

The Schumann Resonances

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The resonances of the earth ionosphere cavity are considered directly as a cavity problem rather than as a limiting case of propagation in a quasi-waveguide with an arbitrary (spherically symmetric) conductivity profile. By using an approximation to a theoretically derived profile, values for the frequencies and Q factors of the lower resonances are computed which compare with experimental results as well as or better than the values derived from previous models. The effects of small changes in the profile are also considered. It is shown that ionic conduction in the lower atmosphere should not be neglected and that the "knee" in the conductivity profile where the transition from ionic to electronic conductivity occurs has a significant, and previously unrecognized, effect, particularly on the Q factors of the resonant modes.

1. Introduction

The existence of resonances of the cavity between the earth and the lower edge of the ionosphere and their excitation by lightning flashes was predicted by Schumann [1952]. Experimental measurements have amply confirmed Schumann's prediction, and the resonant frequencies and Q factors have been determined [Balsler and Wagner, 1960]. The "quality factor," Q , is defined as 2π times the ratio of the stored energy to the energy loss per cycle; it is a measure of the sharpness of the resonance. Theoretical calculations have been performed under various simplifying assumptions [e.g., see Schumann, 1952; Wait, 1964; Galejs, 1961a, b, and 1962; Chapman and Jones, 1964; Jones, 1964]; many of these give reasonable overall agreement with experiment, but none is entirely satisfactory. The most successful models tend to be somewhat artificial and to show great sensitivity to artificially introduced parameters. Careful adjustment of parameters has been particularly necessary to reproduce the observed *increase* of Q with increasing frequency.

It is conventional to approach the resonances as limiting cases of propagation in the quasi-waveguide between earth and ionosphere and to express the properties of the boundaries by surface impedances. The models that give the best results consider a uniformly conducting earth and either an exponential ionospheric conductivity profile, sharply bounded at some (often frequency-dependent) critical height [e.g., see Galejs, 1961a, b 1962], or an ionosphere constructed of several homogeneous layers [e.g., see Jones, 1964]. The effects of finite conductivity below the ionosphere may be treated by a later perturbation calculation.

Treatment of the problem explicitly as a resonant cavity and use of a smooth conductivity-height profile to eliminate the nonphysical surface charge at the lower "edge" of the ionosphere should simplify the analysis and produce better results. This has been

done by using a conductivity profile recently derived theoretically from fundamental considerations by Cole and Pierce [1965].

2. Differential Equations

Maxwell's equations and the constitutive relations in mks units are

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D} \quad \mathbf{D} = K\epsilon_0 \mathbf{E}$$

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0 \quad \mathbf{B} = \mu_0 \mathbf{H}$$

$$\nabla \cdot \mathbf{D} = \rho \quad \mathbf{J} = \sigma \mathbf{E}$$

$$\nabla \cdot \mathbf{B} = 0.$$

Here \mathbf{H} is the magnetic field, \mathbf{E} the electric field, \mathbf{B} the magnetic flux density, \mathbf{D} the displacement, \mathbf{J} the current density, and ρ the charge density. The permittivity and permeability of free space are denoted by ϵ_0 and μ_0 , respectively; K is the local dielectric constant and σ the conductivity.

The wave equation in \mathbf{H} is derived in the usual way by assuming a universal $\exp(i\omega t)$ time dependence, where ω may be complex, taking the curl of the $\nabla \times \mathbf{H}$ equation, and eliminating \mathbf{B} , \mathbf{E} , and \mathbf{D} . We find

$$\nabla^2 \mathbf{H} + \mu_0 K \epsilon_0 \omega^2 \mathbf{H} - i \mu_0 \sigma \omega \mathbf{H} + \frac{\nabla(\sigma + iK\epsilon_0\omega) \times (\nabla \times \mathbf{H})}{\sigma + iK\epsilon_0\omega} = 0.$$

Now, letting $\mu_0 \epsilon_0 = \frac{1}{c^2}$, $k = \frac{\omega}{c}$, and $\eta = \frac{\sigma}{\epsilon_0 c}$, the equation becomes

$$\left[\nabla^2 + Kk^2 - ik\eta + \frac{\nabla(\eta + iKk) \times \nabla \times}{\eta + iKk} \right] \mathbf{H} = 0.$$

For vertical polarization (the conventional case), $\mathbf{H} \cdot \mathbf{r} = 0$, where \mathbf{r} is the radius vector from the center of the earth, and, since $\nabla \cdot \mathbf{H} = 0$, we may expand \mathbf{H} in spherical coordinates as

$$\mathbf{H} = \sum_{lm} \frac{f_{lm}(r)}{r} \mathbf{X}_l^m(\theta, \varphi),$$

where $\mathbf{X}_l^m(\theta, \varphi) = \frac{1}{\sqrt{l(l+1)}} \mathbf{r} \times \nabla Y_l^m(\theta, \varphi)$, and the $Y_l^m(\theta, \varphi)$ are the ordinary spherical harmonics.

