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Irreversible Power and Radiation Resistance of Antennas in Anisotropic Ionized Gases

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In this paper a new theory for the calculation of the radiation resistance of antennas in gyroelectric media is presented. This new theory ensures the irreversibility and finiteness of the radiation resistance.

1. Introduction

In recent years many investigators [e.g., Staras, 1964; Weil and Walsh, 1965] have contributed significantly to the problem of calculating the radiation resistance of a dipole antenna immersed in an electrically anisotropic homogeneous medium whose anisotropy is due to a magnetostatic biasing field $\mathbf{B}_0$. However, the method they have used to calculate the radiation resistance—the so-called conventional method—leads in general to an infinite value for the radiation resistance, and on this account the method and the result it yields are open to question. Some have interpreted this infinity as a type of resonance in the medium, while others have attributed it to the infinitesimal size of the source. It is our contention that the difficulty is of a more basic nature and is due not simply to the size of the source but to the method of calculation. In this paper we show that when the radiation resistance is calculated in accord with the laws of reversibility and irreversibility, the radiation resistance turns out to have a physically reasonable value. Since radiation resistance is on the same footing as circuit resistance in the sense that they both are measures of irreversible power, we require that only the irreversible part of the power be used in calculating radiation resistance.

Our point of departure is the conventional expression for the time-average power P. Using combinations of outgoing and incoming fields we split P into its irreversible part P_{irr} and its reversible part P_{rev} such that $P=P_{\text{irr}}+P_{\text{rev}}$ and show that P_{irr} is free from singularity whereas P_{rev} is not. Thus, for the radiation resistance R we obtain $R=CP_{\text{irr}}$ where C is a constant. This result obeys the required conditions of finiteness and irreversibility. Previous investigators have used the expression $R=CP=C(P_{\text{irr}}+P_{\text{rev}})$ and therein lies the source of their difficulty because this expression, due to the presence of P_{rev}, violates not only the required irreversibility but also the condition of finiteness. We present a general formula for the irreversible power radiated by an arbitrary current distribution varying harmonically in time. We show that this formula reduces to the conventional result when $\mathbf{B}_0=0$ (isotropic media) or when $\mathbf{B}_0=\infty$ (uniaxial media).

To construct an expression for the irreversible part of the power emitted by a source we recall that in the case of an accelerating point electron in vacuum the combination of half the retarded minus half the advanced field is free from singularity, [Dirac, 1938], and corresponds to the irreversible power radiated by the electron [Schwinger, 1949]. We extend this idea of taking a combination field to the case of a monochromatic source radiating into a lossless, homogeneous, gyroelectric medium.

2. Formulation of the Problem

The radiation resistance R of an antenna is defined by the relation

$$R=CP_{\mathrm{irr}} \tag{1}$$

where C is the reciprocal of the squared amplitude II^* of the current $\mathrm{Re}\, Ie^{-i\omega t}$ feeding the antenna and P_{irr} is the irreversible part of the time-average real power P emitted by the antenna. Accordingly, the problem of calculating radiation resistance amounts to the problem of calculating from a knowledge of the antenna current density $\mathrm{Re}\, \mathbf{J}(\mathbf{r})e^{-i\omega t}$ the irreversible part of P.

It is well known that the power P is given by

$$P=-\frac{1}{2}\,\mathrm{Re}\int \mathbf{J}^*(\mathbf{r})\cdot \mathbf{E}_{\mathrm{out}}(\mathbf{r})dV \tag{2}$$

where the integration extends throughout the volume occupied by the antenna current and $\mathbf{E}_{\mathrm{out}}$ denotes the electric vector of the outgoing wave generated by the antenna. However, it has not been recognized that P generally has not only an irreversible part P_{irr} but also a reversible part P_{rev} as displayed by the relation

$$P=P_{\mathrm{irr}}+P_{\mathrm{rev}}. \tag{3}$$

Indeed, the question of reversibility and irreversibility has never before been raised in this connection. This is perhaps due to the fact that the dual nature of P may be overlooked in certain special cases without affecting the value of the radiation resistance. For example, when the antenna is in vacuum or in a simple lossless unbounded medium P_{rev} is identically zero and hence the conventional definition $R=CP$ is indistinguishable from our definition (1). On the other hand, in the case where the ambient medium is a lossless, homogeneous, unbounded gyroelectric medium—the very case we are presently interested in—P_{rev} is not necessarily identically zero and consequently the operation of finding the irreversible part of P is a crucial step in the calculation.

