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Ionospheric Cross Modulation: A Microscopic Theory[1]

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In 1937 Bailey and Martyn proposed a theory of ionospheric cross modulation along the following lines. Radiation emitted by the disturbing transmitter is strongly absorbed in a region of the ionosphere where the wanted wave is refracted and attenuated. Absorption of the disturbing radiation raises the temperature of the ionosphere. If the amplitude of the disturbing radiation varies with time, so will the resulting temperature increase and hence the attenuation suffered by the wanted wave. In the simplest experimental arrangement the wanted wave is initially unmodulated and the disturbing radiation is modulated at a definite audio frequency. In traversing the ionosphere, the wanted wave becomes modulated at the frequency of the disturbing radiation and also at twice this frequency.

The Bailey-Martyn theory makes three quantitative predictions concerning the transferred modulation. The first concerns its dependence on the power of the disturbing radiation, and involves only the assumption that the effect is small enough to warrant a linearized description. The remaining two predictions concern the dependence of the transferred modulation and its phase lag on the modulation frequency. These predictions involve the additional assumption of a unique relation, valid for all values of the modulation frequency, between the mean thermal energy and the mean collision frequency of the electrons. A relation of this kind can exist, however, only for modulation frequencies much less than 150 c/s. Since the experimental range of modulation frequencies extends to nearly 1500 c/s, the foundations of the Bailey-Martyn theory need to be reconsidered.

This paper describes a microscopic theory of ionospheric cross modulation. The velocity-distribution function for the electrons enters explicitly into the theory; it is determined by a differential equation whose form depends on the modulation frequency. Knowing the distribution function, one can calculate the absorption coefficient. The form of the predicted absorption coefficient depends on the assumed form of the electron-molecule interaction law. By numerical methods, we have calculated the transferred modulation and its phase lag as functions of modulation frequency for a few simple interaction laws. The calculations show that the effects of departures from a maxwellian velocity distribution are indeed significant. The predictions are sensitive to the assumed form of the electron-molecule collision law. Although the present theory is still highly idealized, the results obtained suggest that further theoretical and experimental refinements could lead to an experimental determination of the electron-molecule interaction law in the D region.

1. Introduction

1.1. The Bailey-Martyn Theory

The passage of a radio wave through a region of the ionosphere in which it is partially absorbed raises the electron temperature in the region and hence changes its absorption coefficient. The absorption process is accordingly nonlinear: the attenuation suffered by each of several radio waves passing through the same region depends on that suffered by all the others. This is the physical basis of Bailey and Martyn's [1934] theory of ionospheric cross modulation.

In a typical cross-modulation experiment an initially unmodulated carrier wave (the "wanted" wave) traverses a region in which a second wave at a different carrier frequency (the "disturbing" wave) is heavily absorbed. The disturbing wave is amplitude-modulated at a definite frequency. On reception, the wanted wave is found to be modulated at the same frequency and also at twice this frequency.

The main predictions of the Bailey-Martyn theory do not depend on a detailed description of the physical processes involved but follow from a few general assumptions, the first of which is that the phenomenon admits a linearized description. The linearized energy-balance equation for the electron gas has the form

$$\frac{d\epsilon}{dt}=\kappa I-\tau^{-1}\epsilon. \tag{1.1}$$

Here ϵ denotes the difference

$$\epsilon=\frac{3}{2}k(T_e-T_m) \tag{1.2}$$

[1] The first five sections of this paper first appeared in slightly different form in a report dated March 14, 1959, prepared for the U.S. Air Force. A much more complete discussion of the Bailey-Martyn theory and its modifications (sec. 1) and of the kinetic theory of a plasma in an alternating electromagnetic field (sections 2, 3, 4) has subsequently been given by Ginzburg and Gurevich (1960), along with an exceptionally complete bibliography, but the main results of the present paper (sections 5, 6) do not appear to have been published previously. An approach very similar to that used in sections 2, 3 has been outlined by Bayet, Delcroix, and Denisse [1957], but their treatment of the distribution function differs from that of sections 4, 5. We are indebted to the referee, Dr. Julius Cahn, for bringing this paper to our notice.

between the mean thermal energy of the electrons and the mean thermal energy of the molecules, κ denotes the absorption coefficient of the medium, I denotes the intensity of the ambient radiation field, and τ denotes the thermal relaxation time of the electron gas. As (1.1) is linear, we may treat the individual Fourier components of the radiation field separately. The ω-component of (1.1) is

$$i\omega\epsilon_\omega=\kappa I_\omega-\tau^{-1}\epsilon_\omega, \tag{1.3}$$

which has the solution [2]

$$\epsilon_\omega=\frac{\kappa\tau I_\omega}{1+i\omega\tau}=\frac{\kappa\tau e^{-i\varphi}I_\omega}{\sqrt{(1+\tau^2\omega^2)}}, \qquad \varphi=\tan^{-1}(\omega\tau). \tag{1.4}$$

The angle φ represents the phase difference between the ω-component of the radiation field and the ω-component of the electron temperature. In the D and lower E regions, $\tau\approx 100$ sec, so that the attenuation factor $(1+\tau^2\omega^2)^{-1/2}$ is very small except at audio frequencies.

The assumption of linearity also implies that the variable part of the electron-molecule collision frequency $\nu \propto \epsilon$. According to Lorentz's theory, $\kappa \propto \nu$. Hence, setting

$$\kappa=\kappa_0+\int \kappa_\omega e^{i\omega t}d\omega, \qquad \nu=\nu_0+\int \nu_\omega e^{i\omega t}d\omega, \tag{1.5}$$

we have

$$\kappa_\omega \propto \nu_\omega \propto \epsilon_\omega. \tag{1.6}$$

Finally, if the region where appreciable modulation transfer occurs is nearly homogeneous, we have

$$X_\omega \propto \kappa_\omega, \tag{1.7}$$

where X_ω describes the depth and phase of the transferred modulation. From (1.4), (1.6), and (1.7), we have

$$X_\omega=I_\omega T_\omega e^{-i\varphi}; \quad T_\omega=(1+\tau^2\omega^2)^{-1/2}, \quad \phi_\omega=\tan^{-1}(\omega\tau). \tag{1.8}$$

The predicted linear dependence of X_ω on I_ω, the predicted variation of the phase lag φ_ω with ω, and the predicted variation of the coefficient T_ω with ω can all be tested separately.