Substitution of this expansion into the differential equation, with the assumption that η and K are functions of r only, gives the differential equation for the radial function $f_{lm}(r)$

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + Kk^2 - ik\eta - \frac{d}{dr}(\eta + iKk) \frac{d}{dr} \right] f_l(r) = 0,$$

where "m" has been suppressed because it does not enter the operator. The necessary boundary conditions are that the energy flow should be outward from the cavity, downward into the earth, upward into the ionosphere.

If we further specialize to the case $\eta = \eta_e$, a constant, for $r < a$, the boundary condition at $r = a$ may be made explicit in simple form.

For $r < a$, within the earth, $\eta \sim 10^5 \text{ km}^{-1}$, $r \sim 6 \times 10^4 \text{ km}$, and $|k| \sim 10^{-4} \text{ km}^{-1}$ at ELF, where $|k|$ is the magnitude of (complex) k , so the leading terms of the differential equation are

$$\left[\frac{d^2}{dr^2} - ik\eta_e \right] f(r) = 0.$$

The appropriate solution, corresponding to a downward wave that decays with time, is

$$f(r) = A \exp [(ik\eta_e)^{1/2} r],$$

where the square root is chosen to be in the first quadrant for k in the first quadrant.

Just below the earth's surface,

$$\frac{d}{dr} f(a_-) = (ik\eta_e)^{1/2} f(a_-).$$

Assuming the earth-air transition to be sufficiently abrupt that the $\frac{d}{dr}(\eta + iKk)$ term in the radial equation predominates, we may write

$$\frac{d}{dr} \left(\frac{df}{dr} \right) \bigg/ \frac{df}{dr} = \frac{d}{dr} (\eta + iKk) / (\eta + iKk)$$

so that

$$\frac{df}{dr}(a_+) = \frac{\eta(a_+) + ik}{\eta_e + iKk} \frac{df}{dr}(a_-) \approx \frac{\eta(a_+) + ik}{\eta_e} \frac{df}{dr}(a_-).$$

Choosing A so that $f(a_-) = 1$, we have the initial conditions at $r = a_+$

$$f(a_+) = 1$$

$$f'(a_+) = (\eta(a_+) + ik)(ik/\eta_e)^{1/2}.$$

The boundary condition for large r may be expressed by requiring that the outward radial component of the Poynting vector, which may be written as

$$S_r = (\mathbf{E} \times \mathbf{H}^*)_r,$$

$$\sim -[(\eta - \text{Im } k)(\text{Re } f \text{ Re } f' + \text{Im } f \text{ Im } f')]$$

$$+ \text{Re } k(\text{Re } f \text{ Im } f' - \text{Im } f \text{ Re } f'),$$

remain positive. Here * denotes complex conjugation, Re and Im denote real and imaginary part, respectively, and $f' = \frac{df}{dr}$.

In the frequency range, ~ 7 to 100 c/s , appropriate to the lower Schumann resonances, $|k|$ and $1/r$ are of comparable magnitude. At low altitudes, $\eta \ll |k|$; at high altitudes, $\eta \gg |k|$. For likely conductivity profiles, equality occurs at heights of 50 to 60 km. The differential equation is explicitly soluble in these limits, if we assume an exponential profile, $\eta_0 \exp(r/L)$ for η at high altitudes. The solutions are

$$\eta \ll |k| \quad (d^2/dr^2 + k^2)f = 0$$

$$f = A \exp(ikr) + B \exp(-ikr)$$

$$\eta \gg |k| \quad (d^2/dr^2 - ik\eta_0 \exp(r/L) - 1/L) df/dr = 0$$

$$f = \exp(r/2L) [CH_1^{(1)}(2L(-ik\eta_0)^{1/2} \exp(r/2L)) + DH_1^{(2)}(2L(-ik\eta_0)^{1/2} \exp(r/2L))]]$$

where $H_1^{(1)}$ and $H_1^{(2)}$ are the Hankel functions of order one of the first and second kind. If the square root is taken in the upper half plane, $H_1^{(1)}/H_1^{(2)}$ represents the solution that is regular (irregular) at infinity. Only the first is appropriate to this problem; the second represents incident energy. A "matching" of the asymptotic solutions to find the eigenvalue k , and thus resonant frequency and Q , would be highly inaccurate; because both the regular and irregular solutions are constant to first order at the lower end of their domain of applicability, the transition region where $\eta \sim |k|$ is of great importance.

