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Propagation of Pulses in Dispersive Media

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The paper deals with various approximate procedures for calculating the distortion of a pulse after it has propagated through a dispersive channel such as a waveguide. The methods used for evaluating the integrals utilize a stationary phase principle. Both impulsive-type and quasi-monochromatic sources are considered. It is shown that, in most cases, the transient response may be obtained straightforwardly from the shape of the phase versus frequency characteristics of the system. Some attention is given to the complication which arises when the group velocity has an extremum as a function of frequency.

1. Statement of Problem and General Introduction

The problem is to calculate the waveform of an electromagnetic pulse after it has propagated through a linear dispersive channel or medium. The waveform of a component of the electromagnetic field at the input is designated $E_0(t)$ and the transformed waveform at the output is $E(t)$. Thus, $E_0(t)$ is given, while $E(t)$ is sought.

To facilitate the solution, the source field is written as a Fourier integral

$$E_0(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\omega)e^{i\omega t}d\omega, \tag{1}$$

where, for the moment, $g(\omega)$ has no singularities on the real axis of the complex ω plane. The inverse of (1) is given by

$$g(\omega)=\int_{-\infty}^{\infty} E_0(t)e^{-i\omega t}dt, \tag{2}$$

where $g(\omega)$ may be regarded as the spectrum of the source pulse.

After the spectral component $g(\omega)$ exp $(i\omega t)$ passes through the system it will be modified by the transfer function $R(\omega)$ exp $[-i\varphi(\omega)]$ where $R(\omega)$ contains the amplitude and any other slowly varying complex functions of ω. Thus, the output pulse is represented by

$$E(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} R(\omega)g(\omega)e^{i\omega t}e^{-i\varphi(\omega)}\,d\omega. \tag{3}$$

In some cases, the evaluation of this integral may be carried out by analytical means when the functions $g(\omega)$, $R(\omega)$, and $\varphi(\omega)$ have simple algebraic forms. Unfortunately, more than not, the transfer function $R(\omega)$ exp $[-i\varphi(\omega)]$ is some complicated function of frequency which may be given only in numerical form. In such cases, it may be feasible to evaluate the integral by a purely numerical procedure. With the wide availability of the digital computer, this is certainly fashionable at the moment. Consequently, one might say that the problem has been solved and no further discussion is needed. However, it would be a pity if one accepted this answer since all physical insight into the nature of transient processes has been ignored. Furthermore, apart from the economic considerations of using large-scale computers, the direct numerical procedures require very special precautions when the integrand of (3) is highly oscillating. The awkwardness of the situation is compounded when the numerical data for $R(\omega)$ and $\varphi(\omega)$ are given only at a finite number of real frequencies, let alone the possibility that data for a whole frequency range might be missing.

From the above considerations, there seems to be a need to utilize approximate procedures for handling integrals of the type given by (3). Furthermore, by exploiting some of the rapidly varying characteristics of the transfer function, certain approximations can be made which lead to great simplifications.

The point of view adopted here is similar, in many aspects, to those expounded many years ago by Sommerfeld [1914] and Brillouin [1914]. However, in this paper, the emphasis will be on transfer functions which are characteristics of propagation channels such as the earth-ionosphere waveguide. Also, we shall be concerned with source pulses which have spectra covering a wide range of frequencies.

2. The Stationary Phase Method

As a starting point, we write the integral (3) in the following form:

$$J=\int_{-\infty}^{+\infty} G(\omega) \exp\,[-iF(\omega)]d\omega, \tag{4}$$

where $F(\omega)$ is real and contains all the rapidly varying phase terms, while $G(\omega)$ is a slowly varying function which may be complex. In accord with the classical notions of the stationary phase method, the important contributions to the integrand are where the function $F(\omega)$ is stationary. This suggests that we write $F(\omega)$ as a Taylor expansion about the frequency ω_s which is defined by

$$F'(\omega_s)=[\partial F(\omega)/\partial\omega]_{\omega=\omega_s}=0.$$

Thus,

$$F(\omega)=F(\omega_s)+\frac{(\omega-\omega_s)^2}{2}F''(\omega_s)+\frac{(\omega-\omega_s)^3}{6}F'''(\omega_s)+\ldots. \tag{5}$$

For many purposes, the series may be truncated beyond the term containing the second derivative $F''(\omega_s)$. Furthermore, in the physical systems under consideration, the stationary points usually occur in pairs at $\omega=\pm\omega_s$.

The contribution from $+\omega_s$ may be approximated in the following manner:

$$J_+\cong G(\omega_s)e^{-iF(\omega_s)}\int_{-\infty}^{+\infty}\exp\,[-i(\omega-\omega_s)^2F''(\omega_s)/2]\,d\omega, \tag{6}$$

where the slowly varying $G(\omega)$ in (4) is taken outside the integral and replaced by its value at $\omega=\omega_s$. The integral is now easily transformed to the well-known form

$$\int_{-\infty}^{+\infty} e^{-ix^2}dx=(\pi/i)^{1/2}. \tag{7}$$

Thus,

$$J_+\cong G(\omega_s)\left[\frac{2\pi}{iF''(\omega_s)}\right]^{1/2}\exp\,[-iF(\omega_s)]. \tag{8a}$$

In a similar manner, the contribution from $\omega=-\omega_s$ is found to be

$$J_- \cong G^*(\omega_s)\left[\frac{2\pi}{-iF''(\omega_s)}\right]^{1/2} \exp\left[+iF(\omega_s)\right], \tag{8b}$$

where the asterisk denotes a complex conjugate. Writing,

$$G(\omega_s)=|G(\omega_s)| \exp\left[i\hat{g}(\omega_s)\right], \tag{9}$$

it is a simple thing to show the combined contribution is

$$J=J_+ + J_- = 2|G(\omega_s)| \frac{(2\pi)^{1/2}}{|F''(\omega_s)|^{1/2}} \cos\left[F(\omega_s)-\hat{g}(\omega_s)\pm\frac{\pi}{4}\right], \tag{10}$$

where the $+\pi/4$ is to be used when $F''(\omega_s)>0$ and the $-\pi/4$ is to be used when $F''(\omega_s)<0$.

