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Steady-State Heat Conduction in an Exposed Exterior Column of Rectangular Cross Section

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This paper presents a mathematical analysis of two-dimensional steady-state heat conduction in a solid of rectangular cross section, two parallel surfaces of which are exposed to separate temperature environments with respective surface heat transfer coefficients. For the other two parallel surfaces, the temperature environment and surface heat transfer coefficients are assumed to vary as a function of position on these surfaces. A particular application of this analysis has been to determine the temperature distribution in a transverse cross section of exposed concrete columns.

1. Introduction

Modern architectural treatment of exterior columns of high-rise buildings involves the use of columns exposed to temperature extremes on the outside face and relatively constant temperatures on the interior. The temperature extremes give rise to dimensional changes in the column which can best be determined from temperature distribution in the column. In connection with a request from the Federal Housing Administration, an investigation was made for determining the temperature distributions in an exposed exterior column.

This paper presents a mathematical analysis of two-dimensional steady-state heat conduction in an exposed exterior column which may be represented by a solid of rectangular cross section, two parallel surfaces of which are exposed to separate temperature environments with respective surface heat transfer coefficients. For the other two parallel surfaces, the temperature environment and surface heat transfer coefficient are assumed to vary as a function of transverse position on these surfaces.

An analysis for general application is presented, along with the statement and numerical results for four specific problems.

2. Analysis for General Problem

Because the heat-transfer problem in a rectangular solid may have a more general application than that cited in the previous section, the following analysis is based on boundary conditions, arbitrary in nature, which may be adapted to specific applications. Section 3 presents the assumed boundary conditions for heat transfer in an exposed exterior column.

For steady-state, two-dimensional heat conduction in a rectangular homogeneous solid with no flow of heat in the direction of the third dimension, it is assumed that the rate of heat flow at each of the four boundary surfaces is proportional to the temperature difference between the temperature of the surface and that of the ambient adjacent to the surface. The proportionality factor may be prescribed as a function of the distance along the surface. A convenient placement of coordinate axes for the type of problem to be considered is as shown in figure 1, where the boundary conditions at the four surfaces, $x=\pm l$, and $y=\pm a$, are shown adjacent to the surfaces.

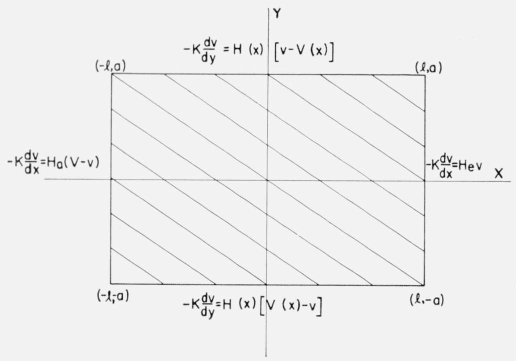

FIGURE 1. *Transverse section of rectangular solid, showing boundary conditions on* x=±1 *and* y=±a.

The temperature potentials V and $V(x)$ are the temperature differences between the temperatures of the ambient adjacent to various surface positions and that of the ambient adjacent to $x=l$; the temperature potential v is the temperature difference between a temperature in the solid and that of the ambient adjacent to $x=l$. The proportionality constants H_a and H_e represent the coefficients of heat transfer between the surface and the ambient at the surfaces $x=-l$, and $x=l$, respectively. For the purposes of this paper the variable heat transfer coefficient $H(x)$ and temperature potential $V(x)$ are considered to be piece-wise continuous on $-l\leq x\leq l$, where $H(x)=H_i(x)$ and $V(x)=V_i(x)$ are continuous on $x_{i-1}\leq x\leq x_i$ and $i=1, 2, 3, \ldots m$; $x_0=-l$, and $x_m=l$.