(i) *Linear dependence of X_ω on I_ω.* If the disturbing wave is modulated to a depth M at a single frequency ω and if P denotes the total power of the disturbing radiation, then

$$I \propto P[(1+\tfrac{1}{2}M^2)+2M\cos\omega t+\tfrac{1}{2}M^2\cos 2\omega t]. \tag{1.9}$$

Ratcliffe and Shaw [1958] verified that $X_\omega \propto P$ over a wide range of P. Huxley et al. [1947, 1948] verified the predicted linear dependence of P_ω on M and showed that the predicted quadratic dependence of $X_{2\omega}$ on M was qualitatively in agreement with experiment. *These experimental results show that a linearized description of ionospheric cross modulation is valid over a wide range of experimental conditions.*

(ii) *The phase lag $\varphi(\omega)$.* Ratcliffe and Shaw [1948] and subsequent workers found that they could secure good agreement between predicted and measured phase lags over a wide range of ω by choosing the parameter τ appropriately. The required values of τ seem to be consistent with values derived by extrapolating laboratory measurements [Huxley, 1959].

(iii) *The coefficient T_ω.* Formula (1.8) for T_ω agrees qualitatively, but not quantitatively, with experiment [Ratcliffe and Shaw, 1948; Huxley, 1950]. The discrepancies between theory and experiment, though not large, appear to be significant. Huxley [1950] suggested that the two spatially separated regions in which cross modulation occurs in a typical experiment—one on the ascending, the other on the descending branch of the wanted ray—may have sufficiently different properties to invalidate the assumption of homogeneity underlying (1.6). Calculations based on a two-center model [Huxley, 1950] do not, however, significantly reduce the discrepancies.

The derivation of formula (1.8) actually rests not only on the assumptions of linearity and homogeneity but also on a third major assumption: that, at any given point in the ionosphere, the quantities κ, ν, and τ depend only on the electron temperature and not explicitly on the modulation frequency. Now, the relation between collision frequency (say) and electron temperature depends in general on the form of the velocity distribution of the electrons. If the distribution is maxwellian, or has any other fixed form depending on a single parameter, it is completely specified by T_e, so that ν becomes a function of T_e. But if the electron temperature does not serve to specify the distribution completely, the relation between ν and T_e may be many valued. In the present problem, the variable part of the distribution changes appreciably in a time of order ω^{-1}. Since this is short compared with the thermal relaxation time τ there is no reason to suppose that the distribution remains accurately mawellian. It is true that if the intensity of the incident radiation field is sufficiently small, the velocity distribution of the electrons will be approximately maxwellian. But the phenomenon of cross modulation depends entirely on the variable component of the velocity distribution, and, as we shall see, this component remains nonmaxwellian even in the limit of vanishing field intensity.

This paper presents a theory of cross modulation in which the electronic velocity-distribution function figures explicitly. Although the present theory is more realistic than the macroscopic theory sketched above, it is oversimplified in one important respect: It treats collisions between electrons and molecules as if they were perfectly elastic. In reality, cooling of the electron gas in the D and E regions results

[2] Equation (1.1) has the same form as that governing the current in an RL circuit; τ represents the time constant of the circuit and I the applied emf.

chiefly from the collisional excitation of molecular nitrogen. Basing themselves on the work of Kovrizhnikh [1960], Caldirola and DeBarbieri [1960, 1964] have recently extended the present theory to allow, at least approximately, for inelastic collisions.

For the sake of simplicity, the following discussion ignores the effects of the earth's magnetic field. This is permissible only if all the carrier frequencies that figure in the discussion are much greater than the gyrofrequency. The modifications required when this condition is not met are straightforward; they are described in the paper by Caldirola and DeBarbieri mentioned in the last paragraph.

2. Reduction of the Boltzmann Equation

The Boltzmann equation for an electron gas in an electromagnetic field has the form

$$\frac{\partial f}{\partial t}+\mathbf{q}\cdot\frac{\partial f}{\partial \mathbf{x}}+\frac{e}{m}\left(\mathbf{E}+\frac{\mathbf{q}}{c}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial \mathbf{q}}=\frac{\delta f}{\delta t}. \tag{2.1}$$

The following discussion of this equation is a generalization of the treatment of a weakly ionized gas in a constant electric field given by Chapman and Cowling [1960, pp. 346–352].

We assume at first that the field is that of a linearly polarized plane wave propagating in the z-direction: [3]

$$E_x=Ee^{i(kz-pt)}, \qquad B_y=\frac{ck}{p}Ee^{i(kz-pt)}. \tag{2.2}$$

The distribution function f depends on the six variables z, $\mathbf{q}$, t. Since $|ck/p|\approx 1$, the magnetic force is of order q/c compared with the electric force. Similarly, the second term on the left side of (2.1) is of order q/c compared with the first term. Neglecting terms of this order, we obtain in place of (2.1)

$$\frac{\partial f}{\partial t}+F_x\frac{\partial f}{\partial u}=\frac{\delta f}{\delta t}, \tag{2.3}$$

where

$$\mathbf{F}=\frac{e}{m}\mathbf{E}. \tag{2.4}$$

In this approximation, f depends on only four arguments:

$$f=f(z;\,u,\,q;\,t). \tag{2.5}$$

We may expand f in Legendre polynomials of the argument u/q. We shall need only the first two terms in the expansion:

