\mathbf{V}}{\partial v}\right)\mathbf{v}/v\right\}\exp\,(i\omega t)\right] \tag{87}$$

and **E** must be replaced by

$$\mathcal{R}\{\,\mathbf{E}\exp\,(i\omega t)\,\}. \tag{88}$$

The second term of (86) is the product of the two harmonically varying quantities (87) and (88) both with angular frequency ω, so that it contains a part with angular frequency 2ω, and a part which is independent of time and given by:

$$\frac{1}{2}\,\mathcal{R}(\mathbf{E}^*\cdot\mathbf{V})+\frac{1}{2}\,\mathcal{R}\left\{(\mathbf{E}^*\cdot\mathbf{v})\left(\mathbf{v}\cdot\frac{\partial\mathbf{V}}{\partial v}\right)/v\right\}. \tag{89}$$

Thus f_{00} must contain a part with angular frequency 2ω and a more slowly varying part.

When the radio wave is first switched on, the function f_{00} changes as energy is imparted to the electrons. But these in turn begin to impart energy to the gas molecules, and eventually a new state of equilibrium is reached. The time constant for this process is of the order of 1 msec [Huxley and Ratcliffe, 1949] and is important in the study of wave interaction. We now assume that the radio wave has been present for a long time, so that the slow variation of f_{00} has ceased, and only the variation of frequency 2ω remains. Now let the time average

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value of f_{00} be $\bar{f}_{00}$. Then $\frac{\partial \bar{f}_{00}}{\partial t}=0$, and the terms in (86) which are independent of time give

$$\frac{\partial_e \bar{f}_{00}}{\partial t}=\frac{1}{2}\mathcal{R}\,(\mathbf{E}^*\cdot\mathbf{V})+\frac{1}{2}\mathcal{R}\left\{(\mathbf{E}^*\cdot\mathbf{v})\left(\mathbf{v}\cdot\frac{\partial \mathbf{V}}{\partial v}\right)/v\right\}. \tag{90}$$

Here $\mathbf{V}$ is given by the first order solution (40) or (42), and (90) is to be solved for the new distribution function $\bar{f}_{00}$. It should be particularly noted that $\mathbf{E}$ is a complex vector, and in general represents an elliptically polarized wave [Budden, 1964b]. A similar equation to (90) has been studied by Sen and Wyller [1960, appendix] but they have taken (88) above to be $\mathbf{E}\cos\omega t$ where $\mathbf{E}$ is a real vector, so that their results apply only to linearly polarized waves.

Next the left-hand side of (90) must be expressed in terms of $\bar{f}_{00}$ and its derivatives with respect to v. To do this it is necessary to make a detailed study of electron-molecule encounters, and it is not now permissible to assume that the molecules are infinitely massive. This is because the electrons are imparting to the molecules some of the energy which they have gained from the radio wave. For $\frac{\partial_e \bar{f}_{00}}{\partial t}$ Sen and Wyller used a formula derived by Chapman and Cowling [1958, sec. 18.71] for electrons in a constant electric field, and this really needs modifying to allow for the harmonic variation of the field with time. Moreover this treatment took no account of molecular rotation caused by the electrons' impact. As far as the author knows, no proper treatment of this phenomenon has yet been given.

The assumption in section 4 that the molecules are infinitely massive would mean that the energy of an electron is unaltered by an encounter. It would then be impossible for the electrons to impart any energy to the gas molecules, so that $\bar{f}_{00}$ could never reach a constant value. But the more exact treatment indicated here shows that there is, on the average, a small amount of energy imparted to a molecule at each encounter. There is then a continuous flow of energy from the wave to the electrons and thence to the gas molecules.

11. Conclusions

This tutorial paper presents a derivation of the modified formulas of magneto-ionic theory, allowing for the dependence of the electron collision frequency on the velocity. No attempt has been made to set out the historical development of the subject nor to give references to all previously published work, and for further information the original paper of Sen and Wyller [1960] should be consulted.

It is concluded that the standard Appleton-Hartree formula can be used without modification for nearly all radio propagation problems in the ionosphere provided that the value used for the collision frequency is the effective value ν_{eff} given by (60). This is $\frac{5}{2}\nu_m$ and is $\frac{5}{3}$ of the average collision frequency. It is this ν_{eff} which is measured in those experiments which study the absorption of radio waves. For those measurements which use wave interaction [for example Fejer, 1955; Barrington and Thrane 1962] a more detailed theory will be needed on the lines indicated near the end of section 10, before any precise interpretation is possible. The effect of the modified formulas on the theory of the reflection of very low-frequency radio waves (10 kc/s to 100 kc/s) from the lowest ionosphere also needs to be studied for those cases where the wave normal is approximately perpendicular to the earth's magnetic field.

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