The partial differential equation for steady-state heat conduction in a homogeneous solid in a rectangular coordinate system is

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0. \tag{1}$$

A solution of (1) satisfying the boundary conditions at $x=l$ and $x=-l$ (fig. 1) is

$$\frac{v}{V}=\frac{h_1+h_1h_2(1-x/l)}{h_1+h_2+2h_1h_2}+\sum_{n=1}\frac{A_nR(\beta_nx/l)\cosh\beta_ny/l}{\cosh\beta_na/l} \tag{2}$$

where due to the similar conditions on $y=\pm a$, it is assumed that $\frac{dv}{dy}=0$ on the center line $y=0$. Symbols used in (2) are:

K=thermal conductivity of the solid
$h_1=H_al/K$
$h_2=H_el/K$
A_n=coefficients to be determined from boundary conditions on $y=a$
$R(\beta_nx/l)=\beta_n\cos\beta_n(1+x/l)+h_1\sin\beta_n(1+x/l)$

where β_n are positive roots of $(\beta^2-h_1h_2)\sin 2\beta-\beta(h_1+h_2)\cos 2\beta=0$.

Substitution of boundary conditions on $y=a$ gives m equations

$$\sum_{n=1}^{\infty}A_nR(\beta_nx/l)\left[\beta_n\tanh\beta_na/l+\frac{lH(x)}{K}\right]=\frac{lH(x)}{K}\left[\frac{V(x)}{V}-\frac{h_1+h_1h_2(1-x/l)}{h_1+h_2+2h_1h_2}\right]. \tag{3}$$

Multiplying each equation by $R(\beta_kx/l)$ and integrating from $x=-l$ to $x=l$, yields

$$\sum_{n=1}^{\infty}\left[\beta_nA_n\tanh\beta_na/l\int_{-l}^{l}R(\beta_nx/l)R(\beta_kx/l)dx+\frac{l}{K}\sum_{i=1}^{m}A_n\int_{x_{i-1}}^{x_i}H(x)R(\beta_nx/l)R(\beta_kx/l)dx\right]$$

$$=\frac{l}{K}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}H(x)R(\beta_kx/l)\times\left[\frac{V(x)}{V}-\frac{h_1+h_2h_1(1-x/l)}{h_1+h_2+2h_1h_2}\right]dx. \tag{4}$$

Using the orthogonal and other related properties of $R(\beta_n x/l)$ as derived in the appendix, (4) may be put in the form of simultaneous equations

$$B_n A_n + \sum_{\substack{k=1\\k\neq n}}^{\infty} C_{n,k} A_k = D_n, \qquad n=1, 2, \ldots \tag{5}$$

where

$$B_n = \beta_n \tanh \beta_n a/l \int_{-l}^{l} R^2(\beta_n x/l)dx + \frac{l}{K}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i} H(x)R^2(\beta_n x/l)dx$$

$$C_{n,k} = \frac{l}{K}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i} H(x)R(\beta_n x/l)R(\beta_k x/l)dx$$

$$D_n = \frac{l}{K}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}\left[\frac{V(x)}{V} - \frac{h_1 + h_1 h_2(1-x/l)}{h_1+h_2+2h_1h_2}\right]\times H(x)R(\beta_n x/l)dx.$$

Of particular interest is the mean temperature in the solid, v_m, given by

$$\frac{v_m}{V} = \frac{1}{2al}\int_{-l}^{l}\int_{0}^{a}\frac{v}{V}dydx$$

$$\frac{v_m}{V} = \frac{h_1(1+h_2)}{h_1+h_2+2h_1h_2} + \frac{l}{2a}\sum_{n}\frac{A_n \tanh \beta_n a/l[\beta_n \sin 2\beta_n + h_1(1-\cos 2\beta_n)]}{\beta_n^2}. \tag{6}$$

3. Temperature Distribution in Exposed Concrete Columns

An interesting specific case for solution of (2) is to consider a transverse section of a rectangular concrete column. Here a portion of the boundary surface is exposed to outside weather conditions or design temperature, another portion of the surface is exposed to the ambient temperature maintained within the building, and another portion of the surface is in contact with the walls of the building. A cross section of the column, and an abutting wall in the region $b<x<d$, is shown in the lower portion of figure 2. The $x=l$ face of the column is assumed to be