Our problem, therefore, is to extract from P an explicit expression for P_{irr} involving the antenna current density $\mathbf{J}(\mathbf{r})$ and the dyadic Green's function of the gyroelectric medium.

3. Time-Reversal Transformation

Since the anisotropy of the gyroelectric medium which surrounds the antenna is produced by a magnetostatic biasing field, the reversal of time must be accompanied by the reversal of the biasing field if Maxwell's equations are to be covariant under the time-reversal transformation. Formally, the time-reversal transformation may be thought of as a transformation from a reference frame K with space coordinates x, y, z, time coordinate t, and biasing field $\mathbf{B}_0$, to a reference frame K' with the same space coordinates, but with time coordinate t', and biasing field $\mathbf{B}_0'$. In transforming from K to K' the space coordinates remain unchanged whereas the time and the biasing field transform according to the rule

$$t\rightarrow t'=-t, \qquad \mathbf{B}_0\rightarrow \mathbf{B}_0'=-\mathbf{B}_0. \tag{4}$$

Since the microscopic Maxwell-Lorentz equations are known to be covariant under this transformation, we demand that the macroscopic Maxwell's equations for a lossless gyroelectric medium be likewise covariant. Using the convention that primed quantities are referred to K' and unprimed ones to K, we display this covariance by stating that in K we have

$$\nabla\times\mathbf{H}(\mathbf{r})=\mathbf{J}(\mathbf{r})-i\omega\epsilon(\mathbf{B}_0)\cdot\mathbf{E}(\mathbf{r}), \qquad \nabla\cdot\epsilon(\mathbf{B}_0)\cdot\mathbf{E}(\mathbf{r})=\rho(\mathbf{r})$$

$$\nabla\times\mathbf{E}(\mathbf{r})=i\omega\mu_0\mathbf{H}(\mathbf{r}), \qquad \nabla\cdot\mathbf{H}(\mathbf{r})=0 \tag{5}$$

and in K' we have

$$\nabla \times \mathbf{H}'(\mathbf{r}) = \mathbf{J}'(\mathbf{r}) - i\omega\epsilon(\mathbf{B}_0') \cdot \mathbf{E}'(\mathbf{r}), \qquad \nabla \cdot \epsilon(\mathbf{B}_0') \cdot \mathbf{E}'(\mathbf{r}) = \rho'(\mathbf{r})$$

$$\nabla \times \mathbf{E}'(\mathbf{r}) = i\omega\mu_0\mathbf{H}'(\mathbf{r}), \qquad \nabla \cdot \mathbf{H}'(\mathbf{r}) = 0. \tag{6}$$

It follows from the covariance of these macroscopic Maxwell equations that the field and source quantities in K are related to those in K' by the transformations:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}'^*(\mathbf{r}), \qquad \mathbf{H}(\mathbf{r}) = -\mathbf{H}'^*(\mathbf{r}), \qquad \mathbf{J}(\mathbf{r}) = -\mathbf{J}'^*(\mathbf{r})$$

$$\rho(\mathbf{r}) = \rho'^*(\mathbf{r}), \qquad \epsilon(\mathbf{B}_0) = \epsilon^*(\mathbf{B}_0'). \tag{7}$$

We can verify that the above transformation law for the dielectric tensor is indeed obeyed when $\mathbf{B}_0' = -\mathbf{B}_0$, by recalling that the dielectric tensor of a lossless gyroelectric medium has the property $\epsilon(\mathbf{B}_0) = \epsilon^*(-\mathbf{B}_0)$.

The physical significance of these transformations (7) may be more easily appreciated in the time domain. For example, in the case of the electric vector we have

$$\mathbf{E}'(\mathbf{r}, t') = \mathrm{Re}\ \mathbf{E}'(\mathbf{r})\ e^{-i\omega t'} = \mathrm{Re}\ E^*(\mathbf{r})\ e^{i\omega t} = \mathrm{Re}\ \mathbf{E}(\mathbf{r})\ e^{-i\omega t} = \mathbf{E}(\mathbf{r}, t) \tag{8}$$

