JOURNAL OF RESEARCH of the National Bureau of Standards—C. Engineering and Instrumentation
Vol. 69C, No. 2, April–June 1965

Common Volume of Two Intersecting Cylinders

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(October 29, 1964)

The volume common to two cylinders of radii $r_1 \leq r_2$ with axes intersecting at angle β is found to be $r_2^3 v(k)/\sin\beta$, where $k = r_1/r_2$ and $v(k)$ may be evaluated (1) as the hypergeometric series

$$2\pi k^2 {}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}; 2; k^2) = 4\pi \sum_1^\infty \binom{\frac{1}{2}}{n}\binom{\frac{1}{2}}{n-1} k^{2n},$$

(2) as the combination of complete elliptic integrals $(8/3)[(1+k^2)E(k)-(1-k^2)K(k)]$ or (3) as the cumulative integral

$$8\int_0^k kE(k)dk.$$

A table of $v(k)$ to 8 decimals over the range $0 \leq k(0.01) \leq 1.00$, including δ_m^2 modified second central differences, is presented. This volume integral was useful in interpreting a gamma-ray albedo experiment involving a collimated source and a collimated detector, and may also be applicable to crossed-beam experiments. Two series useful for k close to unity are provided, one of which involves differencing against the series

$$16/3 = 4\pi \sum_1^\infty \binom{\frac{1}{2}}{n}\binom{\frac{1}{2}}{n-1}.$$

1. Introduction

In crossed-beam experiments [1] [1] using the high-intensity accelerators now becoming available, the "geometrical target," or volume common to the two colliding beams, is a useful parameter for interpreting the measured data. An evaluation of this volume in terms of an infinite series was recently exhumed for possible application to an x-ray free-air ionization chamber having a cylindrical sensitive volume intersected by a pencil of x rays [2]. This evaluation had been used in the analysis of a gamma-ray beam back-scattering experiment [3] for making a theoretical estimate of the single-scattered component of the radiation "seen" by a collimated detector.

Evaluations of the volume common to two circular cylinders of unequal radii with axes intersecting at right angles [4, 5, 6], and of equal radii with axes intersecting at an arbitrary angle [7], have frequently been offered as calculus textbook exercises. However, a combined treatment does not seem to appear in the technical literature in a form convenient for easy application to practical problems. The following results provide formulas, a table, and a graph for such applications.

The series used in [3] is here corrected, expressed in terms of binomial coefficients, and identified as a hypergeometric series. For nearly equal cylinder radii, convergence can be accelerated by use of the difference-series technique [8]. An alternative series for this region, derived from the right-angle elliptic-integral solution [4, 5, 6], is more complicated but also more rapidly convergent.

[1] Figures in brackets indicate the literature references at the end of this paper.

2. Volume Integral

The integral for the common volume of two cylinders of radii $r_1 \leq r_2$ with axes intersecting at angle β (see fig. 1) is found as follows. The cross section parallel to the cylinder axes, at a distance x from them, is a parallelogram of height $2(r_2^2-x^2)^{1/2}$ and base $2(r_1^2-x^2)^{1/2}/\sin\beta$. Hence the volume integral is

$$V(r_1, r_2, \beta) = \int_{-r_1}^{r_1} 2(r_2^2-x^2)^{1/2} \cdot \frac{2(r_1^2-x^2)^{1/2}}{\sin\beta} dx \qquad \text{(1a)}$$

$$= \frac{8}{\sin\beta}\int_0^{r_1} (r_2^2-x^2)^{1/2} \cdot (r_1^2-x^2)^{1/2} dx. \qquad \text{(1b)}$$

3. Common Volume When $r_1 = r_2$

For equal cylinder radii $r_1 = r_2 = r$, the integral in (1b) reduces to the familiar result [7]