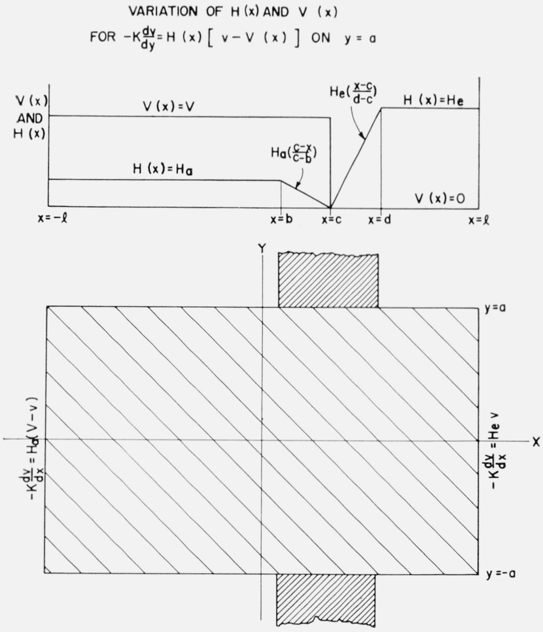

FIGURE 2. *Transverse section of rectangular concrete column with abutting wall separating the inside and outside environment.*
Outside face is located at $x=l$.

the outside face exposed to the outside design temperature which is assumed to be the zero or datum plane temperature from which all other temperature potentials are reckoned; the face $x=-l$ is assumed to be exposed to the indoor temperature potential V.

The upper portion of figure 2 shows the assumed variation of $H(x)$ and $V(x)$ along the surfaces $y=\pm a$. $H(x)$ is assumed constant at H_a in the region $-l\leq x\leq b$; decreases linearly from H_a to zero in going from b to c; increases linearly from zero to H_e in going from c to d; and is constant at H_e in the region $d\leq x\leq l$. $V(x)$ is assumed constant at V from $-l$ to c; and is zero from c to l. In the region of the abutting wall, $b<x<d$, the heat transfer between the wall and the column was assumed to be dependent on the inside and outside temperatures in their region of influence and the surface coefficients decreasing linearly to zero at $x=c$.

$$-K\frac{dv}{dy}=H_a\left(\frac{c-x}{c-b}\right)(v-V)\qquad b<x<c,\, y=a$$

$$-K\frac{dv}{dy}=H_e\left(\frac{x-c}{d-c}\right)v\qquad c<x<d,\, y=a.$$

Expressions for D_n, B_n, and $C_{n,k}$ necessary for substitution in (5) are given in the appendix (A13, A14, A15). The solution of (5) for A_n and its substitution in (2) gives the temperature distribution for values of x and y.

Numerical solutions were obtained, using a digital computer, for a 36 in.×14 in. concrete column for four positions of the abutting wall, based on a 100 deg F temperature difference between the inside and outside. Numerical values used, and assumed as constants, were

Thermal conductivity, $K=1.0$ Btu/hr ft deg F
Inside heat transfer coefficient, $H_a=0.5$ Btu/hr ft^2 deg F
Outside heat transfer coefficient, $H_e=6.0$ Btu/hr ft^2 deg F
Width of column, $2a=14$ in.$=7/6$ ft
Thickness of column, $2l=36$ in.$=3$ ft
Thickness of wall, $d-b=8$ in.$=2/3$ ft
Also it was assumed that $d-c=c-b$.

The inside surfaces of the column were assumed to be separated from the inside of the building by suitable interior finish, giving a nominal value for H_a of 0.5 Btu/hr ft^2 deg F. The value $H_e=6.0$ Btu/hr ft^2 deg F was based on an outside wind velocity of 15 mph.