or

$$\mathbf{E}'(\mathbf{r}, -t) = \mathbf{E}(\mathbf{r}, t) \tag{9}$$

which shows that the past and the future of an electric field are the same. Similarly the past and the future of any of the source or field quantities (except for minus signs) are the same. It is also revealing to consider Poynting's vector. If by $\mathbf{S}'$ and $\mathbf{S}$ we denote the complex Poynting vector in K' and K, respectively, we have the transformation rule

$$\mathbf{S}' = \frac{1}{2}\mathbf{E}'(\mathbf{r}) \times \mathbf{H}'^*(\mathbf{r}) = -\frac{1}{2}\mathbf{E}^*(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) = -\mathbf{S}^* \tag{10}$$

whose real part yields

$$\mathrm{Re}\ \mathbf{S}' = -\mathrm{Re}\ \mathbf{S}. \tag{11}$$

Thus we see that the real part of Poynting's vector changes sign under time-reversal.

Up to the present point of our discussion it appears that in a lossless gyroelectric medium the electromagnetic field and the real part of the complex Poynting's vector are completely reversible. To answer the question of where the irreversibility comes from we must remember that the fields we have been examining are total fields and that Maxwell's equations actually permit two independent wave solutions, one representing an outgoing wave $\mathbf{E}_{\text{out}}$, and the other an incoming wave $\mathbf{E}_{\text{in}}$. In the time domain the two independent solutions are the retarded and the advanced wave, which are related to $\mathbf{E}_{\text{out}}$ and $\mathbf{E}_{\text{in}}$ by

$$\mathbf{E}_{\text{ret}}(\mathbf{r}, t) = \mathrm{Re}\ \mathbf{E}_{\text{out}}(\mathbf{r})\ e^{-i\omega t}$$

$$\mathbf{E}_{\text{adv}}(\mathbf{r}, t) = \mathrm{Re}\ \mathbf{E}_{\text{in}}(\mathbf{r})\ e^{-i\omega t} \tag{12}$$

in K, and by

$$\mathbf{E}'_{\text{ret}}(\mathbf{r}, t') = \text{Re } \mathbf{E}'_{\text{out}}(\mathbf{r})\, e^{-i\omega t'}$$

$$\mathbf{E}'_{\text{adv}}(\mathbf{r}, t') = \text{Re } \mathbf{E}'_{\text{in}}(\mathbf{r})\, e^{-i\omega t'} \tag{13}$$

in K'. Since $\mathbf{E}_{\text{ret}}(\mathbf{r}, t) = \mathbf{E}'_{\text{adv}}(\mathbf{r}, t')$, it follows from (12) and (13) and similar equations for the magnetic fields that

$$\mathbf{E}_{\text{out}}(\mathbf{r}) = \mathbf{E}'^{*}_{\text{in}}(\mathbf{r})$$

$$\mathbf{E}_{\text{in}}(\mathbf{r}) = \mathbf{E}'^{*}_{\text{out}}(\mathbf{r})$$

$$\mathbf{H}_{\text{out}}(\mathbf{r}) = -\mathbf{H}'^{*}_{\text{in}}(\mathbf{r})$$

$$\mathbf{H}_{\text{in}}(\mathbf{r}) = -\mathbf{H}'^{*}_{\text{out}}(\mathbf{r}). \tag{14}$$

Relations (14) show that an outgoing wave in K transforms into an incoming wave in K' and vice versa.

To interpret the result in (14) in terms of Poynting's vector we denote the complex Poynting vectors of the outgoing and incoming waves in K by $\mathbf{S}_{\text{out}}$ and $\mathbf{S}_{\text{in}}$, and in K' by $\mathbf{S}'_{\text{out}}$ and $\mathbf{S}'_{\text{in}}$, respectively, and note that the transformations

$$\begin{aligned}
\mathbf{S}_{\text{out}} &= \frac{1}{2}\mathbf{E}_{\text{out}}(\mathbf{r}) \times \mathbf{H}^{*}_{\text{out}}(\mathbf{r}) = -\frac{1}{2}\mathbf{E}'^{*}_{\text{in}}(\mathbf{r}) \times \mathbf{H}'_{\text{in}} = -\mathbf{S}'^{*}_{\text{in}} \\
\mathbf{S}_{\text{in}} &= \frac{1}{2}\mathbf{E}_{\text{in}}(\mathbf{r}) \times \mathbf{H}^{*}_{\text{in}}(\mathbf{r}) = -\frac{1}{2}\mathbf{E}'^{*}_{\text{out}}(\mathbf{r}) \times \mathbf{H}'_{\text{out}}(\mathbf{r}) = -\mathbf{S}'^{*}_{\text{out}}
\end{aligned} \tag{15}$$

lead to

$$\text{Re}(\mathbf{S}_{\text{out}} \mp \mathbf{S}_{\text{in}}) = \text{Re}(-\mathbf{S}'^{*}_{\text{in}} \pm \mathbf{S}'^{*}_{\text{out}}) = \pm\,\text{Re}(\mathbf{S}'_{\text{out}} \mp \mathbf{S}'_{\text{in}}). \tag{16}$$