$$V(r, \beta) = \frac{8}{\sin\beta}\int_0^r (r^2-x^2)dx \qquad \text{(2)}$$

$$= \frac{16r^3}{3\sin\beta}. \qquad \text{(3)}$$

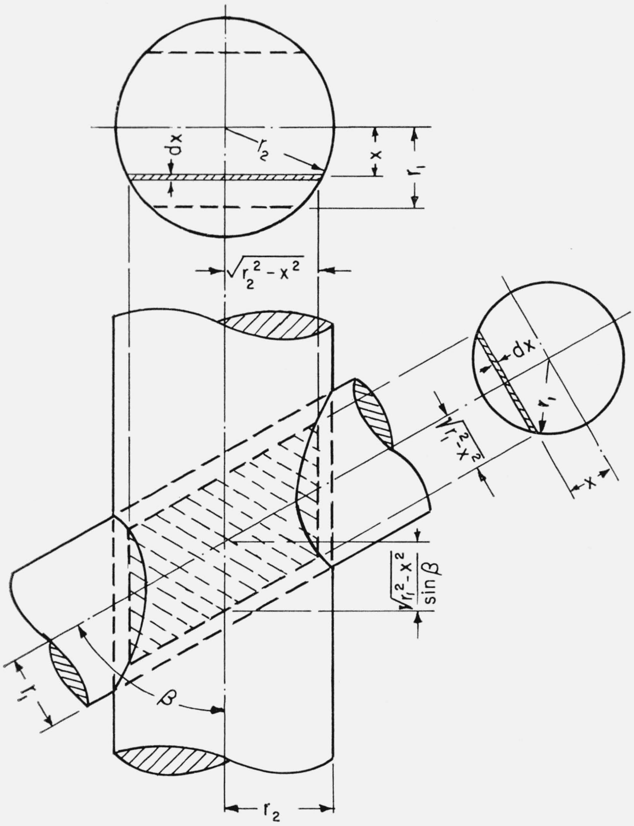

FIGURE 1. *Three-view sketch of the common volume of cylinders with radii* r_1 *and* r_2 *axially intersecting at angle* β.

The area of the shaded parallelogram parallel to the plane of the axes in the lower left view comprises the integrand in eq (1a) and is integrated over the range $-r_1 \leq x \leq r_1$ shown in the other two views.

4. Common Volume When $r_1 \leq r_2$

4.1. Series Solution

The factor $(r_2^2 - x^2)^{1/2}$ in the integral in (1b) may be expanded as a power series [9, p. 2, eq 5.3] in x/r_2, since $x \leq r_1 \leq r_2$. The volume integral then becomes

$$V(r_1, r_2, \beta) = \frac{8r_2}{\sin\beta} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \int_0^{r_1} \left(-\frac{x^2}{r_2^2}\right)^n (r_1^2 - x^2)^{1/2} dx \quad (4)$$

which may be integrated term by term.

The resulting series solution is

$$V(r_1, r_2, \beta) = \frac{4\pi r_2^3}{\sin\beta} \left\{ \frac{1}{2}\left(\frac{r_1}{r_2}\right)^2 - \frac{1}{4}\left(\frac{1}{2}\right)^2 \left(\frac{r_1}{r_2}\right)^4 - \frac{3}{6}\left(\frac{1}{2 \cdot 4}\right)^2 \left(\frac{r_1}{r_2}\right)^6 - \frac{5}{8}\left(\frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2 \left(\frac{r_1}{r_2}\right)^8 - \frac{7}{10}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}\right)^2 \left(\frac{r_1}{r_2}\right)^{10} - \cdots \right\}$$

$$= \frac{4\pi r_2^3}{\sin\beta} \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \binom{\frac{1}{2}}{n-1} \left(\frac{r_1}{r_2}\right)^{2n}. \quad (5)$$

This same result may be obtained by casting the integral in (1b) in the form

$$V(r_1, r_2, \beta) = (r_2^3/\sin\beta) 4 \left(\frac{r_1}{r_2}\right)^2 \times \int_0^1 t^{-1/2}(1-t)^{1/2} \left(1 - \frac{r_1^2}{r_2^2} t\right)^{1/2} dt, \quad (6)$$

where $t = x^2/r_1^2$, which is now recognizable as the integral representation of the hypergeometric series [10]

$$V(r_1, r_2, \beta) = (r_2^3/\sin\beta) 2\pi \left(\frac{r_1}{r_2}\right)^2 {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 2; \frac{r_1^2}{r_2^2}\right) \quad (7)$$

identifiable with (5).

Since (5) is somewhat slowly convergent when $r_1 \simeq r_2$, under some circumstances it may be advantageous to difference this series against a $1/\pi$-series (16) discussed in the appendix, giving

$$V(r_1, r_2, \beta) = \frac{4\pi r_2^3}{\sin\beta} \left\{ \frac{4}{3\pi} - \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \binom{\frac{1}{2}}{n-1} \left[1 - \left(\frac{r_1}{r_2}\right)^{2n}\right] \right\}. \quad (8)$$

The convergence rate of the series-term in (8) is not improved over that of (5). However, for $r_1 \simeq r_2$ this sum is small compared to the constant term $4/3\pi$, identifiable with the equal-radii solution (3), hence resulting in higher precision of $V(r_1, r_2, \beta)$ for the same number of terms. An alternative series solution for this region is given at the end of the following section.