For any k a solution—which does not in general satisfy the boundary condition at large r —may be found by numerical integration of the radial differential equation, starting from the known boundary condition at $r = a_+$. For k from an eigenvalue, S_r will become negative above some height; an inward flux of energy is necessary to support the imposed variation of \mathbf{H} with φ , θ , and t . By repeated integrations with successive refinements of the trial value of k , systematically chosen to maximize the height at which S_r becomes negative, we may approximate the eigenvalue k_l to any desired degree of accuracy. Restated, the height h_c at which S_r becomes negative, considered as a function of complex k , has a pole at the eigenvalue k_l . We are finding this pole by a simple ascent method.

Having found k_l , the resonant frequency and Q may then be computed as

$$f = c \operatorname{Re} k / 2\pi$$

$$Q = \operatorname{Re} k / (2 \operatorname{Im} k).$$

The result for $\eta \gg |k|$ strongly resembles one given by Wait [1962] for propagating waves in the flat-earth case. If the dropped terms $k^2 - l(l+1)/r^2$ are retained, with r approximated by the earth radius a , the order of the Hankel function is $[1 - 4L^2(k^2 - l(l+1)/a^2)]^{1/2}$. Since the modified Bessel function K is defined by $K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$, our solution differs from Wait's only in that k^2 is replaced by $k^2 - l(l+1)/a^2$, a familiar average correction for spherical geometry.

3. Calculations

The conductivity profile derived by Cole and Pierce leads to a profile for η that is reasonably well fitted by an equation of the form $\eta(h) = A \exp(h/\alpha) + B \exp(h/\beta)$, where $h = r - a$; the error is certainly of less significance than that due to the assumption of spherical symmetry. The values of A , B , α , and β derived from the suggested approximation give

$$\eta(h) = 7.5 \times 10^{-8} \exp(h/6.4) + 2.3 \times 10^{-12} \exp(h/3.0) \text{ km}^{-1};$$

nearly as good a fit may be obtained with somewhat different parameters. The two terms represent ionic and electronic contributions to the conductivity.

The value of η_e should be between 3×10^3 and $2 \times 10^6 \text{ km}^{-1}$; computed from the conductivities of

average earth and sea water, respectively. A reasonable average is probably about 10^5 km^{-1} .

The radial equation was numerically integrated by using a high speed computer optimizing the complex variable k by successive approximation. This was done for the η -profile above, referred to as I, and for two others:

$$\text{II } \eta = 5.0 \times 10^{-8} \exp(h/6.4) + 2.3 \times 10^{-12} \exp(h/3.0) \text{ km}^{-1}$$

$$\text{III } \eta = 5.0 \times 10^{-8} \exp(h/6.4) + 2.3 \times 10^{-13} \exp(h/2.7) \text{ km}^{-1}.$$

Profile II gives reduced conductivity in the ionic region and displaces the "knee" of the curve, while III has a slightly steeper onset of electronic conduction with an essentially unmoved knee. The value $\eta_e = 10^5 \text{ km}^{-1}$ was used in most calculations; values of 10^4 and 10^6 were also used for profile I, $l=3$, to check sensitivity to this parameter.

These profiles and the resulting mode frequencies and Q factors, together with those of several of the better previous models of various types, and experimental data are given in figure 1 and table 1.

Note that the experimentally observed increase in Q with increasing frequency is reproduced in each case, I, II, and III, which has not been true of previous models. Also of interest is the considerable effect, particularly on Q but also on frequency, of a small change in the conductivity in the ionic region (profile II). In previous models, this conductivity has been omitted or, at best, treated as a perturbation affecting Q but not frequency.

The results are quite insensitive to earth conductivity; with profile I, $l=3$, a factor of 10 change in η_e changes f by 0.2 percent, Q by 0.5 percent, both quite insignificant.