When the integral has additional saddle points, the contribution from them must be added to those from $\pm\omega_s$.

3. Pulse Propagation in an Ideal Waveguide

The applicability of the simple stationary phase method is best illustrated by an example. The model we shall choose is a vertical electric dipole sitting on the bottom surface of a parallel plate waveguide. In terms of cylindrical coordinates (ρ, ϕ, z), the dipole of length ds is located at the origin, while the perfectly conducting walls are defined by $z=0$ and $z=h$. When the dipole current varies as $I(\omega)\exp(i\omega t)$, the magnetic field has only a ϕ component given by [Wait, 1962]

$$H_\phi(\omega)=-\frac{I(\omega)ds e^{i\omega t}}{4h} i \frac{\partial}{\partial\rho} \sum_{n=0,1,2,\ldots}^{\infty} \epsilon_n H_0^{(2)}(kS_n\rho)\cos(kC_n z) \tag{11}$$

where $\epsilon_0=1$, $\epsilon_n=2(n\neq 0)$,

$$S_n=(1-C_n^2)^{1/2}, \qquad C_n=\pi n/(kh),$$

and $k=\omega/c$. The function $H_0^{(2)}$ is the Hankel function of the second kind of order zero. Now, in this particular problem, the input function is the waveform $i_0(t)$ of the current in the source dipole and the output is the magnetic field $h_\phi(t)$ as a function of time t. These are related to the time-harmonic quantities $I(\omega)$ and $H_\phi(\omega)$ in an analogous manner to (2) and (3). Specifically,

$$h_\phi(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} H_\phi(\omega)e^{i\omega t}d\omega \tag{12}$$

and

$$I(\omega)=\int_{-\infty}^{\infty} i_0(t)e^{-i\omega t}dt. \tag{13}$$

For this example, we shall choose $i_0(t)$ to be a suddenly applied impulse (e.g., an idealized lightning discharge). Thus,

$$i_0(t)=I_0\delta(t), \tag{14}$$

where $\delta(t)$ is the unit impulse function. Thus, from (13), it follows that:

$$I(\omega)=I_0, \tag{15}$$

which, in effect, says that the source has a uniform spectrum over all frequencies.

The transient response of the waveguide channel for this impulsive current may thus be written

$$h_\phi(t)=\frac{I_0 ds}{4\pi h}\frac{\partial}{\partial\rho}\sum_{n=0}^{\infty}\epsilon_n V_n(\rho,\, t)\cos\left(\frac{\pi n z}{h}\right), \tag{16}$$

where

$$V_n(\rho,\, t)=-\frac{i}{2}\int_{-\infty}^{+\infty} e^{i\omega t}H_0^{(2)}\left[\left[\omega^2-\left(\frac{\pi nc}{h}\right)^2\right]^{1/2}\frac{\rho}{c}\right]d\omega. \tag{17}$$

The integral $V_n(\rho,\, t)$ will now be evaluated by the stationary phase method and the result will be compared with the known exact result. The central assumption is that the Hankel function may be replaced by the first term of its asymptotic expansion. Thus,

$$V_n(\rho,\, t)=-\frac{i}{2}\int_{-\infty}^{+\infty} G(\omega)e^{-iF(\omega)}d\omega, \tag{18}$$

where

$$G(\omega)=\left(\frac{2i}{\pi}\right)^{1/2}\left[\left[\omega^2-\left(\frac{\pi nc}{h}\right)^2\right]^{1/2}\frac{\rho}{c}\right]^{-1/4}, \tag{19}$$

and

$$F(\omega)=\left[\omega^2-\left(\frac{\pi nc}{h}\right)^2\right]^{1/2}\frac{\rho}{c}-\omega t. \tag{20}$$

The radical in the latter expression is to be chosen so that the real part is positive if $|\omega|>|\pi nc/h|$ and the imaginary part is negative if $|\omega|<|\pi nc/h|$. Physically, these conditions amount to saying that only outgoing propagating modes are permitted for $\rho\rightarrow\infty$, while the cutoff modes are damped in the positive ρ-direction.