4.2. Elliptic Integral Solution

An alternative solution of the integral in (1b) may be obtained as a combination of complete elliptic integrals [11] of the first and second kinds, $K(k)$ and $E(k)$. Applying formulas (219.11) and (361.03) from Byrd and Friedman [12] the result is found to be[2]

$$V(r_1, r_2, \beta) = \frac{8r_2^3}{3\sin\beta} [(1+k^2)E(k) - (1-k^2)K(k)] \quad (9)$$

where

$$k = r_1/r_2.$$

Except for the angle factor $1/\sin\beta$ this result is the standard textbook solution [5, 6] for cylinders intersecting at right angles. Also, the formulation in (9) is related to the indefinite integral [12, eq (611.01)]

$$\int_0^k kE(k)dk = \frac{1}{3}[(1+k^2)E(k) - (1-k^2)K(k)]. \quad (10)$$

[2] This integral is part of the "*G* factor" used for interpreting gas scattering experiments in which a circular-aperture detector views a gas target transversed by a cylindrical beam. In this context this elliptic integral solution has been given by E. A. Silverstein, Nucl. Instr. and Meth. **4**, 53 (1959) and by D. F. Herring and K. W. Jones, Nucl. Instr. and Meth. **30**, 88 (1964).

A partial check on (9) is obtained by expanding $E(k)$ and $K(k)$ as power series in k according to reference [9], eqs (773.1) and (774.1). Combining like powers of k and substituting back r_1/r_2 for k the resulting series is identical with (5). An additional check is provided by the identity of eqs (9) and (3) in the limit as $k\rightarrow 1$. Also, eq (9) can be obtained from (7) by use of the tenth Gauss recursion formula on page 9 of reference [10].

For k close to unity, a series which converges more rapidly than (8) may now be derived by substituting in (9) the series in equations (773.3) and (774.3) in reference [9] for $K(k)$ and $E(k)$. The first few terms of this series are

$$
\begin{aligned}
V(r_1, r_2, \beta) = (r_2^3/\sin\beta)\cdot\frac{16}{3}\Bigg\{ & 1-\frac{3}{4}k'^2 \\
& -\frac{1}{2^2\cdot 4}\left(\ln\frac{4}{k'}-\frac{1}{4}\right)k'^4-2\cdot\frac{1^2\cdot 3}{2^2\cdot 4^2\cdot 6}\left(\ln\frac{4}{k'}-\frac{11}{12}\right)k'^6 \\
& -3\cdot\frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6^2\cdot 8}\left(\ln\frac{4}{k'}-\frac{43}{40}\right)k'^8 \\
& -4\cdot\frac{1^2\cdot 3^2\cdot 5^2\cdot 7}{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot 10}\left(\ln\frac{4}{k'}-\frac{967}{840}\right)k'^{10}-\ldots\Bigg\}
\end{aligned}
\tag{11}
$$

where $k'^2=1-k^2=1-(r_1/r_2)^2$. Using only the terms given in (11) the sum for $r_1/r_2=0.90$, without the factor $(r_3^3/\sin\beta)$, gives 4.49991482 as compared with the exact value of 4.49991288 . . ., and the convergence improves as r_1/r_2 goes toward unity. The series in (11) may be obtained in general form, if desired, by use of the appropriate transformation [13] on the hypergeometric series given in (7).

5. Numerical Results

In table 1, the dimensionless factor

$$v(k)=\frac{\sin\beta}{r_2^3}V(r_1, r_2, \beta), \tag{12a}$$

$$=4\pi\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}\binom{\frac{1}{2}}{n-1}k^{2n}, \tag{12b}$$

$$=2\pi k^2\,{}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}; 2; k^2), \tag{12c}$$

$$=8\int_0^k kE(k)dk, \tag{12d}$$

$$=\tfrac{8}{3}[(1+k^2)E(k)-(1-k^2)K(k)], \tag{12e}$$

where $k=r_1/r_2$ is tabulated to 8 decimal places for $0\leq k(0.01)\leq 1.00$, computed using (12e) and $K(k)$ and $E(k)$ from [11]. Hence, for many practical applications, the common volume of two cylinders with radii $r_1\leq r_2$ and axes intersecting at angle β may be computed as

$$V(r_1, r_2, \beta)=\frac{r_2^3}{\sin\beta}v(k) \tag{13}$$

in which values of $v(k)$ are interpolated from table 1.