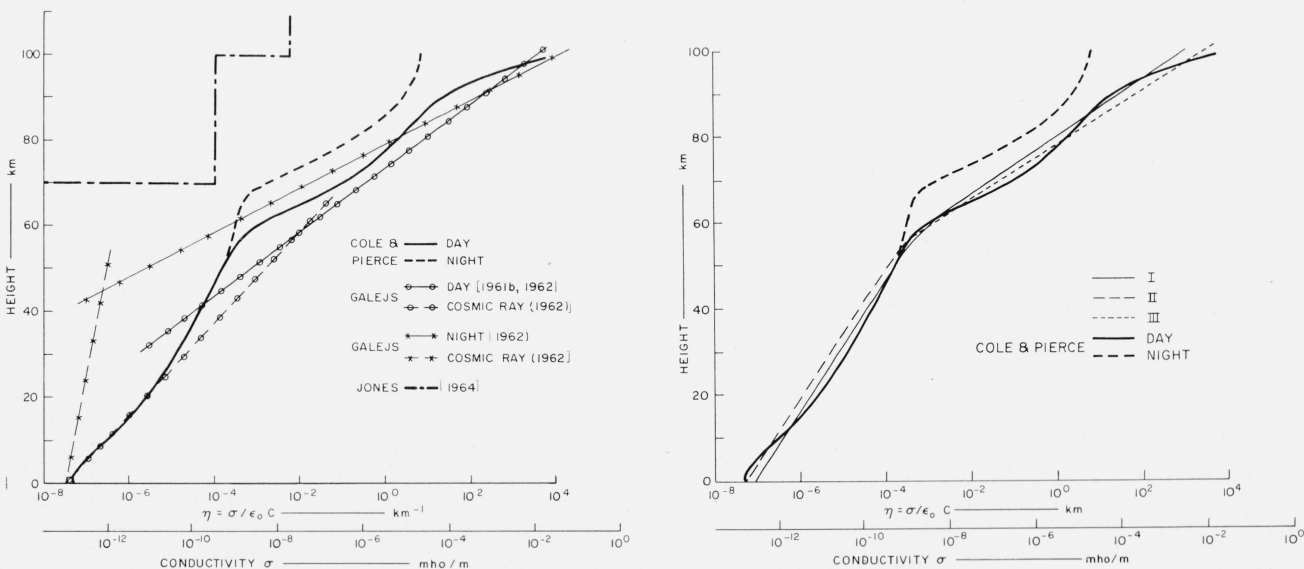


FIGURE 1. Conductivity profiles.

TABLE 1. Frequencies and Q factors of the cavity modes

	<i>l</i> =1	<i>l</i> =2	<i>l</i> =3	<i>l</i> =4	<i>l</i> =5	Profile
Measured (1)	<i>f</i> =7.8 <i>Q</i> =4.0	14.1 4.5	20.3 5.0	26.3 5.5	32.5 6.0	
Lossless	<i>f</i> =10.6	18.3	25.9	33.5	41.1	
Constant <i>h'</i> (2)	^a <i>f</i> =8.3 ^a <i>Q</i> =7.6	14.2	20.0	25.8	31.7 ^b 5.8	$\eta = 1.6 \times 10^{-10} \exp(h/3.25) \text{ km}^{-1} h > h' = 50 \text{ km}$ $0 < h < h'$
$\sigma(h') = \omega \epsilon_0$ (2)	^a <i>f</i> =8.0 ^a <i>Q</i> =3.8	13.8	19.5	25.4	31.3 ^b 4.8	$\eta = 1.6 \times 10^{-10} \exp(h/3.25) \text{ km}^{-1} h > h'$
"Average" (3)	^a <i>f</i> =8.05 ^a <i>Q</i> =6.75 ^{a,c} 4.5	14.25	20.5	26.8 ^b 5.7 ^{b,c} 4.7	$\eta = 1.6 \times 10^{-10} \exp(h/3.25) \text{ km}^{-1} h' = 45 \text{ km (Day)}$ $\eta = 8.1 \times 10^{-16} \exp(h/2.25) \text{ km}^{-1} h' = 60 \text{ km (Night)}$
Two Layer (4)	<i>f</i> =7.90 <i>Q</i> =4.7	14.13 6.0	20.3 6.9	26.4 7.5	32.6 8.2	$\eta = 0$ $0 < h < 70 \text{ km}$ $\begin{cases} 1 \times 10^4 \text{ km}^{-1} & 70 < h < 100 \text{ km} \\ 4.9 \times 10^2 \text{ km}^{-1} & h > 100 \end{cases}$
I	<i>f</i> =7.43 <i>Q</i> =4.1	19.4 4.7	31.5 5.1	$\eta = 7.5 \times 10^{-8} \exp(h/6.4) + 2.3 \times 10^{-12} \exp(h/3.0) \text{ km}^{-1}$ $\eta_e = 10^2 \text{ km}^{-1}$ Same, $\eta_e = 10^6, 10^4$
II	<i>f</i> =7.55 <i>Q</i> =4.5	19.6 5.3	31.9 5.6	$\eta = 5.0 \times 10^{-8} \exp(h/6.4) + 2.3 \times 10^{-12} \exp(h/3.0) \text{ km}^{-1}$ $\eta_e = 10^2 \text{ km}^{-1}$
III	<i>f</i> =7.71 <i>Q</i> =4.6	13.9 5.1	20.0 5.4	26.2 5.6	32.4 5.8	$\eta = 5.0 \times 10^{-8} \exp(h/6.4) + 2.2 \times 10^{-13} \exp(h/2.7) \text{ km}^{-1}$ $\eta_e = 10^2 \text{ km}^{-1}$