The stationary phase point, obtained by applying $F'(\omega_s)=0$ to (20), is given by

$$\omega_s=(\pi nc/h)ct/X$$

where

$$X=[(ct)^2-\rho^2]^{1/2}.$$

It is also a simple matter to show that

$$F(\omega_s)=-\pi nX/h, \qquad F''(\omega_s)=-hX^3/(\pi nd^2c^2),$$

and, finally,

$$V_n(\rho,\, t)\cong\frac{2c}{X}\cos\left(\frac{\pi nX}{h}\right)u(t'), \tag{21}$$

where $u(t')=1$ for $t'>0, =0$ for $t'<0$, and $t'=t-(\rho/c)$. This is to be compared with the exact value given by [McLachlan and Humbert, 1941; Budden, 1951]

$$V_n(\rho, t)=\frac{2c}{X}\left[\cos\left(\frac{\pi nX}{h}\right)-\frac{2}{\pi}\sin\left(\frac{\pi nX}{h}\right)\log_e\frac{ct+X}{\rho}\right]u(t'). \tag{22}$$

When $X<<ct$ or ρ, and if $\rho>>h$, the term containing the logarithm is negligible. On close inspection of the matter, it is seen that these are just the conditions for the stationary phase method to be valid. It is also worth noting that (21), for $h=\infty$, may be checked by comparing it with pair 912.3 in Campbell's and Foster's [1948] tables.

The transient magnetic field, within the stationary phase approximation, is found from (16) and (21) to be

$$h_\phi(t)\cong\frac{I_0ds}{2\pi h}\frac{c\rho}{X^3}\sum_{n=0}^{\infty}\epsilon_n\left[\cos\left(\frac{\pi nX}{h}\right)+\left(\frac{\pi nX}{h}\right)\sin\left(\frac{\pi nX}{h}\right)\right]\times\cos\left(\frac{\pi nz}{h}\right)u(t') \tag{23}$$

which is valid when $X<<ct$ or ρ and $\rho>>h$.

It is clear from (23) that the individual modes, for $n>0$, vary with time in a quasi-sinusoidal manner with a quasi half-period which is approximately given by

$$\frac{T}{2}=\frac{h}{nc}\frac{1}{[1+(2\rho/ct')]^{1/2}}\text{ seconds for } n>0. \tag{24}$$

This shows that the apparent period of the oscillation increases slowly with time. It is also interesting to note that the higher order modes have shorter oscillation periods. Of course, in the case of the zero-order mode (i.e., $n=0$), the magnetic field varies with time monotonically as $[(ct)^2-\rho^2]^{-3/2}$.

4. Application to More Realistic Models

As indicated above, the application of the stationary phase method is quite straightforward when applied to transient calculations in conventional waveguides with no wall losses. On the other hand, if the waveguide walls have finite conductivity, the problem may be very difficult if highly precise results are desired. Fortunately, for most cases of practical interest, the attenuation of the modes is accounted for by a multiplicative factor which, in the frequency plane, may be designated by $\exp[-\alpha_n(\omega)\rho]$ where $\alpha_n(\omega)$ is the attenuation coefficient as a function of frequency. Within the approximations of the stationary phase approach, the transient response of the individual modes is then simply multiplied by $\exp[-\alpha_n(\omega_s)\rho]$ where ω_s is the stationary phase point as computed for the lossless case. This perturbation technique has been applied on earlier occasions to the computation of transient waveforms in the earth-ionosphere waveguide [Wait, 1958]. Apart from the plausibility of the approach for small losses, a mathematical justification has been given by Gajewski [1958] for metallic walled waveguides at centimetric wavelengths.

In order to discuss the full significance of the stationary phase method in mode theory, it is desirable to return to (4). When the integrand represents an individual mode, the rapidly varying phase function $F(\omega)$ may be written

$$F(\omega)=\frac{\omega\rho}{v_p(\omega)}-\omega t \tag{25}$$

under the assumption that the phase characteristics of the source and other factors are sufficiently slowly varying to be lumped into $G(\omega)$. Here, it is understood that $v_p(\omega)$ is the phase velocity of a given mode. The stationary phase condition may now be written

$$F'(\omega_s)=\frac{\rho}{v_g(\omega_s)}-t=0, \tag{26}$$

where $v_g(\omega_s)=1\Big/\left[\frac{\partial}{\partial\omega}\frac{\omega}{v_p(\omega)}\right]_{\omega=\omega_s}$ is, by definition, the group velocity. Often it is convenient to use (26) to estimate the stationary point ω_s from given data on $v_g(\omega)$ plotted as a function of ω. In a similar manner, one may wish to estimate the second derivative $F''(\omega_s)$ which occurs in the denominator of (10). In this case,

$$F''(\omega_s)=\rho\left[\frac{\partial}{\partial\omega}\frac{1}{v_g(\omega)}\right]_{\omega=\omega_s} \tag{27}$$

which, as indicated, is the derivative evaluated at $\omega=\omega_s$.

In the preceding development, it has been tacitly assumed that the phase function $F(\omega)$ may be adequately approximated by the first three terms in a Taylor expansion of $F(\omega)$. Clearly, this is inadequate when the second derivative $F''(\omega_s)$ vanishes. This would lead to particular difficulty in trying to evaluate (10). As an alternative for situations of this type, we develop $F(\omega)$ in an expansion about the point ω_g which is defined by the solution

$$F''(\omega)=\frac{\partial^2}{\partial\omega^2}F(\omega)=0 \tag{28}$$

or

$$\frac{\partial}{\partial\omega}\frac{1}{v_g(\omega)}=-\frac{1}{v_g^2(\omega)}\frac{\partial v_g(\omega)}{\partial\omega}=0. \tag{29}$$

This condition corresponds to the situation where the group velocity, as a function of frequency, has a minimum or a maximum. In the case of practical waveguides in radio technology, $v_g(\omega)$ is usually a smooth monotonic function and the condition is not satisfied for any finite frequency. However, there are numerous examples in seismology [Ewing, et al., 1957] and, in certain cases, in terrestrial radio waveguides [Wait, 1962] where the group velocity has a maximum or minimum. For example, an analytical model of the earth-ionosphere waveguide would indicate the group velocity, for the dominant mode, has a maximum at a frequency ($\omega_g/2\pi$) of about 22 kc/s for typical daytime conditions [Wait and Spies, 1964] (i.e., where the phase velocity versus frequency curve has maximum curvature).