$$f=f^{(0)}(q,\,t)+iFe^{i(kz-pt)}uf^{(1)}(q,\,t). \tag{2.6}$$

If the number density n of electrons is sufficiently small compared with the number density of molecules, we may neglect electron-electron and electron-ion collisions. The collision term in the Boltzmann equation, represented by $\delta f/\delta t$, is then a linear functional of f:

$$\frac{\delta f}{\delta t}=\Lambda(f). \tag{2.7}$$

Lorentz showed that

$$\Lambda[ug(q)]=-\nu(q)ug(q)+O\left(\frac{m}{M}\right), \tag{2.8}$$

where m/M is the electron-molecule mass ratio, ν is the velocity-dependent collision frequency, and g is an arbitrary function. Davydov [1935] derived the important formula

$$\Lambda[g(q)]=\frac{m}{M}\frac{1}{q^2}\frac{d}{dq}\left[q^3\nu(q)\left(1+\frac{kT_m}{m}\frac{1}{q}\frac{d}{dq}\right)g(q)\right]+O\left[\left(\frac{m}{M}\right)^2\right]. \tag{2.9}$$

By inserting (2.6) into (2.3) and using the formulae of Lorentz and Davydov to evaluate the right side of the resulting equation, we obtain differential equations for the functions $f^{(0)}$ and $f^{(1)}$.[4]

The equation for $f^{(1)}$ is

$$uf^{(1)}(p+i\nu)+\frac{\partial f^{(0)}}{\partial u}=0 \tag{2.10}$$

or

$$f^{(1)}=-\frac{1}{p+i\nu}\frac{1}{q}\frac{\partial f^{(0)}}{\partial q}. \tag{2.11}$$

The left side of the equation for $f^{(0)}$ has the form

$$\frac{\partial f^{(0)}}{\partial t}+\left\langle F_x\frac{\partial}{\partial u}(iF_xuf^{(1)})\right\rangle. \tag{2.12}$$

Here the brackets indicate a double average: over direction in velocity space (to eliminate terms containing Legendre polynomials $P_k(u/q)$ with $k>0$), and over time (to eliminate radio-frequency fluctuations in $f^{(0)}$, which are of no physical interest). Since we need to preserve audio-frequency variations of $f^{(0)}$, the time-averaging must be over an interval that is short compared with the reciprocal of the highest modulation frequency of interest as well as long compared with the reciprocal of the carrier frequency p; thus we must have $p>>\omega$. Application of the well-known rule

$$\langle \mathrm{Re}\,Ae^{ipt}\cdot\mathrm{Re}\,Be^{ipt}\rangle_t=\tfrac{1}{2}\,\mathrm{Re}\,\overline{A}B, \tag{2.13}$$

[3] We adopt the convention that when a real quantity, such as E_x or f, is represented by a complex expression, as in (2.2) and (2.6), the real part of this expression is to be understood. Note, however, that the quantities E, F, k, $f^{(0)}$, etc., are all complex, and that both the real and imaginary parts of an equation like (2.11) are significant.

[4] The differential equation for $f^{(k)}$, the coefficient of $P_k(u/q)$ in the Legendre expansion of f, involves only the functions $f^{(j)}$ with $j\leq k$, so that the sequence of approximations may be terminated at any point. By contrast, the equations for the moments of f of order k involve moments of order $k+1$.

where $\overline{A}$ denotes the complex conjugate of A, gives

$$\left\langle F_x \frac{\partial}{\partial u}(iF_x u f^{(1)}) \right\rangle_t$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{ \overline{F}\left(\frac{-i}{p+i\nu}\right) F \frac{\partial}{\partial u}\left(\frac{u}{q}\frac{\partial f^{(0)}}{\partial q}\right)\right\}$$

$$= -\frac{1}{2}\frac{\nu}{p^2+\nu^2}|F|^2\left[\frac{1}{q}\frac{\partial f^{(0)}}{\partial q} + \frac{u^2}{q}\frac{\partial}{\partial q}\left(\frac{1}{q}\frac{\partial f^{(0)}}{\partial q}\right)\right]. \quad (2.14)$$

Since

$$u^2 = q^2\left[\frac{1}{3} + \frac{2}{3}P_2\left(\frac{u}{q}\right)\right],$$

averaging over direction in velocity space gives

$$\left\langle F_x \frac{\partial}{\partial u}(iF_x u f^{(1)}) \right\rangle$$

$$= -\frac{1}{2}\frac{\nu}{p^2+\nu^2}|F|^2\left[\frac{1}{q}\frac{\partial f^{(0)}}{\partial q} + \frac{1}{3}q\frac{\partial}{\partial q}\left(\frac{1}{q}\frac{\partial f^{(0)}}{\partial q}\right)\right]$$

$$= -\frac{1}{6}\frac{\nu}{p^2+\nu^2}|F|^2\frac{1}{q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial f^{(0)}}{\partial q}\right). \quad (2.15)$$

Combining this result with (2.9) we have, finally,

$$\frac{\partial f^{(0)}}{\partial t} - \frac{m}{M}\frac{1}{q^2}\frac{\partial}{\partial q}\left\{ q^3\nu(q)\left[f^{(0)} + \frac{\Theta}{q}\frac{\partial f^{(0)}}{\partial q}\right]\right\} = 0, \quad (2.16)$$

where

$$\Theta = \frac{1}{6}\frac{M}{m}\frac{|F|^2}{p^2+\nu^2} + \frac{kT_m}{m}. \quad (2.17)$$

We may immediately generalize these results to an arbitrary radiation field. The electric vector at a given point may be written in the form

$$\mathbf{E} = \sum_j \mathbf{E}_j, \quad (2.18)$$

where each Fourier component $\mathbf{E}_j$ represents the electric field of a linearly polarized plane wave. In place of (2.6) we have

$$f = f^{(0)}(q,t) + \sum_j iF_j(\hat{\mathbf{k}}_j\cdot\mathbf{q})f_j^{(1)}(q,t)\exp(i\mathbf{k}_j\cdot\mathbf{x} - p_j t) \quad (2.19)$$

and in place of (2.10),

$$f_j^{(1)} = -\frac{1}{p_j+i\nu}\frac{1}{q}\frac{\partial f^{(0)}}{\partial q}. \quad (2.20)$$

Finally, the function $f^{(0)}$ satisfies (2.16) with Θ given by

$$\Theta = \frac{1}{6}\frac{M}{m}\sum_j \frac{|F_j|^2}{p_j^2+\nu^2} + \frac{kT_m}{m} \quad (2.21)$$

instead of by (2.17).