TABLE 1. *Values of* v(k), *defined in eq (12a–e), over the range* $0\leq$ k*(0.01)* $\leq$ *1.00, valid to the 8D given*

Modified second central differences δ_m^2 are provided for interpolation using auxiliary tables.

k	$v(k)$	δ_m^2	k	$v(k)$	δ_m^2
0.00	0.0000 0000	+12 5665			
.01	0.0006 2831	12 5656	0.51	1.5792 3625	+9 8789
.02	0.0025 1315	12 5627	.52	1.6394 2033	9 7607
.03	0.0056 5423	12 5580	.53	1.7005 8043	9 6391
.04	0.0100 5109	12 5514	.54	1.7627 0439	9 5142
.05	0.0157 0305	12 5429	.55	1.8257 7969	9 3854
.06	0.0226 0928	12 5325	.56	1.8897 9346	9 2530
.07	0.0307 6874	12 5203	.57	1.9547 3247	9 1167
.08	0.0401 8019	12 5061	.58	2.0205 8308	8 9766
.09	0.0508 4222	12 4900	.59	2.0873 3128	8 8323
.10	0.0627 5321	12 4720	.60	2.1549 6262	8 6837
.11	0.0759 1138	12 4520	.61	2.2234 6226	8 5308
.12	0.0903 1471	12 4302	.62	2.2928 1490	8 3734
.13	0.1059 6104	12 4065	.63	2.3630 0478	8 2112
.14	0.1228 4797	12 3807	.64	2.4340 1570	8 0441
.15	0.1409 7294	12 3530	.65	2.5058 3093	7 8719
.16	0.1603 3316	12 3234	.66	2.5784 3325	7 6943
.17	0.1809 2569	12 2917	.67	2.6518 0490	7 5113
.18	0.2027 4735	12 2581	.68	2.7259 2757	7 3222
.19	0.2257 9478	12 2224	.69	2.8007 8235	7 1272
.20	0.2500 6441	12 1848	.70	2.8763 4974	6 9258
.21	0.2755 5248	12 1451	.71	2.9526 0957	6 7174
.22	0.3022 5502	12 1034	.72	3.0295 4102	6 5020
.23	0.3301 6786	12 0595	.73	3.1071 2253	6 2791
.24	0.3592 8660	12 0136	.74	3.1853 3179	6 0481
.25	0.3896 0667	11 9656	.75	3.2641 4571	5 8086
.26	0.4211 2326	11 9154	.76	3.3435 4031	5 5600
.27	0.4538 3135	11 8632	.77	3.4234 9073	5 3016
.28	0.4877 2572	11 8087	.78	3.5039 7111	5 0328
.29	0.5228 0092	11 7520	.79	3.5849 5457	4 7527
.30	0.5590 5129	11 6932	.80	3.6664 1308	4 4605
.31	0.5964 7093	11 6320	.81	3.7483 1740	4 1550
.32	0.6350 5373	11 5686	.82	3.8306 3696	3 8350
.33	0.6747 9336	11 5029	.83	3.9133 3972	3 4992
.34	0.7156 8323	11 4348	.84	3.9963 9206	3 1456
.35	0.7577 1654	11 3644	.85	4.0797 5861	2 7725
.36	0.8008 8626	11 2915	.86	4.1634 0199	2 3774
.37	0.8451 8508	11 2163	.87	4.2472 8265	1 9574
.38	0.8906 0549	11 1385	.88	4.3313 5853	1 5089
.39	0.9371 3970	11 0582	.89	4.4155 8469	1 0276
.40	0.9847 7969	10 9753	.90	4.4999 1288	+ 5075
.41	1.0335 1716	10 8899	.91	4.5842 9096	− 586
.42	1.0833 4356	10 8017	.92	4.6686 6213	6811
.43	1.1342 5009	10 7108	.93	4.7529 6390	1 3730
.44	1.1862 2764	10 6171	.94	4.8371 2665	2 1554
.45	1.2392 6686	10 5207	.95	4.9210 7158	3 0581
.46	1.2933 5809	10 4213	.96	5.0047 0740	4 1308
.47	1.3484 9141	10 3191	.97	5.0879 2495	5 4606
.48	1.4046 5657	10 2137	.98	5.1705 8678	−7 1824
.49	1.4618 4304	10 1053	.99	5.2525 0293	----
.50	1.5200 3999	+9 9937	1.00	5.3333 3333	----

Modified second central differences δ_m^2 are included for interpolation by Everett's formula

$$
\begin{aligned}
v(k_i+p\Delta k)=(1-p)v(k_i)+pv(k_{i+1})+E_2(p)\delta^2_{m,\,i} \\
+F_2(p)\delta^2_{m,\,i+1}+\ldots,
\end{aligned}
\tag{14}
$$

where p is the interpolation fraction of the interval of tabulation Δk, and $\delta^2_{m,i}$, $\delta^2_{m,i+1}$ are the modified second differences at the tabular points i and $i+1$ and were evaluated from the second and fourth differences according to

$$\delta_m^2=\delta^2-0.184\delta^4.$$

Everett's coefficients $E_2(p)$ and $F_2(p)$ are available in standard tables [14, 15] and are identical with Lagrangian interpolation coefficients A^4_{-1} and A^4_2 [16, table 25.1].