Instead of (5), we now write

$$F(\omega)=F(\omega_g)+(\omega-\omega_g)F'(\omega_g)+(\omega-\omega_g)^3F'''(\omega_g)/6+\ldots\,, \tag{30}$$

where the second derivative term is absent on account of

$$F''(\omega_g)=0.$$

When the series in (30) is now truncated beyond the third term, it is a simple matter to show that

$$J_+\cong 2G(\omega_g)e^{-iF(\omega_g)}\int_0^\infty \cos\left(\Omega F'+\frac{\Omega^3F'''}{6}\right)d\Omega, \tag{31}$$

where

$$\Omega = \omega - \omega_g, \ F' = F'(\omega_g), \text{ and } F''' = F'''(\omega_g).$$

When $F''' > 0$, the approximate expression for J_+ may be written in the following form:

$$J_+ \cong 2\pi^{1/2} G(\omega_g) \left(\frac{2}{F'''}\right)^{1/3} e^{-iF(\omega_g)} v \left[F' \left(\frac{2}{F'''}\right)^{1/3}\right], \tag{32}$$

where $v(Z)$ is the Airy integral defined by

$$v(Z) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\,[Zs + (s^3/3)]\,ds. \tag{33}$$

In a similar manner, we find that the contribution from $\omega = -\omega_g$ is given by

$$J_- \cong 2\pi^{1/2} G^*(\omega_g) \left(\frac{2}{F'''}\right)^{1/3} e^{+iF(\omega_g)} v \left[F' \left(\frac{2}{F'''}\right)^{1/3}\right]. \tag{34}$$

As a result,

$$J \cong J_+ + J_- \cong \frac{4\pi^{1/2}|G(\omega_g)|}{|F'''|^{1/3}} \cos\,[F(\omega_g) - \hat{g}(\omega_g)]\, v \left[F' \left(\frac{2}{F'''}\right)^{1/3}\right], \tag{35}$$

where, in analogy to (9),

$$G(\omega_g) = |G(\omega_g)| \exp\,[i\hat{g}(\omega_g)].$$

A very similar expression holds for $F''' < 0$. Then, it is convenient to write

$$J \cong -\frac{4\pi^{1/2}|G(\omega_g)|}{|F'''|^{1/3}} \cos\,[F(\omega_g) - \hat{g}(\omega_g)]\, v \left[-F' \left(\frac{2}{-F'''}\right)^{1/3}\right] \tag{36}$$

When this is applied to a waveguide mode,

$$F(\omega_g) = \frac{\omega_g \rho}{v_p(\omega_g)} - \omega_g t, \tag{37}$$

where ω_g is defined as the solution of

$$\frac{\partial}{\partial \omega} v_g(\omega) = 0.$$

Also, for this situation,

$$F' = \frac{\rho}{v_g(\omega_g)} - t,$$

and

$$F''' = \rho \left[\frac{\partial^2}{\partial \omega^2} \frac{1}{v_g(\omega)}\right]_{\omega = \omega_g}$$

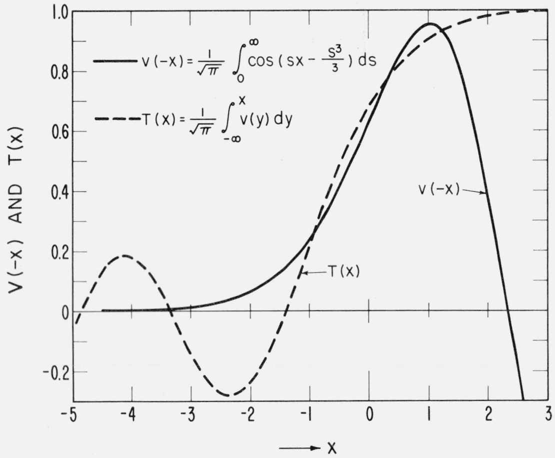

FIGURE 1. *The Airy function and a related integral.*

From the above, it is evident that the transient response of a waveguide mode is an oscillatory function with a constant frequency $\omega_g/2\pi$. It has an amplitude which is proportional to $|F'''|^{-1/3}$ and the envelope is proportional to the Airy function $v(-x)$ where $x=-F'(2/F''')^{1/3}$. It is understood that the transient response so calculated is valid only when x has a magnitude not large compared with unity. If this is not the case, higher order terms in the expansion in (30) should be considered. To show the manner in which the envelope of the pulse varies with time in this region, the function $v(-x)$ is plotted in figure 1 as a function of x from -5 to $+3$. For a waveguide mode,

$$x=\left[t-\frac{\rho}{v_g(\omega_g)}\right]\left(\frac{2}{F'''}\right)^{1/3} \tag{38}$$

which is positive when $t>\rho/v_g(\omega_g)$ since F''' is positive when the group velocity has a maximum (at $\omega=\omega_g$). Thus, in a sense, the curve of $v(-x)$ versus x in figure 1 may be regarded as the time history of the envelope of the leading portion of the pulse response of the waveguide. In this case, time increases from left to right. On the other hand, if the group velocity has a minimum, F''' is negative and then we should note that

$$x=-\left[t-\frac{\rho}{v_g(\omega_g)}\right]\left(\frac{2}{-F'''}\right)^{1/3}, \tag{39}$$

which indicates that time increases from right to left.