3. The Dispersion Relation

The electric current density produced by the radiation field (2.2) is given by

$$j_x = \frac{e}{c}\int u f d^3\mathbf{q} = \frac{ieF_x}{c}\int u^2 f^{(1)}(q)d^3\mathbf{q}$$

$$= \frac{4\pi ie}{3c}F_x\int q^4 f^{(1)}dq = -\frac{4\pi ie}{3c}F_x\int\frac{q^3}{p+i\nu}\frac{df^{(0)}}{dq}dq$$

$$= i\frac{e}{c}F_x\int f^{(0)}d\left(\frac{4\pi}{3}\frac{q^3}{p+i\nu}\right), \quad (3.1)$$

$$j_y = j_z = 0. \quad (3.1')$$

From Maxwell's equation,

$$\mathrm{curl}\,\mathbf{B} = \frac{1}{c}\dot{\mathbf{E}} + 4\pi\mathbf{j}, \quad (3.2)$$

and (3.1) we obtain

$$(ck)^2 = p^2 - \frac{4\pi e^2 p}{m}\int f^{(0)}d\left(\frac{4\pi}{3}\frac{q^3}{p+i\nu}\right)$$

$$= p^2 - \frac{4\pi e^2}{m}\int f^{(0)}d\left(\frac{4\pi}{3}\frac{p^2q^2}{p^2+\nu^2}\right)$$

$$+ i\frac{4\pi e^2}{m}\int f^{(0)}d\left(\frac{4\pi}{3}\frac{p\nu q^3}{p^2+\nu^2}\right). \quad (3.3)$$

In order to write this relation in a more compact form we introduce the plasma frequency p_0, defined by

$$p_0^2 = \frac{4\pi ne^2}{m}, \quad (3.4)$$

and use the abbreviation

$$\langle\Phi\rangle = \frac{1}{n}\int f^{(0)}d\left[\frac{4\pi}{3}q^3\Phi(q)\right], \quad (3.5)$$

where n denotes the electron density. The dispersion relation (3.3) then takes the form

$$(ck)^2 = p^2 - p_0^2\left\langle\frac{p^2}{p^2+\nu^2}\right\rangle + ip_0^2\left\langle\frac{p\nu}{p^2+\nu^2}\right\rangle. \quad (3.6)$$

The absorption coefficient κ is defined by

$$k = k_0 + \tfrac{1}{2}i\kappa, \quad (3.7)$$

where k_0 and κ are both real. The following approximate formulae, valid under the conditions stated, are often convenient:

$$c\kappa=\frac{p_0^2}{p}\left\langle\frac{p\nu}{p^2+\nu^2}\right\rangle \quad (p_0<<p), \tag{3.8}$$

$$c\kappa=\left(\frac{p_0}{p}\right)^2\langle\nu\rangle \quad (p_0<<p, \nu<<p). \tag{3.9}$$

The results of this section were first obtained by Lorentz.

4. Absorption of Unmodulated Radio Waves

When $\omega=0$, (2.16) reduces to an ordinary first-order differential equation, whose solution is

$$f^{(0)}=nC\exp\left\{-\int_0^q\Theta^{-1}qdq\right\}, \tag{4.1}$$

where Θ is given by (2.21) and the constant C is determined by the normalization condition

$$4\pi\int_0^\infty f^{(0)}q^2dq=n. \tag{4.2}$$

Formula (4.1) is valid for all values of the carrier frequencies p_j occuring in formula (2.21) for Θ. We recall that $f^{(0)}$ represents the isotropic part of the distribution function averaged over a time interval long compared with all the characteristic times p_j^{-1}. If the radiation field has only a single Fourier component, $f^{(0)}$ represents the isotropic part of the distribution function averaged over a single period.

Formula (4.1) remains valid in the limit $p=0$ ($\mathbf{E}$=const.). In this case

$$\Theta=\frac{1}{6}\frac{M}{m}\frac{|F|^2}{\nu^2(q)}+\frac{kT_m}{m} \qquad (p=0) \tag{4.3}$$

and (4.1) coincides with a formula derived by Chapman and Cowling [1960; eq (13), p.350].

When the carrier frequency is very large, Θ is given by

$$\Theta=\frac{1}{6}\frac{M}{m}\frac{|F|^2}{p^2}+\frac{kT_m}{m} \qquad (p>>\nu): \tag{4.4}$$

$f(^\circ)$ is accordingly maxwellian, the electron temperature being given by Lorentz's formula,

$$kT_e=kT_m+\frac{1}{6}\frac{M|F|^2}{p^2}. \tag{4.5}$$

If the molecules are maxwellian (collision frequency independent of velocity), the distribution function is maxwellian for all values of the carrier frequency. Otherwise the distribution function departs markedly from the maxwellian form when $\Theta>> kT_m/m$ and $p\approx\nu$.

If the molecules are rigid elastic spheres ($\nu=q/l$, l=const.) the integral in (4.1) may be evaluated explicitly; one obtains

$$f^{(0)}=C'(\bar{q}^2+\bar{q}^2+A)^A e^{-\bar{q}^2}, \tag{4.6}$$

where

$$\bar{q}^2=\frac{mq^2}{2kT_m}, \qquad \bar{p}^2=\frac{ml^2p^2}{2kT_m}, \qquad A=\frac{M|F|^2}{2kT_m}. \tag{4.7}$$

5. The Variable Part of $f^{(0)}$

In this section we derive the equations governing small departures from the steady-state velocity distributions discussed in the preceding section.