(1) Balsler and Wagner [1960].

(2) Galejs [1961b].

(3) Galejs [1962].

(4) Jones [1964].

^a At *f*=10.

^b At *f*=30.

^c With "cosmic ray" ionization, η exponential below *h'* $\eta = 3.76 \times 10^{-8} \text{ km}^{-1}$ at *h*=0, and continuous at *h'*.

A typical modeform is shown in figure 2. Although drawn for the *l*=1 mode with profile I, these profiles are all but indistinguishable from those for other modes and conductivity profiles. Energy and loss density profiles derived from this are given in figure 3. The energy densities are $\frac{1}{2} \mu_0 \mathbf{H} \cdot \mathbf{H}^*$ and $\frac{1}{2} \epsilon_0 \mathbf{E} \cdot \mathbf{E}^*$; the loss density is $|\mathbf{E} \cdot \mathbf{J}^*|$. Note that the loss density is significant only near the height where $\eta = |k|$ (for $\text{Im } k \ll \text{Re } k$ this is $\omega = \sigma/\epsilon_0$).

4. Importance of the Knee of the Conductivity Profile

From the definition of *Q* as 2π times the ratio of stored energy to energy loss per cycle, we have

$$Q = \frac{\omega[\epsilon_0 \int \mathbf{E} \cdot \mathbf{E}^* dV + \mu_0 \int \mathbf{H} \cdot \mathbf{H}^* dV]}{2 \int \mathbf{E} \cdot \mathbf{J}^* dV}$$

In the cavity, $r \approx a$, the earth radius, and

$$H \approx \frac{f}{a} \mathbf{X}_l = f/a(l(l-1))^{-1/2} \frac{\partial}{\partial \theta} Y_l(\theta, \varphi) \hat{e}_\theta$$

$$E_r \approx \frac{1}{a} f/a(\sigma + i\epsilon_0\omega)^{-1} \left(\frac{\partial^2}{\partial \theta^2} + \text{ctn } \theta \frac{\partial}{\partial \theta} \right) (l(l+1))^{-1/2} Y_l(\theta, \varphi)$$

$$= \frac{c\mu}{a^2} \frac{f}{\eta + ik} (l(l+1))^{1/2} Y_l(\theta, \varphi)$$

$\mathbf{E} \approx E_r \hat{e}_r$ at low altitudes

$$J_r = \sigma E_r \approx \frac{\eta f}{a^2(\eta + ik)} (l(l+1))^{1/2} Y_l(\theta, \varphi)$$

The spherical harmonics are normalized to

$$\int \mathbf{X}_l^* \cdot \mathbf{X}_l d\Omega = \int Y_l^* Y_l d\Omega = 1,$$

so that (neglecting field penetration into the earth)

$$Q \approx \frac{\omega}{2c} \frac{l(l+1) \int_0^\infty |f|^2 / (\eta^2 + |k|^2) dh + a^2 \int_0^\infty |f|^2 dh}{l(l+1) \int_0^\infty |f|^2 / (\eta^2 + |k|^2) dh}$$

Now if we approximate η by $\eta_0 \exp(r/L)$ near h_0 , where $\eta = |k|$, and observe that $|f| \approx 1$ for $0 < h < h^*$, and $f \approx 0$ elsewhere, we have upon performing the integrals

$$Q_l \approx \frac{1}{\pi L_l} \left(h_{0l} + \frac{(\text{Re } k_l a)^2}{l(l+1)} h_l^* \right),$$

where $\text{Re } k \approx |k|$ has been used.