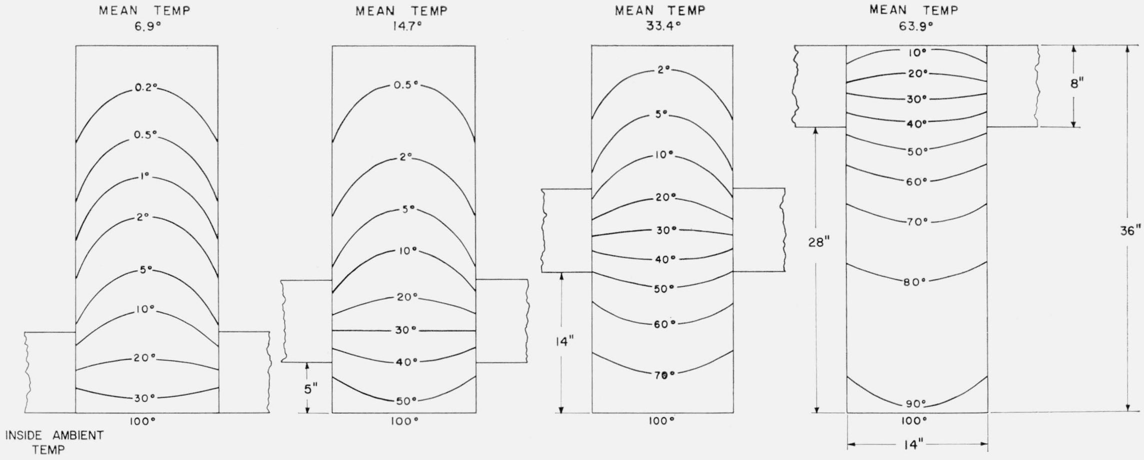

FIGURE 3. *Temperature potential distributions in 36-in. by 14-in. concrete columns for four positions of the abutting walls, based on a 100-deg temperature difference between the inside and outside and the constants cited in the text.*

Figure 3 shows temperature potential isotherms plotted within the column cross section for four positions of the abutting wall. Mean temperatures were computed from (6). Figure 4 shows mean temperatures over the width of the columns plotted against distance measured from the inside surface of the column. Positions of the abutting wall are shown for clarity.

Temperatures in degrees Fahrenheit can be expressed in terms of the temperature potentials, v, of figure 3 by means of the relation

$$t,\ ^{\circ}\mathrm{F}=t_e+(v/100)(t_a-t_e)$$

where t_a is the inside air temperature, and t_e is the outside air temperature, in degrees Fahrenheit.

4. Discussion

The analysis of section 2 assumes similar boundary conditions on the surfaces, $y=\pm a$. For the consideration of a more general case where these conditions are not similar, it is convenient to make a translation of the x–axis of figure 1 to the lower side and rewrite the summation term in (2)

$$\frac{\left[A_n \cosh \beta_n y/l+E_n \cosh \beta_n \left(\frac{2a-y}{l}\right)\right] R(\beta_n x/l)}{\cosh \beta_n 2a/l}.$$

Substitution of boundary conditions at $y=0$ and $y=2a$ will give two sets of simultaneous equations similar to (5), both containing the coefficients A_n and E_n. Solution for A_n and E_n may be obtained by iteration; for example, let $E_n=0$ in one set and solve for A_n, substitute A_n in the second set and solve for E_n, which is substituted in the first set. This process may be continued until A_n and E_n are invariant.

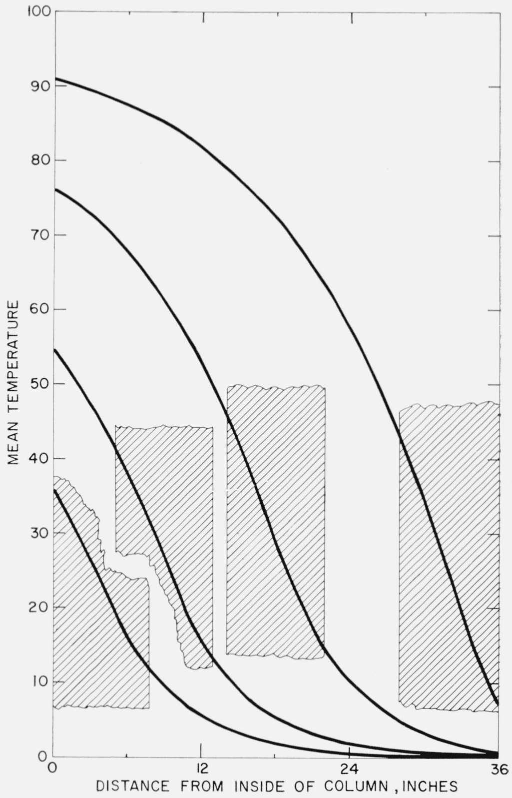

FIGURE 4. *Mean temperatures over the width of the columns against distance measured from the inside surface of the columns.*

Positions of the abutting walls are shown.