This equation demonstrates that the combination $\text{Re}(\mathbf{S}_{\text{out}} - \mathbf{S}_{\text{in}})$ is irreversible and $\text{Re}(\mathbf{S}_{\text{out}} + \mathbf{S}_{\text{in}})$ is reversible under the time reversal transformation (4).

From the complex Poynting's vector theorem, which in the present case of a lossless gyroelectric medium has the form

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^{*}) = -\mathbf{J}^{*} \cdot \mathbf{E} + i\omega\mu_0 \mathbf{H} \cdot \mathbf{H}^{*} - i\omega \mathbf{E} \cdot \epsilon(\mathbf{B}_0) \cdot \mathbf{E}^{*}, \tag{17}$$

we get

$$\text{Re}\,\nabla \cdot (\mathbf{E} \times \mathbf{H}^{*}) = -\text{Re}\,\mathbf{J}^{*} \cdot \mathbf{E} \tag{18}$$

which on integration yields

$$\text{Re}\int \mathbf{n} \cdot \mathbf{S}\, da = -\frac{1}{2}\,\text{Re}\int \mathbf{J}^{*}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\, dV \tag{19}$$

where the volume integration extends throughout the volume occupied by the current and the surface integral extends over any closed surface with outward normal $\mathbf{n}$, enclosing the current.

Applying this relation (19) to $\mathbf{S}_{\text{out}}$ and $\mathbf{S}_{\text{in}}$ we obtain the relations

$$\begin{aligned}\operatorname{Re}\int \mathbf{n}\cdot\mathbf{S}_{\text{out}}\,da &= -\frac{1}{2}\operatorname{Re}\int \mathbf{J}^*(\mathbf{r})\cdot\mathbf{E}_{\text{out}}(\mathbf{r})\,dV\\ \operatorname{Re}\int \mathbf{n}\cdot\mathbf{S}_{\text{in}}\,da &= -\frac{1}{2}\operatorname{Re}\int \mathbf{J}^*(\mathbf{r})\cdot\mathbf{E}_{\text{in}}(\mathbf{r})\,dV\end{aligned} \tag{20}$$

whose difference and sum yield

$$\operatorname{Re}\int \mathbf{n}\cdot(\mathbf{S}_{\text{out}}\mp\mathbf{S}_{\text{in}})\,da = -\frac{1}{2}\operatorname{Re}\int \mathbf{J}^*(\mathbf{r})\cdot[\mathbf{E}_{\text{out}}(\mathbf{r})\mp\mathbf{E}_{\text{in}}(\mathbf{r})]\,dV. \tag{21}$$

Since by (16) $\operatorname{Re}(\mathbf{S}_{\text{out}}-\mathbf{S}_{\text{in}})$ is irreversible and $\operatorname{Re}(\mathbf{S}_{\text{out}}+\mathbf{S}_{\text{in}})$ is reversible, we see from (21) that

$$\operatorname{Re}\int \mathbf{J}^*(\mathbf{r})\cdot[\mathbf{E}_{\text{out}}(\mathbf{r})-\mathbf{E}_{\text{in}}(\mathbf{r})]\,dV \tag{22}$$

expresses irreversible power and

$$\operatorname{Re}\int \mathbf{J}^*(\mathbf{r})\cdot[\mathbf{E}_{\text{out}}(\mathbf{r})+\mathbf{E}_{\text{in}}(\mathbf{r})]\,dV \tag{23}$$

expresses reversible power. The irreversibility of (22) and the reversibility of (23) can also be made evident by submitting them to the field transformations of (14) and the current density transformation of (7).