Note in figure 1 that the points where $|k| = \eta$ for $l=1$ to 5 ($|k| \sim 2$ to $7 \times 10^{-4} \text{ km}^{-1}$) lie on the knee of the curve. This has two effects: the effective value of *L* decreases as *l* increases, and the increase in h_0 as *l* increases is greater than it would be for a single exponential fitting the upper portion of the conductivity profile. Both tend to produce an increase in *Q* with increasing *l*. Therefore neglect of ionic conductivity, which is responsible for the knee, is not justified. Treating it as a perturbation to compute corrections to the *Q*s, as has been done before [Galejs, 1962], is an

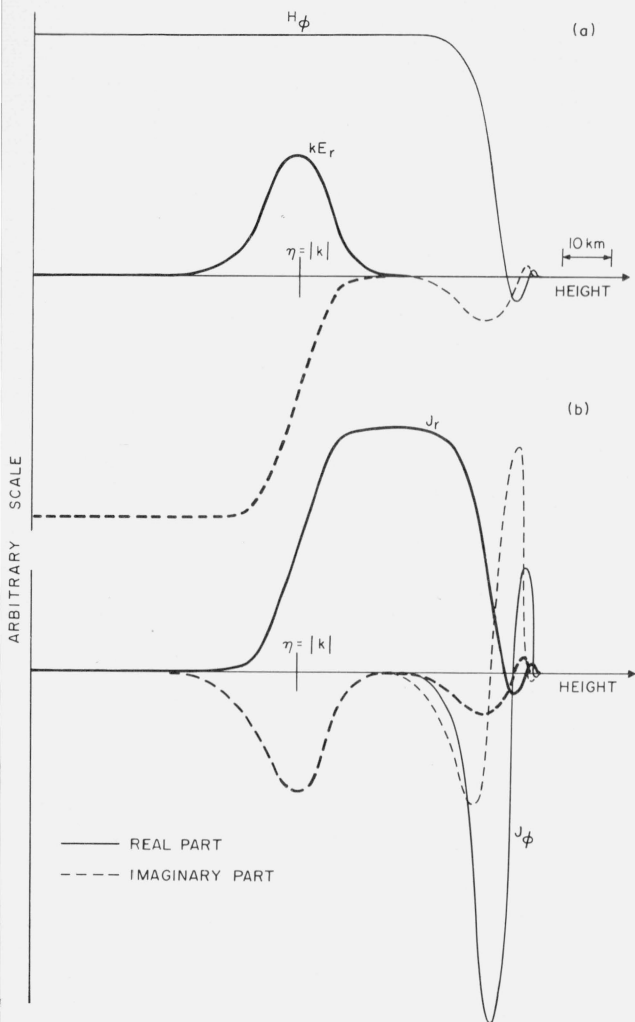


FIGURE 2. Approximate radial dependence of fields.

improvement, but the changes in the resonant frequencies (clearly not negligible: compare cases I and II) are all but impossible to calculate.

5. Conclusion

The values of the resonant frequencies and of the Q_s of the cavity resonances found by the method of this paper compare with experimental values as well as or better than those of previous models. Although further variation of parameters would almost certainly produce more accurate results, the worth of such calculations is questionable. First, the experimental values, particularly those for Q_s , are not that well established, and the resonant frequencies exhibit considerable temporal variation. Second, the assumption of spherical symmetry is clearly invalid because the conductivity profiles change temporally (diurnally, seasonally, etc.) and with geographic location; these influences may be expected to have an effect that

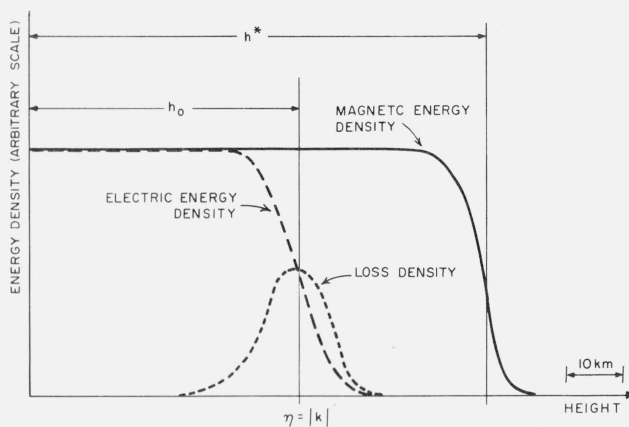


FIGURE 3. Approximate radial dependence of energy and loss densities.

should not be neglected. Also, anisotropy of the ionosphere due to the earth's magnetic field will have some effects.

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