Let

$$\Theta(q,t)=\Theta_0(q)[1+\theta(q,t)]. \tag{5.1}$$

The function Θ_0 coincides with the time-independent function that was called Θ in the preceding section. We define new dimensionless variables x, X, φ:

$$dx=\frac{qdq}{\Theta_0}, \quad X=\frac{1}{2}\frac{q^2}{\Theta_0},$$

$$\varphi(x,t)=4\pi n^{-1}\int_0^q q^2f^{(0)}(q,t)dq. \tag{5.2}$$

Note that

$$\frac{dx}{X}=\frac{dX}{X}+\frac{d\Theta_0}{\Theta_0}. \tag{5.3}$$

The function φ is the *probability distribution function* of the dimensionless velocity variable x;

$$\varphi(a,t)=\Pr\{x\leq a\}. \tag{5.4}$$

It follows that

$$\varphi(0,t)=0, \qquad \varphi(\infty,t)=1. \tag{5.5}$$

The partial derivative $\partial\varphi/\partial x$ is the *probability density* associated with x. In terms of the new variables (2.16) takes the form

$$\frac{\partial\varphi}{\partial t}=\frac{2m}{M}\nu\left\{X\Theta_0\frac{\partial}{\partial x}\left(\Theta_0^{-1}\frac{\partial\varphi}{\partial x}\right)+\left(X-\frac{1}{2}\right)\frac{\partial\varphi}{\partial x}+\theta\left[-\frac{1}{2}\frac{\partial\varphi}{\partial x}+X\Theta_0\frac{\partial}{\partial x}\left(\Theta_0^{-1}\frac{\partial\varphi}{\partial x}\right)\right]\right\}. \tag{5.6}$$

So far we have not made any assumptions about the magnitude of the function $\theta(q,t)$. In order to reduce (5.6) to an ordinary differential equation we now set

$$\varphi=\varphi_0(x)+\varphi_1(x,t) \tag{5.7}$$

and assume that θ and φ_1 are so small that their product may be neglected in (5.6). This will be true if at least one of the conditions $\Theta\approx kT_m/m$,

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$M<<1$, where M denotes the modulation depth, is satisfied. Having linearized (5.6), we can deal separately with the Fourier components of the radiation field, just as in the elementary theory of section 1. Let

$$\theta(x,t)=\theta_\omega(x)e^{i\omega t}, \qquad \varphi_1(x,t)=\varphi_\omega(x)e^{i\omega t}. \quad (5.8)$$

It is convenient to replace the modulation frequency ω by a dimensionless variable and to separate out the velocity dependence of the function $\nu(x)$. We accordingly define

$$\eta(x)=\frac{\nu}{\nu(x)}, \quad \bar{\nu}=\int_0^\infty \nu(x)d\varphi_0, \quad (5.9)$$

$$\alpha=\frac{\omega}{2(m/M)\bar{\nu}}. \quad (5.10)$$

Finally, since φ_0 satisfies the equation

$$X\Theta_0\frac{\partial}{\partial x}\frac{1}{\Theta_0}\frac{\partial\varphi_0}{\partial x}+\left(X-\frac{1}{2}\right)\frac{\partial\varphi_0}{\partial x}=0, \quad (5.11)$$

we have

$$\frac{\partial\varphi_0}{\partial x}=C\Theta_0^{3/2}X^{1/2}e^{-x}, \quad (5.12)$$

where C is determined by the normalization condition $\varphi_0(\infty)=1$. Hence

$$-X\frac{\partial\varphi_0}{\partial x}=-C(\Theta_0X)^{3/2}e^{-x}. \quad (5.13)$$

Omitting second-order terms in (5.6) and using (5.9), (5.10), and (5.13), we obtain

$$X\Theta_0\frac{\partial}{\partial x}\left(\Theta_0^{-1}\frac{\partial\varphi_\omega}{\partial x}\right)+\left(X-\frac{1}{2}\right)\frac{\partial\varphi_\omega}{\partial x}+i\alpha\eta\varphi_\omega=\theta_\omega C\Theta_0^{3/2}X^{3/2}e^{-x}. \quad (5.14)$$

Since φ and φ_0 both satisfy the boundary conditions (5.5), φ_ω must satisfy the boundary conditions

$$\varphi_\omega(0)=\varphi_\omega(\infty)=0. \quad (5.15)$$

Equations (5.14) and (5.15) together with the dispersion relation (3.6) represent the formal solution of our problem. The functions Θ_0 and θ are defined by the radiation field. Given Θ_0, one finds the function $x(q)$ and $X(q)$ from (5.2). Equation (5.14) with the boundary conditions (5.15) may be integrated numerically by the method described in the appendix. Finally, knowing φ_ω, one can calculate the variable part of the absorption coefficient from the dispersion relation.

Equation (5.14) assumes a simpler form when $\Theta\approx kT_m/m$, so that the unperturbed distribution function is nearly maxwellian. We may then write

$$X=x, \qquad \Theta_0=\text{const.}, \qquad \theta_\omega=\text{const.}, \quad (5.16)$$

so that (5.12)becomes

$$\frac{\partial\varphi_0}{\partial x}=\frac{2}{\sqrt{\pi}}x^{1/2}e^{-x}. \quad (5.17)$$

Setting

$$\varphi_\omega(x)=-\frac{2}{\sqrt{\pi}}\theta_\omega y(x), \quad (5.18)$$

we obtain in place of (5.14)

$$xy''+(x-\tfrac{1}{2})y'-i\alpha\eta y=-x^{3/2}e^{-x}. \quad (5.19)$$

The boundary conditions are

$$y(0)=y(\infty)=0. \quad (5.20)$$

In the numerical work described in the next section the function $\eta(x)$ is taken to have the form

$$\eta(x)=\frac{2(r+\frac{1}{2})!}{\sqrt{\pi}}x^{-r}. \quad (5.21)$$

6. Some Numerical Results

Miss Cara Joy Hughes employed the IBM 7094 computer at the Harvard University Computing Center to integrate (5.19) with the boundary conditions (5.20) and with $\eta(x)$ given by (5.21). Integrations were carried out for every pair of parameter values (α,r) with α in the set (0.1, 0.2, 0.5, 1, 2, 5, 10, 15) and r in the set (−.5, .5, 1, 1.5). The integration procedure is described in the appendix.