Recalling expression (2) for P and using the algebraic identity

$$\mathbf{E}_{\text{out}} = \frac{1}{2}(\mathbf{E}_{\text{out}}-\mathbf{E}_{\text{in}}) + \frac{1}{2}(\mathbf{E}_{\text{out}}+\mathbf{E}_{\text{in}}) \tag{24}$$

we find that

$$\begin{aligned}P = &-\frac{1}{4}\operatorname{Re}\int \mathbf{J}^*(\mathbf{r})\cdot[\mathbf{E}_{\text{out}}(\mathbf{r})-\mathbf{E}_{\text{in}}(\mathbf{r})]\,dV\\ &-\frac{1}{4}\operatorname{Re}\int \mathbf{J}^*(\mathbf{r})\cdot[\mathbf{E}_{\text{out}}(\mathbf{r})+\mathbf{E}_{\text{in}}(\mathbf{r})]\,dV.\end{aligned} \tag{25}$$

Thus we split P into its irreversible part (first term on right) and its reversible part (second term on right).

Consequently we can write

$$P = P_{\text{irr}} + P_{\text{rev}} \tag{26}$$

where P_{irr} and P_{rev} are given by

$$P_{\text{irr}} = -\frac{1}{4}\operatorname{Re}\int \mathbf{J}^*(\mathbf{r})\cdot[\mathbf{E}_{\text{out}}(\mathbf{r})-\mathbf{E}_{\text{in}}(\mathbf{r})]\,dV \tag{27}$$

$$P_{\text{rev}} = -\frac{1}{4}\operatorname{Re}\int \mathbf{J}^*(\mathbf{r})\cdot[\mathbf{E}_{\text{out}}(\mathbf{r})+\mathbf{E}_{\text{in}}(\mathbf{r})]\,dV \tag{28}$$

or equivalently by

$$P_{\text{irr}} = \frac{1}{2}\operatorname{Re}\int \mathbf{n}\cdot(\mathbf{S}_{\text{out}}-\mathbf{S}_{\text{in}})\,da \tag{29}$$

$$P_{\text{rev}} = \frac{1}{2}\operatorname{Re}\int \mathbf{n}\cdot(\mathbf{S}_{\text{out}}+\mathbf{S}_{\text{in}})\,da. \tag{30}$$

To find P_{irr} we must use either expression (27) or expression (29). These expressions are noteworthy because they place in evidence the necessity of including the incoming field or the incoming Poynting's vector.

4. Regular Behavior of the Difference Field

Since we have

$$P = P_{\text{irr}} + P_{\text{rev}} \tag{31}$$

and since P is not free from singularity [Staras, 1964; Weil and Walsh, 1965], the question of how the possible singularity in P is distributed between the terms P_{irr} and P_{rev} arises. Clearly, to answer this query one examines expressions (27) and (28) for P_{irr} and P_{rev}. Here we make the reasonable assumption that the volume integral of $\mathbf{J}$ itself is bounded, and therefore the determining quantities to be examined are the difference field $\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}}$, and the sum field $\mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}$.

We know that $\mathbf{E}_{\text{out}}$ and $\mathbf{E}_{\text{in}}$ satisfy the same differential equation, i.e.,

$$\nabla \times \nabla \times \mathbf{E}_{\text{out}} - \omega^2 \mu_0 \boldsymbol{\epsilon} \cdot \mathbf{E}_{\text{out}} = i\omega\mu_0 \mathbf{J} \tag{32}$$

$$\nabla \times \nabla \times \mathbf{E}_{\text{in}} - \omega^2 \mu_0 \boldsymbol{\epsilon} \cdot \mathbf{E}_{\text{in}} = i\omega\mu_0 \mathbf{J}. \tag{33}$$

However, at infinity the boundary condition on $\mathbf{E}_{\text{out}}$ is different from the boundary condition on $\mathbf{E}_{\text{in}}$, and therefore $\mathbf{E}_{\text{out}}$ and $\mathbf{E}_{\text{in}}$ cannot be the same function.

Taking the difference of (32) and (33) we obtain the homogeneous equation for the difference field

$$\nabla \times \nabla \times (\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}}) - \omega^2 \mu_0 \boldsymbol{\epsilon} \cdot (\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}}) = 0 \tag{34}$$

which holds everywhere. This equation implies that $\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}}$ is free from singularity. Using this result and recalling that $\mathbf{E}_{\text{out}} \neq \mathbf{E}_{\text{in}}$, we see from (27) that P_{irr} is finite.