5. The Quasi-Monochromatic Pulse

The approaches mentioned above are applicable to propagation of the broadband pulses in a dispersive channel. In these cases, the spectral characteristics of the source are sufficiently broad that they may be lumped into $G(\omega)$ and taken outside the integral in either (4) or (31). However, when the source has most of its energy in a relatively narrow band of frequencies, another approach is desirable.

An important practical situation corresponds to choosing the source to be of the quasi-monochromatic form

$$E_0(t)=A(t)e^{i\omega_0 t}, \tag{40}$$

where ω_0 is the (angular) carrier frequency and $A(t)$ is a specified well-behaved function which may be called the modulation envelope. The corresponding spectrum, using (2), is

$$g(\omega)=\int_{-\infty}^{\infty} A(t)e^{i(\omega_0-\omega)t}dt. \tag{41}$$

Using (3), the response of the dispersive propagation channel is then to be obtained from

$$E(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}A(\hat{t})e^{i[\omega_0-\omega)\hat{t}+\omega t-\varphi(\omega)]}R(\omega)d\omega d\hat{t}, \tag{42}$$

where $R(\omega)e^{-i\varphi(\omega)}$ is the complex transfer function of the channel. The phase function $\varphi(\omega)$ is now expanded in a series about the carrier frequency ω_0. Thus,

$$\varphi(\omega)=\varphi(\omega_0)+\Omega\varphi'(\omega_0)+(\Omega^2/2)\varphi''(\omega_0)+(\Omega^3/6)\varphi'''(\omega_0), \tag{43}$$

where $\Omega=\omega-\omega_0$. It is tacitly assumed that higher order terms in Ω may be neglected.[1]

We now seek a method to carry out the ω integration in (42). To facilitate this step, it is convenient to replace $\exp\,[-i(\Omega^3/6)\varphi'']$ by a suitable integral representation which involves only the first power in Ω in the exponent. This is obtained by noting that the definition of the Airy function $v(y)$ of argument y may be written as a Fourier transform [Wait, 1962]

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-ix^3/3}e^{-ixy}dx=\frac{1}{\pi^{1/2}}v(y). \tag{44}$$

The inverse of this transform is

$$\frac{1}{\pi^{1/2}}\int_{-\infty}^{+\infty}v(y)e^{ixy}dy=e^{-ix^3/3}. \tag{45}$$

Using this result, it is then readily found that

$$\exp\left[-i\frac{\Omega^3\varphi'''(\omega_0)}{6}\right]=\frac{1}{\pi^{1/2}}\int_{-\infty}^{+\infty}v(y)\exp\,[i(\varphi'''/2)^{1/3}y\Omega]dy, \tag{46}$$

which is the desired form.

On using (43) and (46), it is now seen that (42) takes the rather ominous-looking form

$$E(t)=e^{i\omega_0 t-i\varphi(\omega_0)}\frac{1}{2\pi^{3/2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}A(\hat{t})v(y)$$
$$\times\exp\left[i\Omega[t-\hat{t}-\varphi'+(\varphi'''/2)^{1/3}y]-\frac{1}{2}i\Omega^2\varphi''\right]dyd\hat{t}d\Omega. \tag{47}$$

We now introduce a new variable z such that

$$\left(\Omega+\frac{\hat{t}-t+\varphi'-(\varphi'''/2)^{1/3}y}{\varphi''}\right)(\varphi'')^{1/2}=\pi^{1/2}z. \tag{48}$$

The exponential term in the integrand of (47) then becomes

$$\exp\left[-i(\pi z^2/2)+i\frac{(\hat{t}-t+\varphi'-(\varphi'''/2)^{1/3}y)^2}{2\varphi''}\right]. \tag{49}$$

Now, since we are dealing with frequencies ω near ω_0, the limits of the new variable z may be re-

[1] It is also assumed here that $R(\omega)$ is slowly varying compared with $\exp\,[-i\varphi(\omega)]$ in the vicinity of ω_0. Thus, it may be taken out of the integral and replaced by $R(\omega_0)$. In most of what follows, we shall set $R(\omega_0)=1$ for the sake of convenience.

garded as $-\infty$ and $+\infty$. Thus, using the result that

$$\int_{-\infty}^{+\infty} \exp\,[i\pi z^2/2]dz = 2^{1/2} \exp\,(i\pi/4), \tag{50}$$

we obtain

$$E(t) = e^{i\omega_0 t - i\varphi(\omega_0)} \frac{1}{(2i)^{1/2}\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(\hat{t}) \exp\,[i(\hat{t} - t + \varphi' - (\varphi'''/2)^{1/3}y)^2/(2\varphi'')] \times \frac{1}{\sqrt{\varphi''(\omega_0)}} v(y) dy d\hat{t}. \tag{51}$$

Then, on introducing another variable u defined by

$$\frac{(\hat{t} - t + \varphi' - (\varphi'''/2)^{1/3}y)^2}{2\varphi''} = \frac{\pi}{2} u^2,$$

it is seen that (51) becomes

$$E(t) = \exp\,[i(\omega_0 t - \varphi(\omega_0) - (\pi/4))]$$

$$\times \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v(y) A[t - \varphi' + (\varphi'''/2)^{1/3}y + (\pi\varphi'')^{1/2}u] \exp\,(i\pi u^2/2) du dy. \tag{52}$$