For $r=0$ the solution of (5.19) is

$$y=\frac{x^{3/2}e^{-x}}{1+i\alpha} \qquad (r=0). \quad (6.1)$$

The complete distribution function $f^{(0)}$ corresponding to this solution is maxwellian with a variable electron temperature. The factor $(1+i\alpha)^{-1}$ in (6.1) corresponds to the factor $(1+i\omega\tau)^{-1}$ in (1.4).

Figures 1 and 2 show the modulus and argument of the function

$$\tilde{y}=(1+i\alpha)y \quad (6.2)$$

for $r=-0.5$, 1.5 and for $\alpha=2$, 5. In both figures the curves corresponding to $r=-0.5$ and $r=1.5$ differ markedly from the curve for the maxwellian case $r=0$. For $r=-0.5$ the peak of the function $|\tilde{y}(x)|$ occurs at a smaller value of x than for $r=0$, while for $r=1.5$ it occurs at a larger value. In the four cases with $r\neq 0$ the function arg $\tilde{y}$ changes sign near $x=1.5$. For large positive values of x, arg $\tilde{y}$ is positive for $r=1.5$, negative for $r=-0.5$.

According to (3.9), the absorption coefficient κ varies directly as the mean collision frequency (the mean being defined by (3.5)) if the carrier

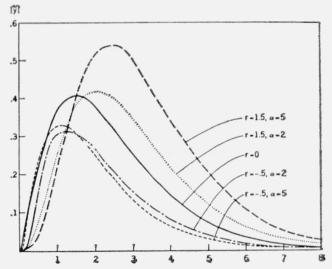

FIGURE 1. *The function* $|\tilde{y}|$ *(see eqs 6.2 and 5.18) for representative values of* r *and* α.

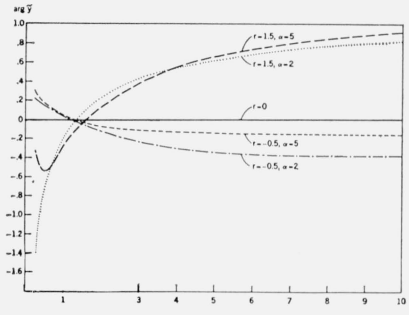

FIGURE 2. *The function arg* $\tilde{y}$ *(see eqs 6.2 and 5.18) for representative values of* r *and* α.

frequency is sufficiently high. By (3.5), (5.2), (5.7), (5.8), (5.18), and (5.21),

$$\langle \nu_\omega \rangle \propto \int \frac{d\varphi_\omega}{dq}\, q^{-2} d(q^3 \nu) \propto (2r+3) \int d\varphi_\omega x^r$$

$$\propto -(2r+3) r \int_0^\infty \varphi_\omega x^{r-1} dx \propto -(2r+3) r \theta \omega \int_0^\infty y x^{r-1} dx. \tag{6.3}$$

Just as in section 1, the assumptions of linearity and homogeneity imply that $X_\omega \propto \kappa_\omega$ (see (1.7)). Hence

$$X_\omega \propto (2r+3) r \theta_\omega \int_0^\infty y x^{r-1} dx. \tag{6.4}$$

Note that X_ω vanishes for maxwellian molecules ($r=0$). This means that the elementary theory becomes rigorously valid in the limit when the phenomenon it describes disappears.

By analogy with (1.8), we write

$$\int_0^\infty y x^{r-1} dx = T(\alpha) e^{-i\varphi(\alpha)}. \tag{6.5}$$

The functions $\phi(\alpha)$ and $T(\alpha)$ are shown in figures 3 and 4 for $r=-0.5$, 0, 0.5, 1, 1.5. For $r \geq 0$ the curves $\phi(\alpha)$ have roughly the same shape. Compressing the horizontal scale of the curve for $r=0.5$ by 25 percent would bring it into near coincidence with the curve for $r=0$, but for $r=1$, 1.5, the scaling factor increases markedly with x. For $r=-0.5$, the