On the other hand, taking the sum of (32) and (33) we see that the sum field satisfies the inhomogeneous equation

$$\nabla \times \nabla \times (\mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}) - \omega^2 \mu_0 \boldsymbol{\epsilon} \cdot (\mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}) = 2\, i\omega\mu_0 \mathbf{J} \tag{35}$$

which exhibits the singular behavior of the sum field. Since it is the sum field that carries the singularity, we see from (28) that the singularity appears in P_{rev}.

5. Irreversible Power in Terms of the Dyadic Green's Function

To reduce expression (27) for the irreversible power to a simpler computational form we introduce the dyadic Green's functions which relate $\mathbf{E}_{\text{out}}$ and $\mathbf{E}_{\text{in}}$ to the source current density $\mathbf{J}$.

From the linearity of Maxwell's equations we can write

$$\mathbf{E}_{\text{out}}(\mathbf{r}, \pm \mathbf{B}_0) = i\omega\mu_0 \int \boldsymbol{\Gamma}_{\text{out}}(\mathbf{r}, \mathbf{r}', \pm \mathbf{B}_0) \cdot \mathbf{J}(\mathbf{r}')\, dV' \tag{36}$$

$$\mathbf{E}_{\text{in}}(\mathbf{r}, \pm \mathbf{B}_0) = i\omega\mu_0 \int \boldsymbol{\Gamma}_{\text{in}}(\mathbf{r}, \mathbf{r}', \pm \mathbf{B}_0) \cdot \mathbf{J}(\mathbf{r}')\, dV' \tag{37}$$

where $\boldsymbol{\Gamma}_{\text{out}}$ and $\boldsymbol{\Gamma}_{\text{in}}$ are the dyadic Green's functions which satisfy

$$\nabla \times \nabla \times \boldsymbol{\Gamma}_{\text{out}}(\pm \mathbf{B}_0) - \omega^2 \mu_0 \boldsymbol{\epsilon}(\pm \mathbf{B}_0) \cdot \boldsymbol{\Gamma}_{\text{out}}(\pm \mathbf{B}_0) = \mathbf{u}\delta(\mathbf{r} - \mathbf{r}') \tag{38}$$

$$\nabla \times \nabla \times \boldsymbol{\Gamma}_{\text{in}}(\pm \mathbf{B}_0) - \omega^2 \mu_0 \boldsymbol{\epsilon}(\pm \mathbf{B}_0) \cdot \boldsymbol{\Gamma}_{\text{in}}(\pm \mathbf{B}_0) = \mathbf{u}\delta(\mathbf{r} - \mathbf{r}') \tag{39}$$

where $\mathbf{u}$ = unit dyadic. Although $\boldsymbol{\Gamma}_{\text{out}}$ and $\boldsymbol{\Gamma}_{\text{in}}$ satisfy the same differential equation, they are not identically equal because they must satisfy different boundary conditions at infinity. Since $\boldsymbol{\epsilon}(-\mathbf{B}_0)=\boldsymbol{\epsilon}^*(\mathbf{B}_0)$ we find by taking the conjugate complex of (39) that

$$\nabla \times \nabla \times \boldsymbol{\Gamma}^*_{\text{in}}(\mp \mathbf{B}_0) - \omega^2\mu_0 \boldsymbol{\epsilon}(\pm \mathbf{B}_0) \cdot \boldsymbol{\Gamma}^*_{\text{in}}(\mp \mathbf{B}_0) = \mathbf{u}\delta(\mathbf{r}-\mathbf{r}'). \tag{40}$$

Comparing (38) with (40) we see that $\boldsymbol{\Gamma}_{\text{out}}(\pm \mathbf{B}_0)$ and $\boldsymbol{\Gamma}^*_{\text{in}}(\mp \mathbf{B}_0)$ satisfy the same equation. In the special case where $\mathbf{B}_0=0$ we know that $\boldsymbol{\Gamma}_{\text{out}}=\boldsymbol{\Gamma}^*_{\text{in}}$, and thus we conclude that

$$\boldsymbol{\Gamma}^*_{\text{in}}(\mp \mathbf{B}_0) = \boldsymbol{\Gamma}_{\text{out}}(\pm \mathbf{B}_0) \tag{41}$$

holds for all values of $\mathbf{B}_0$.