This general expression, although it appears to be rather complicated, may be used as the starting point for a number of investigations dealing with the effect of dispersion of pulses. For example, if it is permissible to set $\varphi''' = 0$, (52) reduces to

$$E(t) = \frac{1}{2}(1 - i)e^{i\omega_0 t - i\varphi(\omega_0)} \int_{-\infty}^{+\infty} A[t - \varphi' + (\pi\varphi'')^{1/2}u] \exp\,(i\pi u^2/2) du \tag{53}$$

by virtue of the identity

$$\int_{-\infty}^{+\infty} v(y) dy = \pi^{1/2}, \tag{54}$$

which is a special case of (45) when $x = 0$. Equation (53) is identical to the special case derived by Ginzburg [1964] who neglects φ''' at the outset. If, *in addition*, $\varphi'' = 0$, it is seen that

$$E(t) = e^{i\omega_0 t - i\varphi(\omega_0)} A[t - \varphi'], \tag{55}$$

which is the well-known result for a nondispersive channel. Here, the pulse envelope has been delayed by $\varphi'(\omega_0)$ seconds but has not changed its shape. On the other hand, the carrier phase is determined by $\varphi(\omega_0)$. The quantities φ' and φ are referred to often as the group delay and the phase delay, respectively.

6. Discussion of Pulse Distortion

Equation (53) may be used to study the distortion of the pulse shape, provided the φ''' term is negligible. For example, if the original pulse has a rectangular shape such that

$$A(t) = 0 \text{ for } t < -T_0/2$$

$$= 1 \text{ for } t > -T_0/2 \text{ and } t < T_0/2$$

$$= 0 \text{ for } t > T_0/2, \tag{56}$$

it readily follows that:

$$E(t) = \frac{1-i}{2} e^{i\omega_0 t - i\varphi(\omega_0)} \int_{u_1}^{u_2} \exp(i\pi u^2/2) du, \tag{57}$$

where

$$u_1 = \frac{-\theta}{[\pi\varphi''(\omega_0)]^{1/2}}, \qquad u_2 = \frac{T_0 - \theta}{[\pi\varphi''(\omega_0)]^{1/2}},$$

and $\theta = (T_0/2) + t - \varphi'(\omega_0)$. Here, θ is the time measured from the instant $-(T_0/2) + \varphi'(\omega_0)$. If the dispersion is sufficiently small, u_1 and u_2 become infinite and (57) reduces to (55), as it must.

To illustrate the influence of the dispersion, it is assumed that the signal duration T_0 is large compared with $[\pi\varphi''(\omega_0)]^{1/2}$. Then, the form of the leading edge of the signal may be written

$$|E(t)| \cong \frac{1}{2^{1/2}} \left| \int_{u_1}^{\infty} \exp(i\pi u^2/2) du \right|, \tag{58}$$

where $u_1 = -\theta/[\pi\varphi''(\omega_0)]^{1/2}$. Using readily available tabulated data for the above Fresnel integral, $E(t)$ is plotted as a function of $-u_1$ in figure 2. The abscissa, in this case, may be regarded as a time scale where zero corresponds to the time $t = \varphi'(\omega_0)$ which is the total group delay. As indicated, some energy arrives before this time for an actual dispersive medium. The time for the pulse to actually build up to its final value may be described conveniently by a parameter t_b. This is defined as the time from $\theta = 0$ for the pulse envelope $|E(t)|$ to approach within 5 percent of unity. From inspection of figure 2, it is seen that

$$t_b \cong 4[\pi\varphi''(\omega_0)]^{1/2} \text{ seconds.} \tag{59}$$

A typical value of t_b for daytime propagation in the earth-ionosphere waveguide of height 70 km, at 15 kc/s, for a range of 5000 km is 300 μsec. The parameter t_b is proportional to the square root of range and it depends in a rather complicated manner on other geometrical parameters. However, it is only slightly dependent on the nature of the ionospheric model assumed once the reflecting height has been specified.

It is interesting to note that the distortion of the envelope shape indicated in figure 2 is very similar to the spatial pattern resulting from diffraction at a semi-infinite screen. In general, for the waveguide case, decreasing the carrier frequency towards cutoff causes the envelope function to spread out increasingly (i.e., t_b becomes larger). Also, as may be seen from a detailed study of the complex form given by (56), the instantaneous frequency shows a modification from the carrier ω_0. This may be described as "angle modulation."

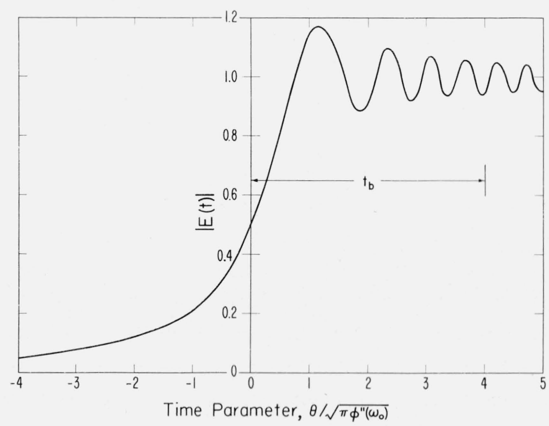

FIGURE 2. *The leading edge of the envelope of a step-modulated carrier in a dispersive medium.*

Using essentially the present second-order theory (i.e., neglecting φ''', etc.), a number of papers have appeared which deal with distortion of pulses both in acoustic [Pearson, 1953; Proud et al., 1956] and electromagnetic waveguides [Elliott, 1957; Karbowiak, 1957; Kotvun, 1958]. A rather disconcerting aspect of Elliott's [1957] work is that his calculated envelope patterns do not have the oscillations near the final buildup region of the pulse. The existence of these "Fresnel wiggles" has been recently confirmed by Ito [1964] who carried out some very convincing experiments in an electromagnetic microwave guide at frequencies of 7.8 and 8.2 Gc/s, which were respectively 1.19 f_c and 1.25 f_c where f_c is the cutoff frequency. Similar effects were observed by Walther [1961] who worked with acoustic waves in a water-filled channel.