The general behavior of $v(k)$ is shown by the curve in figure 2. Values of $v(k)$ can be taken directly from this curve for use in rough calculations where only two- or three-figure accuracy is required.

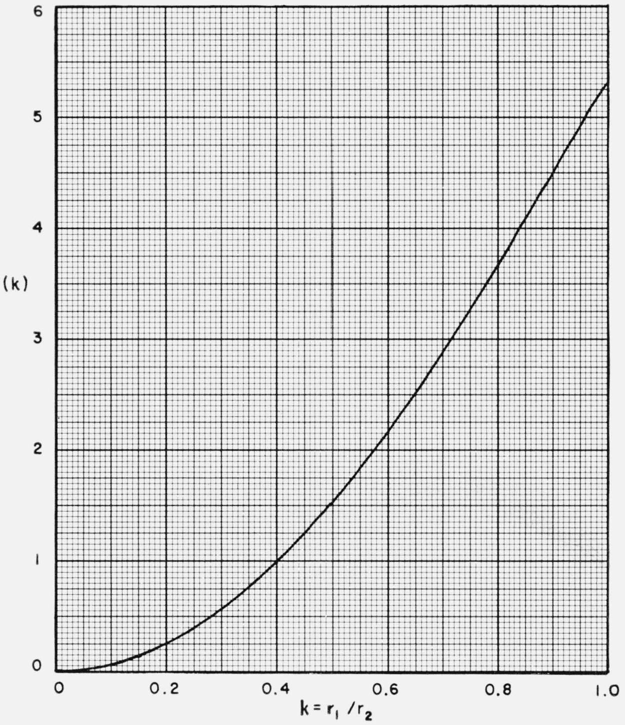

FIGURE 2. *Graph of* v(k) *showing the general behavior of the function and suitable for rough calculations.*

6. Appendix. Two Series for $1/\pi$

The series evaluation (5) for $V(r_1, r_2, \beta)$ contains a factor of π and the formula (3) for $V(r, \beta)$ does not. Thus, for $r_1 = r_2 = r$, the right-hand side of (5) can be equated to the right-hand side of (3) to form

$$\frac{16r^3}{3 \sin \beta} = \frac{4\pi r^3}{\sin \beta} \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \binom{\frac{1}{2}}{n-1} \tag{15}$$

from which

$$\frac{4}{3\pi} = \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \binom{\frac{1}{2}}{n-1} = \frac{1}{2} - \frac{1}{4}\left(\frac{1}{2}\right)^2 - \frac{3}{6}\left(\frac{1}{2 \cdot 4}\right)^2 - \frac{5}{8}\left(\frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2 - \dots \tag{16}$$

This series can now be used to form the difference-series in eq (8).

An additional $1/\pi$-series, which also does not appear in standard compilations of series [17], can be obtained by combining (16) with a series discussed by Bromwich [18], [17, eq 274]

$$\frac{4}{\pi} = \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n-1}^2 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2 + \dots \tag{17}$$

to form

$$\frac{4}{3\pi} + \frac{4}{\pi} = \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n-1} \left[\binom{\frac{1}{2}}{n} + \binom{\frac{1}{2}}{n-1}\right]. \tag{18}$$

Using the addition theorem

$$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n} \tag{19}$$

the result is

$$\frac{16}{3\pi} = \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n-1} \binom{\frac{3}{2}}{n} = 1 \cdot \frac{3}{2} + \frac{1}{2} \cdot \frac{3 \cdot 1}{2 \cdot 4} + \frac{1}{2 \cdot 4} \cdot \frac{3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{3 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{3 \cdot 1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} + \dots \tag{20}$$

The author thanks E. Hayward for suggesting the problem, and P. Lamperti and L. W. B. Jolley for their stimulating interest in the cylinder intersection and $1/\pi$ series, respectively. The author is also indebted to F. W. J. Olver, I. A. Stegun, L. F. Epstein, and A. Fletcher for suggestions and comments, and especially to Mrs. Ruth Capuano for computing table 1.

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(Paper 69C2–193)