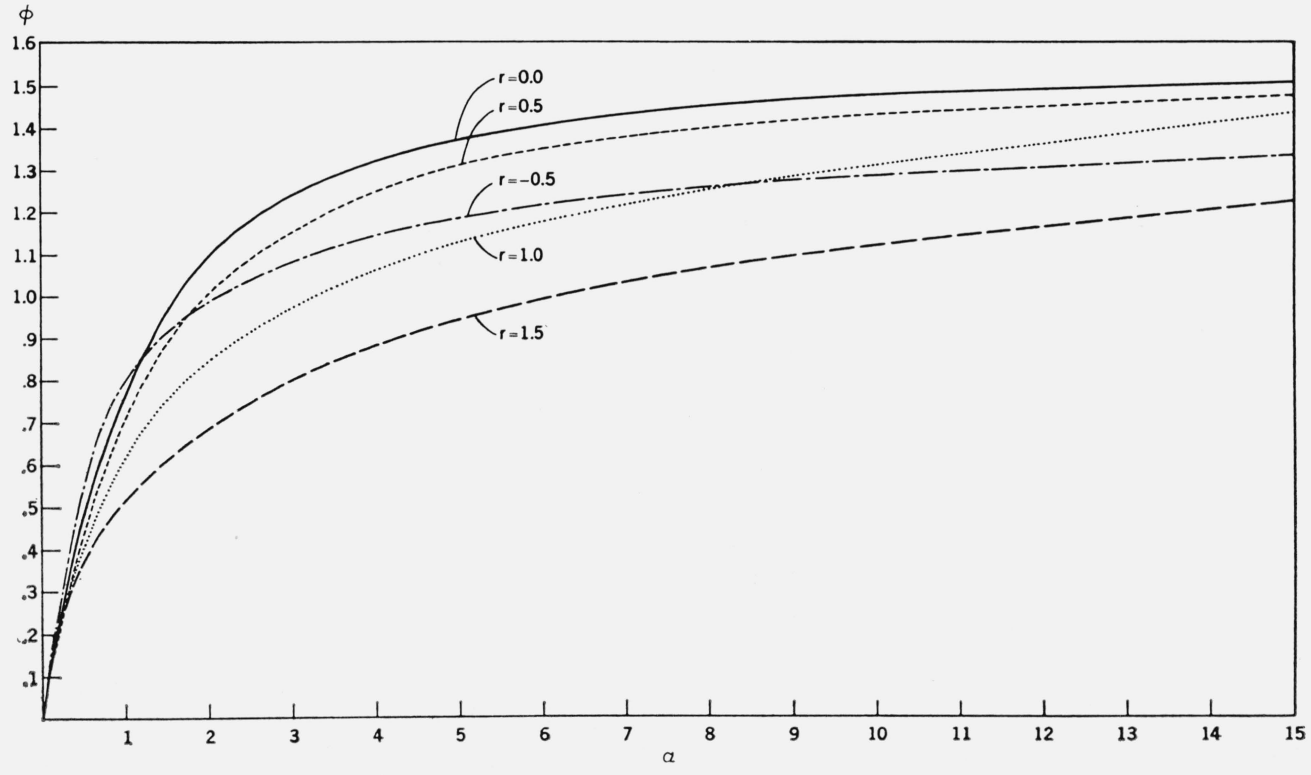

FIGURE 3. *Phase lag of the transferred modulation (see eq 6.5) as a function of modulation frequency for various collision laws.*

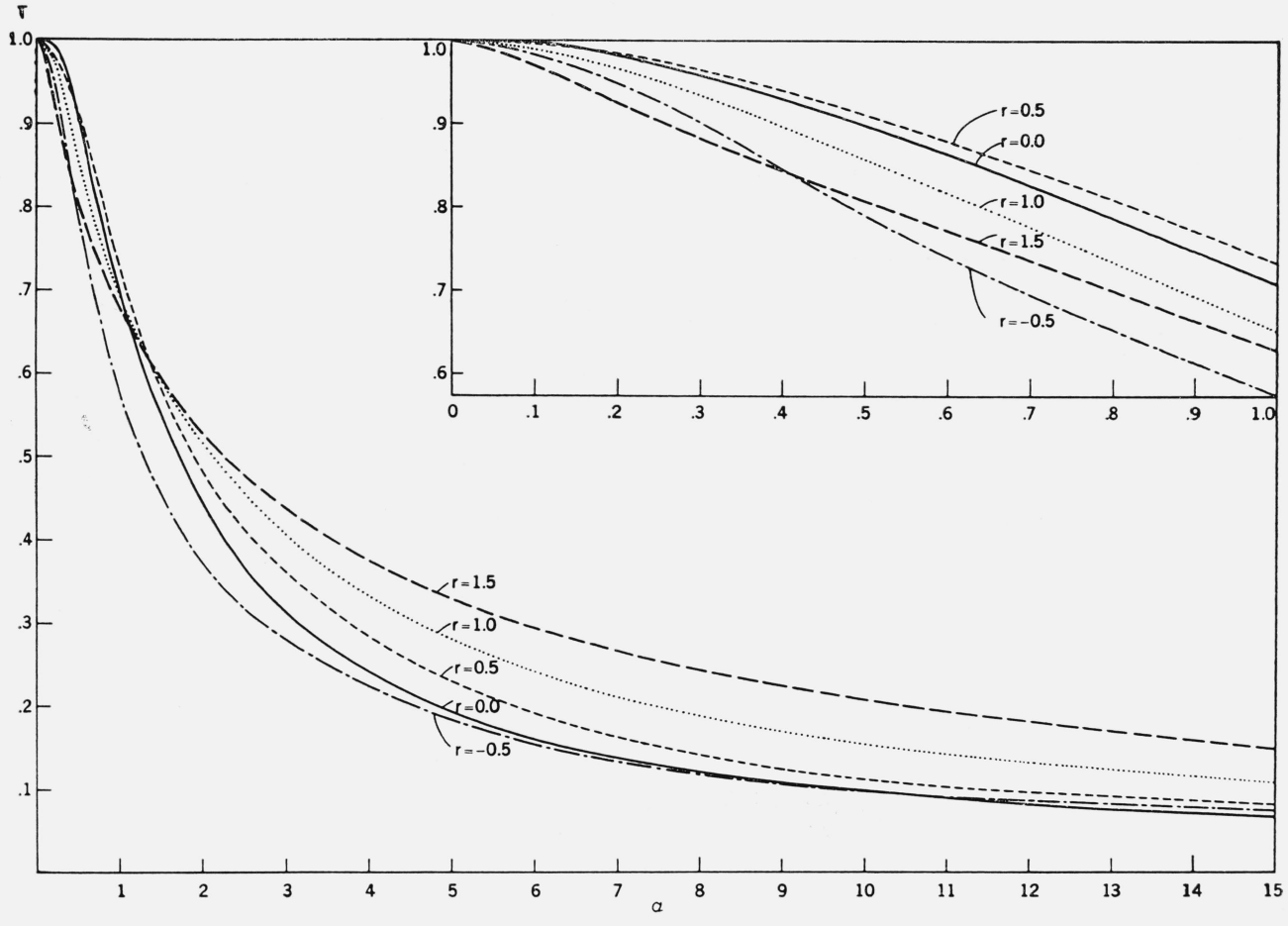

FIGURE 4. *Amplitude of the transferred modulation (see eq 6.5) as a function of modulation frequency for various collision laws.*

phase lag at first increases more rapidly with increasing modulation frequency than in the maxwellian case, then more slowly. The shapes of the curves for $r=-0.5$ and $r=0$ are entirely different.