The assumption was made that a sufficiently good approximation to the solution of the infinite set of equations (5) in the infinitely many unknowns A_n could be obtained by solving the first forty equations for A_n, $n=1, 2, \ldots, 40$. Solution of the first twenty equations for A_n, $n=1, 2\ldots, 20$ gave results which differed from that for forty A_n by less than 0.1 °F. It therefore appeared that convergence of the approximating sequence of A_n's to the true solution of (5) was sufficiently rapid to justify our assumption.

A computer program in Fortran IV language is available for the problem as set forth in section 3. Data input to this program allows any possible variation in the numerical values for the thermal conductivity, surface coefficients, and dimensions.

4. Appendix. Orthogonal and Other Related Properties of $R(\beta_n x/l)$

To satisfy the boundary condition in (2) at $x=l$, β_n must be the roots of

$$(\beta^2-h_1h_2)\sin 2\beta-\beta(h_1+h_2)\cos 2\beta=0. \tag{A1}$$

Without proof in this paper, the roots β_n will be assumed real and positive. Proof of the real nature of these roots is similar to that for roots of an expression given in Carslaw and Jaeger.[1] It then becomes expedient to develop the orthogonal properties of

$$R(\beta_n x/l)=\beta_n\cos\beta_n(1+x/l)+h_1\sin\beta_n(1+x/l) \tag{A2}$$

where $-l<x<l$. Several expressions useful for later developments are cited below:

$$\sin^2 2\beta=\frac{\beta^2(h_1+h_2)^2}{(\beta^2+h_1^2)(\beta^2+h_2^2)} \tag{A3}$$

$$\cos^2 2\beta=\frac{(\beta^2-h_1h_2)^2}{(\beta^2+h_1^2)(\beta^2+h_2^2)} \tag{A4}$$

$$R^2(\beta x/l)=\frac{\beta^2+h_1^2}{2}+\frac{\beta^2-h_1^2}{2}\cos 2\beta(1+x/l)+h_1\beta\sin 2\beta(1+x/l). \tag{A5}$$

$$R(\beta_n x/l)R(\beta_k x/l)=\frac{L_1(x)+L_2(x)}{2} \tag{A6}$$

$$L_1(x)=(\beta_n\beta_k-h_1^2)\cos\phi(1+x/l)+h_1\phi\sin\phi(1+x/l)$$

$$L_2(x)=(\beta_n\beta_k+h_1^2)\cos\psi(1+x/l)-h_1\psi\sin\psi(1+x/l)$$

$$\phi=\beta_n+\beta_k, \qquad \psi=\beta_n-\beta_k$$

$$M_0(x)=\int R(\beta_n x/l)dx=\frac{l}{\beta_n}[\beta_n\sin\beta_n(1+x/l)-h_1\cos\beta_n(1+x/l)] \tag{A7}$$

$$N_0(x)=\int R^2(\beta_n x/l)dx=\frac{x(\beta_n^2+h_1^2)}{2}+\frac{l(\beta_n^2-h_1^2)}{4\beta n}\sin 2\beta_n(1+x/l)-\frac{h_1 l}{2}\cos 2\beta_n(1+x/l) \tag{A8}$$

$$P_0(x)=\int L_1(x)dx=\frac{l}{\phi}[(\beta_n\beta_k-h_1^2)\sin\phi(1+x/l)-h_1\phi\cos\phi(1+x/l)] \tag{A9}$$

$$Q_0(x)=\int L_2(x)dx=\frac{l}{\psi}[(\beta_n\beta_k+h_1^2)\sin\psi(1+x/l)+h_1\psi\cos\psi(1+x/l)]. \tag{A10}$$

[1] Carslaw, H. S., and Jaeger, J. C., Conduction of Heat in Solids, 2d ed., p. 114 (Oxford University Press, London, 1959).