With the aid of relation (41) we obtain from (36) and (37) the following expression for the difference field:

$$\mathbf{E}_{\text{out}}(\mathbf{r},\mathbf{B}_0) - \mathbf{E}_{\text{in}}(\mathbf{r},\mathbf{B}_0) = i\omega\mu_0 \int [\boldsymbol{\Gamma}_{\text{out}}(\mathbf{r},\mathbf{r}',\mathbf{B}_0) - \boldsymbol{\Gamma}^*_{\text{out}}(\mathbf{r},\mathbf{r}',-\mathbf{B}_0)] \cdot \mathbf{J}(\mathbf{r}')dV'. \tag{42}$$

Substituting this expression into (27) we see that the irreversible power is given by

$$P_{\text{irr}} = \frac{\omega\mu_0}{4} \text{Im} \int \mathbf{J}^*(\mathbf{r}) \cdot [\boldsymbol{\Gamma}_{\text{out}}(\mathbf{r},\mathbf{r}',\mathbf{B}_0) - \boldsymbol{\Gamma}^*_{\text{out}}(\mathbf{r},\mathbf{r}',-\mathbf{B}_0)] \cdot \mathbf{J}(\mathbf{r}')\, dV dV'. \tag{43}$$

To simplify further this relation we note that

$$\boldsymbol{\Gamma}_{\text{out}}(\mathbf{B}_0) - \boldsymbol{\Gamma}^*_{\text{out}}(-\mathbf{B}_0) = \text{Re}\,[\boldsymbol{\Gamma}_{\text{out}}(\mathbf{B}_0) - \boldsymbol{\Gamma}_{\text{out}}(-\mathbf{B}_0)] + i\, \text{Im}\,[\boldsymbol{\Gamma}_{\text{out}}(\mathbf{B}_0) + \boldsymbol{\Gamma}_{\text{out}}(-\mathbf{B}_0)] \tag{44}$$

and accordingly write P_{irr} in the following form

$$\begin{aligned} P_{\text{irr}} &= \frac{\omega\mu_0}{4} \text{Re} \int \mathbf{J}^*(\mathbf{r}) \cdot \text{Im}\,[\boldsymbol{\Gamma}_{\text{out}}(\mathbf{r},\mathbf{r}',\mathbf{B}_0) + \boldsymbol{\Gamma}_{\text{out}}(\mathbf{r},\mathbf{r}',-\mathbf{B}_0)] \cdot \mathbf{J}(\mathbf{r}')\, dV dV' \\ &+ \frac{\omega\mu_0}{4} \text{Im} \int \mathbf{J}^*(\mathbf{r}) \cdot \text{Re}[\boldsymbol{\Gamma}_{\text{out}}(\mathbf{r},\mathbf{r}',\mathbf{B}_0) - \boldsymbol{\Gamma}_{\text{out}}(\mathbf{r},\mathbf{r}',-\mathbf{B}_0)] \cdot \mathbf{J}(\mathbf{r}')\, dV dV'. \end{aligned} \tag{45}$$

This is the desired expression for the irreversible power. It shows that from a knowledge of $\mathbf{J}$ and $\boldsymbol{\Gamma}_{\text{out}}$ we can find P_{irr}, and from a knowledge of P_{irr} we can find in turn the radiation resistance by the formula $R=CP_{\text{irr}}$.

For the sake of completeness, we also include the expression for the reversible power:

$$\begin{aligned} P_{\text{rev}} &= \frac{\omega\mu_0}{4} \text{Im} \int \mathbf{J}^*(\mathbf{r}) \cdot \text{Re}\,[\boldsymbol{\Gamma}_{\text{out}}(\mathbf{r},\mathbf{r}',\mathbf{B}_0) + \boldsymbol{\Gamma}_{\text{out}}(\mathbf{r},\mathbf{r}',-\mathbf{B}_0)] \cdot \mathbf{J}(\mathbf{r}')\, dV dV' \\ &+ \frac{\omega\mu_0}{4} \text{Re} \int \mathbf{J}^*(\mathbf{r}) \cdot \text{Im}\,[\boldsymbol{\Gamma}_{\text{out}}(\mathbf{r},\mathbf{r}',\mathbf{B}_0) - \boldsymbol{\Gamma}_{\text{out}}(\mathbf{r},\mathbf{r}',-\mathbf{B}_0)] \cdot \mathbf{J}(\mathbf{r}')\, dV dV'. \end{aligned} \tag{46}$$