If one were to use the phase-lag curve for $r=0$ to analyze experimental results relating to a hypothetical gas in which the electron-molecule collisions were elastic and were characterized by a velocity-independent free path (the case $r=0.5$), one would over estimate the mean collision frequency $\overline{\nu}$ by 25 percent. If a higher value of r were appropriate, the error could be much greater.

Turning now to the coefficient of transferred modulation $T(\alpha)$ (fig. 4), we see that the curvature of the function $T(\alpha)$ increases with increasing r over the entire range $-0.5 \leq r \leq 1.5$. Compressing the horizontal scale of the curve for $r=0.5$ by 25 percent would make it fall off more steeply at small and moderate values of α than the curve for $r=0$. In general, one cannot devise a horizontal scale transformation that, for a given value of $r \neq 0$, will make both $T(\alpha)$ and $\varphi(\alpha)$ assume the forms appropriate to the case $r=0$.

To sum up, the numerical calculations show that significant departures from the predictions of the elementary theory may be expected. Moreover, values of the mean collision frequency derived by using the elementary theory to analyze cross-modulation data may be significantly in error. On the positive side, the sensitivity of the predictions to changes in the collision law suggests that further theoretical and experimental refinements could ultimately lead to an accurate experimental determination not only of the collision frequency but of the form of the collision law.

7. Appendix

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The following paragraphs describe the method used to integrate (5.19). The same method can be applied to the more general equation (5.14).

Consider the inhomogeneous second-order linear differential equation

$$a(x)y''+b(x)y'+c(x)y=d(x). \tag{A.1}$$

We may approximate (A.1) by the set of coupled difference equations

$$a_i\Delta^2 y_i+b_i\Delta y_i+c_i y_i=d_i \qquad (i=1, \ldots, n-1) \tag{A.2}$$

where

$$a_i=a(x_i), \text{ etc.} \tag{A.3}$$

$$x_i=x_0+ih \tag{A.4}$$

$$\Delta y_i=\frac{y_{i+1}-y_{i-1}}{2h}, \quad \Delta^2 y_i=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}, \tag{A.5}$$

(A.2) is a set of $(n-1)$ equations for the $(n+1)$ variables $y_0, y_1, \ldots, y_n$. The two boundary conditions provide two additional equations, so that in general the set of difference equations together with the boundary conditions have a unique solution.

In the problem at hand, y_0 and y_n are given:

$$y_0=y_n=0. \tag{A.6}$$

The equations (A.2) thus have the form

$$\left(\frac{a_2}{h^2}+\frac{b_2}{2h}\right)y_2+\left(-\frac{2a_1}{h^2}+c_1\right)y_1=d_1$$

$$\left(\frac{a_{i+1}}{h^2}+\frac{b_i}{2h}\right)y_{i+1}+\left(-\frac{2a_i}{h^2}+c_i\right)y_i$$

$$+\left(\frac{a_{i-1}}{h^2}-\frac{b_{i-1}}{2h}\right)y_{i-1}=d_i \qquad (2\leq i\leq n-2)$$

$$\left(-\frac{2a_{n-1}}{h^2}+c_{n-1}\right)y_{n-1}+\left(\frac{a_{n-2}}{h^2}-\frac{b_{n-2}}{2h}\right)y_{n-2}=d_{n-1}. \tag{A.7}$$

In matrix notation,

$$MY=D \tag{A.8}$$

where M is a tridiagonal square matrix of order $(n-1)$, and Y, D are column matrices.

To solve (A.8), we write M in the form

$$M=LR \tag{A.9}$$

where L and R have the forms

$$L=\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ l_2 & 1 & 0 & . & . & . & 0 \\ 0 & l_3 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & l_{n-1} & 1 \end{bmatrix}$$

$$R=\begin{bmatrix} k_1 & r_1 & 0 & . & . & . & 0 \\ 0 & k_2 & r_2 & . & . & . & 0 \\ 0 & 0 & k_3 & r_3 & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & k_{n-1} \end{bmatrix} \tag{A.10}$$

The coefficients l_i, k_i, r_i must satisfy the equations

$$m_{1,1}=k_1, \; m_{1,2}=r_1, \; m_{i,i-1}=l_i k_{i-1}, \; m_{i,i}=l_i r_{i-1}+k_i,$$

$$m_{i,i+1}=r_i \qquad (2\leq i\leq n-1) \tag{A.11}$$

which may clearly be solved in serial order for k_1, r_1, l_2, k_2, r_2, etc. Thus L and R are completely specified by M.

Let

$$RY=U. \tag{A.12}$$

Then (A.8) becomes

$$LU=D \tag{A.13}$$

or, again equating matrix elements,

$$u_1=d_1, \; l_i u_{i-1}+u_i=d_i \qquad (2\leq i\leq n-1) \tag{A.13'}$$

which can be solved serially for the u_i. Having found U, we then determine Y from (A.12), giving

$$k_i y_i+r_i y_{i+1}=u_i \qquad (1\leq i\leq n-2)$$

$$k_{n-1}y_{n-1}=u_{n-1} \tag{A.12'}$$

which can be solved in *reverse* serial order, beginning with the last equation in the set. The solution is now complete.

One can estimate the accuracy of the procedure described, as applied to the differential equation (5.19), by comparing numerical solutions of this equation for $\eta=1$ with the exact solutions (6.1). The numerical solutions were carried out for a grid spacing $h=0.05$ and with the approximate boundary condition $y(x=25)=0$ in place of $y(x=\infty)=0$.

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