7. Complications at an Extremum of the Group Velocity

The generalization of the above treatment for the rectangular pulse to allow for the third derivative $\varphi'''(\omega_0)$ may be readily obtained from (52). Thus, using the special form of the source envelope given by (56), it is a straightforward matter to show that

$$E(t)=\frac{1-i}{2\pi^{1/2}}\,e^{i\omega_0 t-i\varphi(\omega_0)}\int_{-\infty}^{+\infty}\left\{F\left[\frac{T_0-\theta-(\varphi'''/2)^{1/3}y}{(\pi\varphi'')^{1/2}}\right]+F\left[\frac{\theta+(\varphi'''/2)^{1/3}y}{(\pi\varphi'')^{1/2}}\right]\right\}v(y)dy, \tag{60}$$

where

$$F(u)=\int_0^u e^{i\pi u^2/2}du$$

is the standard Fresnel integral. The primes over φ indicate differentiation evaluated at the carrier frequency $\omega_0/2\pi$.

To study the leading edge of the signal, we again choose T_0 to be effectively infinite. Thus, (60) becomes, for the leading edge of the field,

$$E(t)=e^{i\omega_0 t-i\varphi(\omega_0)}\left[\frac{1}{2}+\frac{1-i}{2\pi^{1/2}}\int_{-\infty}^{+\infty}F\left[\frac{\theta+(\varphi'''/2)^{1/3}y}{(\pi\varphi'')^{1/2}}\right]v(y)dy\right], \tag{61}$$

whereas, for the trailing edge

$$E(t)=e^{i\omega_0 t-i\varphi(\omega_0)}\left[\frac{1}{2}+\frac{1-i}{2\pi^{1/2}}\int_{-\infty}^{+\infty}F\left[\frac{T_0-\theta-(\varphi'''/2)^{1/3}y}{(\pi\varphi'')^{1/2}}\right]v(y)dy\right]. \tag{62}$$

As indicated by (61) and (62), the leading edge and the trailing edge of the signal envelope have a different form. This asymmetry is a result of the φ''' term in the argument of the Fresnel integrals. To characterize completely the envelope function, we need to introduce another parameter β defined by

$$\beta=[\varphi'''(\omega_0)/2]^{1/3}[\pi''(\omega_0)]^{-1/2}, \tag{63}$$

in addition to the "buildup" time t_b. Gershman [1952] has introduced the β parameter in connection with a rather qualitative discussion of reflection of an rf pulse from the ionosphere near its critical frequency ω_{cr}. For example, if $\omega_0/\omega_{cr}=0.9$, he estimates that β is from 0.23 to 0.3 for the E layer and from 0.14 to 0.21 for the F layer. Thus, it is of considerable importance to know if the signal envelope shown in figure 2, for $\beta=0$, is significantly modified for finite β values of this order. In an example, Gershman shows that near the steepest portion of the leading and trailing edges of the signal the envelope is not modified by more than 2 percent over that shown in figure 2. However, differences are somewhat greater in the "forerunner" or "tail" regions. A proper understanding of this phenomenon awaits a detailed numerical study of the integral in (60).

An interesting limiting case which may be studied without difficulty is when $(\pi\varphi'')^{1/2}$ is effectively zero (which occurs at an extreme of the group velocity). In this case, it is not difficult to show that (61), for $\varphi'''(\omega_0) < 0$, reduces to

$$E(t) = e^{i\omega_0 t - i\varphi(\omega_0)}\, T\left[\theta\left(\frac{2}{-\varphi'''}\right)^{1/3}\right], \tag{64}$$

while, for $\varphi'''(\omega_0) > 0$,

$$E(t) = e^{i\omega_0 t - i\varphi(\omega_0)}\left\{1 - T\left[-\theta\left(\frac{2}{\varphi'''}\right)^{1/3}\right]\right\}, \tag{65}$$

where

$$T(x) = \frac{1}{\pi^{1/2}}\int_{-\infty}^{x} v(y)\,dy\cdot \tag{66}$$

As usual, θ is the time measured from the instant $-(T_0/2) + \varphi'(\omega_0)$ which, in the absence of dispersion, is the leading edge of the signal. Thus, in (64), the function T characterizes the shape of the envelope of the signal when φ''' is negative (i.e., group velocity has a minimum at ω_0). The function $T(x)$ is shown plotted in figure 1 for a range of x from -5 to $+3$ where the abscissa is proportional to time from left to right. On the other hand, in (65), the function $1-T$ characterizes the shape of the signal envelope when φ''' is positive (i.e., group velocity has a maximum at ω_0). However, in this case, the abscissa in figure 1 corresponds to time moving from right to left.