By use of (A3) and (A4), it can be shown that $P_0(l)+Q_0(l)=P_0(-l)+Q_0(-l)=0$; i.e., the integral $\int_{-l}^{l} R(\beta_n x/l)R(\beta_k x/l)dx=0$, showing that $R(\beta_n x/l)$ and $R(\beta_k x/l)$ are orthogonal to each other in the interval $-l<x<l$. Also, using (A3) and (A4),

$$N_0(l)=\frac{l}{2}\frac{(\beta_n^2+h_1^2)(\beta_n^2+h_2^2+h_2)}{(\beta_n^2+h_2^2)}$$

$$-N_0(-l)=\frac{l}{2}(\beta_n^2+h_1^2+h_1).$$

Assuming $H_i(x)$ and $V_i(x)$ to be polynomials in x, then the recurrence

$$M_p(x)=\int x^p R(\beta x/l)dx=xM_{p-1}(x)-\int M_{p-1}(x)dx \quad \text{(A11)}$$

and similar relationships for $N_p(x)$, $P_p(x)$ and $Q_p(x)$ may be established for determining B_n, D_n, and $C_{n,k}$ in (5). Assuming a resulting polynomial of the form $S+Tx+Ux^2$ in the solution for D_n, the indefinite integral becomes

$$\int(S+Tx+Ux^2)R(\beta_n x/l)dx=\left(S+Tx+Ux^2-\frac{2l^2U}{\beta_n^2}\right)M_0(x)+\frac{l^2}{\beta_n^2}(T+2Ux)R(\beta_n x/l). \quad \text{(A12)}$$

The solution for D_n for the specific case illustrated in figure 2 takes the form

$$D_n=\frac{h_1h_2}{h_1+h_2+2h_1h_2}\sum_{i=1}^{4}\int_{x_{i-1}}^{x_i}(S_i+T_ix+U_ix^2)R(\beta_n x/l)dx$$

where

$S_1=1+h_1$	$T_1=h_1/l$	$U_1=0$
$S_2=cS_1/(c-b)$	$T_2=(cT_1-S_1)/(c-b)$	$U_2=-T_1/(c-b)$
$S_3=-cS_4/(d-c)$	$T_3=(S_4-cT_4)/(d-c)$	$U_3=T_4/(d-c)$
$S_4=-(1+h_2)$	$T_4=h_2/l$	$U_4=0.$

The function in the region $-l<x<l$ is a linear, piecewise continuous function for which the integrals go to zero at the integration limits $x=-l$ and $x=l$, and elsewhere at the other limits only the coefficients of l^2/β_n^2 in (A12) need be considered,

$$D_n=\frac{lh_1h_2}{\beta_n^2(h_1+h_2+2h_1h_2)}[2\{U_2M_0(b)+(U_3-U_2)M_0(c)-U_3M_0(d)\}$$
$$+(T_1+T_2-2U_2b)R(\beta_n b/l)-(T_4-T_3-2U_3d)R(\beta_n d/l)$$
$$+\{T_3-T_2+2c(U_2-U_3)\}R(\beta_n c/l)]. \quad \text{(A13)}$$

By a similar analysis

$$\frac{B_n}{l}=\frac{(\beta_n\tanh\beta_n a/l+h_2)(\beta_n^2+h_1^2)(\beta_n^2+h_2^2+h_2)+(\beta_n\tanh\beta_n a/l+h_1)(\beta_n+h_2^2)(\beta_n+h_1^2+h_1)}{2(\beta_n^2+h_2^2)}$$
$$+\frac{1}{4\beta_n^2}\left[\frac{h_2l}{(d-c)}\{R^2(\beta_n d/l)-R^2(\beta_n c/l)\}-\frac{h_1l}{(c-b)}\{R^2(\beta_n c/l)-R^2(\beta_n b/l)\}\right]$$
$$+\frac{(\beta_n^2+h_1^2)}{4}\left[\frac{h_1(c+b)}{l}-\frac{h_2(d+c)}{l}\right] \quad \text{(A14)}$$

$$\frac{C_{n,k}}{l}=\frac{h_2l}{2(d-c)}\left[\frac{L_1(d)-L_1(c)}{\phi^2}+\frac{L_2(d)-L_2(c)}{\psi^2}\right]+\frac{h_1l}{2(c-b)}\left[\frac{L_1(c)-L_1(b)}{\phi^2}+\frac{L_2(c)-L_2(b)}{\psi^2}\right]. \quad \text{(A15)}$$

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