6. Dipole Antenna

Let us now consider the case of a dipole antenna in a gyroelectric medium. For a dipole antenna situated at the origin of the coordinates we have

$$\mathbf{J}(\mathbf{r}) = -i\omega\mathbf{p}\delta(\mathbf{r}). \tag{47}$$

Substituting this expression into (45) we see that the irreversible power radiated by the dipole is given by

$$P_{\text{irr}}=\frac{\omega^3\mu_0}{4}\,\mathbf{p}\cdot\operatorname{Im}\left[\boldsymbol{\Gamma}_{\text{out}}(0,0,\mathbf{B}_0)+\boldsymbol{\Gamma}_{\text{out}}(0,0,-\mathbf{B}_0)\right]\cdot\mathbf{p} \tag{48}$$

which is clearly an even function of the biasing field $\mathbf{B}_0$ as it should be.

If we had used expression (2), as one would do in the conventional method, we would have obtained

$$P=\frac{\omega^3\mu_0}{2}\,\mathbf{p}\cdot\operatorname{Im}\boldsymbol{\Gamma}_{\text{out}}(0,0,\mathbf{B}_0)\cdot\mathbf{p}. \tag{49}$$

Evidently expressions (48) and (49) are different. However, in the extreme cases where $\mathbf{B}_0=0$ (isotropic media) or where $\mathbf{B}_0=\infty$ (uniaxial crystals), the conventional expression (49) becomes identically equal to our expression (48). Hence, when $\mathbf{B}_0=\infty$ or when $\mathbf{B}_0=0$, it is permissible to use expression (49) for the calculation of the radiation resistance [Kogelnik, 1960; Kuehl, 1962], but otherwise it is not. When $\mathbf{B}_0$ has a finite value, expression (49) cannot be used for this purpose because it does not represent irreversible power. When $\mathbf{B}_0$ is finite one must use expression (48).

7. Conclusions

From the above discussion we can draw the following conclusions. For an antenna in a gyroelectric medium, the time average power P emitted by the antenna has an irreversible part P_{irr} and a reversible part P_{rev}. The irreversible part of the power does not change sign under the time-reversal transformation and hence represents the time-average real power that is absorbed by the sphere at infinity; the reversible part of the power, on the other hand, changes sign under the time-reversal transformation and accordingly has the nature of reactive power. When the biasing field $\mathbf{B}_0$ is zero or infinite P_{rev} is identically zero, and there is no difference between P and P_{irr}. However, for finite values of $\mathbf{B}_0$ there is a profound difference. This means that for finite values of $\mathbf{B}_0$ the radiation resistance R, which is a measure of the power absorbed by the sphere at infinity, must involve only the irreversible part of the power.

Although the resulting electromagnetic field of an antenna in a gyroelectric medium consists of only the outwardly moving wave, it is expedient to introduce the inwardly moving wave because the difference of the outgoing and incoming waves yields P_{irr} and their sum yields P_{rev}. Moreover, consideration of the difference and sum fields shows that P_{irr} is free from singularity, whereas P_{rev} is not.

By means of the dyadic Green's function the explicit representations of P_{irr} and P_{rev} take a particularly simple form. Indeed, they require only a knowledge of two dyadic Green's functions, one of which can be obtained from the other by reversing the sign of the biasing field.

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8. References

Dirac, P. A. M. (1938), Classical theory of radiating electrons, Proc Roy. Soc. (London) Ser. A **167**, 148–169.

Kogelnik, H. (1960), The radiation resistance of an elementary dipole in anisotropic plasmas, Proc. Fourth Intern. Conf. on Ionization Phenomena in Gases (North-Holland Publishing Co., Amsterdam).

Kuehl, H. H. (1962), Electromagnetic radiation from an electric dipole in a cold anisotropic plasma, Phys. Fluids **5**, No. 9, 1095–1103.

Schwinger, J. (1949), On the classical radiation of accelerated electrons, Phys. Rev. **75**, No. 12, 1912–1925.

Staras, H. (1964), The impedance of an electric dipole in a magnetoionic medium, IEEE Trans. Ant. Prop. **AP–12**, No. 6, 695–702.

Weil, H., and D. Walsh (1965), Radiation resistance of an elementary loop antenna in a magnetoionic medium, IEEE Trans. Ant. Prop. **AP–13**, No. 1, 21–27.

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