Asymptotic expansions for the function $T(x)$ may be readily obtained by making use of formulas quoted by Antosiewicz [1964] who also gives some additional numerical data of closely related functions. Retaining only the leading term in the asymptotic representation, it follows that

$$T(x) \sim 1 - \frac{1}{2\pi^{1/2}x^{3/4}}\exp\left(-\frac{2}{3}x^{3/2}\right) \text{ for } x >> 1$$

and

$$T(x) \sim \frac{1}{\pi^{1/2}(-x)^{3/4}}\cos\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right) \text{ for } -x >> 1.$$

This points up the marked asymmetry of the envelope function about the origin. As also indicated in figure 1, $T(x)$ is highly oscillatory for negative x (i.e., in the "forerunner" region) whereas it is monotonic for positive x (i.e., in the "posterior" region).

The shape of the signal envelope at the trailing edge for the case $\varphi''(\omega_0) = 0$ is found in a very similar manner to the above. Thus, for $\varphi'''(\omega_0) < 0$, (62) becomes

$$E(t) = e^{i\omega_0 t - i\varphi(\omega_0)}\left\{1 - T\left[\left(\theta - T_0\right)\left(\frac{2}{-\varphi'''}\right)^{1/3}\right]\right\} \tag{67}$$

while, for $\varphi'''(\omega_0) > 0$,

$$E(t) = e^{i\omega_0 t - i\varphi(\omega_0)}\, T\left[\left(\theta - T_0\right)\left(\frac{2}{\varphi'''}\right)^{1/3}\right] \tag{68}$$

The assumption in the above of effectively infinitely long pulse length requires here that $T_0\,|\,2/\varphi'''\,|^{1/3} >> 1$. In this case, it is necessary to suitably superimpose the responses in (64), (65), (67), and (68). An interesting consequence of the superposition is that the response or signal envelope is not symmetrical about the center of the pulse. This is to be contrasted with the situation exemplified by (57) which leads to a symmetrical envelope. In fact, in general, it may be asserted that distortion resulting from even-ordered derivatives of $\varphi(\omega)$ leads to symmetrical pulse distortion. Asymmetry is introduced when any of the odd-ordered derivatives of $\varphi(\omega)$ contribute to the final response.

8. The Forerunner

In the preceding discussion of quasi-monochromatic pulse propagation, it is tacitly assumed that the spectrum of the signal has a predominant peak at the carrier frequency $\omega_0/2\pi$. This is usually justified for the main body of the signal provided the dispersion is well behaved in the vicinity of ω_0 (i.e., no strong absorption bands near ω_0). However, in any dispersive channel such as a waveguide, some energy always arrives immediately after the instant $t=d/c$ where d is the distance between the source and the observer and c is the velocity of light. While this part of the signal usually has a very small energy content compared with the main signal occurring in the region at $t=\varphi'(\omega_0)$, it is of interest to make a quantitative estimate of this "precursor" as it is sometimes called. This is readily accomplished by applying the stationary phase method directly to (42) where we regard $g(\omega)$ to be slowly varying compared with other factors in the integrand. Obviously, this will be valid only at very small times in the response where the very-high frequencies dominate. The integral to evaluate these has the same form as (4) and the result (10) may be used directly. For example, if the source $E_0(t)$ is given by

$$\begin{aligned} \mathrm{E}_0(t) &= \sin \omega_0 t \text{ for } t>0 \\ &= 0 \text{ for } t<0 \end{aligned} \tag{69}$$

then,

$$g(\omega)=\int_0^\infty \sin \omega_0 t e^{-i\omega t} dt = +\frac{\omega_0}{\omega_0^2-\omega^2}, \tag{70}$$

and the "precursor" is given by

$$E(t) \cong \left(\frac{2}{\pi}\right)^{1/2} \frac{\omega_0}{|\omega_0^2-\omega_s^2|} \frac{1}{|\varphi''(\omega_s)|^{1/2}} \cos\left[\varphi(\omega_s)-\omega_s t \pm \pi/4\right], \tag{71}$$

where $+\pi/4$ is to be used for $\varphi''>0$ and $-\pi/4$ for $\varphi''<0$, and where ω_s is the saddle point determined by

$$\varphi'(\omega_s)=0.$$

Expressing this result in terms of waveguide mode parameters,

$$E(t) \cong \left(\frac{2}{\pi}\right)^{1/2} \frac{\omega_0}{\omega_0^2-\omega_s^2} \frac{1}{\left|d\dfrac{\partial 1/v_g(\omega)}{\partial\omega}\right|_{\omega=\omega_s}} \cos \omega_s \left[\left(t-\frac{d}{v_p(\omega_s)}\right) \pm \frac{\pi}{4}\right], \tag{72}$$

where $-\pi/4$ is to be used for $\partial(1/v_g)/\partial\omega>0$, and $+\pi/4$ for $\partial(1/v_g)/\partial\omega<0$. It is understood that this result for $E(t)$ is to be used only to estimate the early forerunner response. That is, in the time region following $t=d/c$ seconds but before the arrival of the main body of the signal near $t=\varphi'(\omega_0)$ $=d/v_g(\omega_0)$. Because $\omega_s \gg \omega_0$ in this early forerunner region, (72) is usually of negligible magnitude compared with the field amplitude in the main body of the signal.

9. Concluding Remarks

In this paper, an attempt has been made to collect together various methods for handling transient calculations for a dispersive channel. The approach has been to employ simplifications at the expense of rigor. Like most asymptotic results, the indicated formulas should be applied to a given problem only when accompanied with a certain amount of caution.

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