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Superimposed Birefractory Plates

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A study is made of the relations among the parameters involved in the passage of a beam of plane-polarized light through a pair of birefractory plates. For a pair of plates having known phase lags and a known angle between their principal axes, the angle of inclination necessary for a plane-polarized beam to emerge plane-polarized is determined. In addition, the conditions for producing circularly polarized light are developed and presented in a simple and convenient form. It turns out that the "combination" quarter-wave plate shows a striking similarity in several respects, to the ordinary quarter-wave plate. It is shown how the "combination" plates can be used to construct several novel polarimetric half-shades.

1. Introduction

Many optical instruments employ polarized light and associated birefractory plates as essential elements in their operation. For example a pair of quarter-wave plates is used in photoelastic stress analysis in order to observe a system of isochromatic lines in a stressed plate irrespective of the azimuth of the plate [1, 2].[1] Exact quarter-wave compensators are especially needed for devices such as the one developed by Goranson and Adams [3] (based on the classical method first fully described by Friedel [4]) which was used initially to measure birefringence in stressed glass, and later for determining the effect of pressure on the release of birefringence in some transparent materials [5]. Jerrard [6] has developed a similar instrument. The Friedel method has also been adapted in constructing an interference microscope [7].

The full advantage of the extraordinarily precise Friedel methods is realized only when using a compensator the optical retardation of which is almost exactly one-quarter wavelength. There are various procedures for constructing precise quarter-wave compensators. One of these is to utilize a suitable combination of two birefractory plates of arbitrary phase lag [8, 2]. A consideration of the properties of such a two plate combination brings out some relations which are of great interest and also provides a method for constructing such compensators.

2. Combining Two Plates

When a beam of plane-polarized light passes through a birefractory plate the emerging beam, as is well known, will be found in general to be eliptically polarized. In treating this problem and the problem of passing the beam through two superimposed plates we shall follow the development of Goranson and Adams [3] using their notation with some additions (see table 1). For the passage of plane-polarized light through a single plate, the four quantities, ϵ', θ_1', Φ_I and $\underline{\mathbf{i}}_1$ which are defined in table 1 are related by the equations [2]

$$\sin 2\epsilon' = \sin 2\underline{\mathbf{i}}_1 \sin \Phi_\mathrm{I}, \tag{1}$$

$$\cos 2\underline{\mathbf{i}}_1 = \cos 2\epsilon' \cos 2\theta_1', \tag{2}$$

$$\cot \Phi_\mathrm{I} = \cot 2\epsilon' \sin 2\theta_1', \tag{3}$$

$$\tan 2\theta_1' = \tan 2\underline{\mathbf{i}}_1 \cos \Phi_\mathrm{I}. \tag{4}$$

[2] Equations (1), (2), and (3) above have been obtained from eqs (6), (7), and (8) in the paper by Goranson and Adams [3] using the substitutions which they have indicated. Equation (4) above has been derived from eq (5) of Goranson and Adams following their procedures and making appropriate substitutions.

TABLE 1. *Definitions of the symbols used in the text* [a]

Plates I, II—are the first and second doubly refracting plate in the path of a beam of plane-polarized light.

α_1' and γ_1'—are the vibrational directions of the faster and slower wave respectively in plate I. These directions form a set of mutually perpendicular reference axes.

α_2' and γ_2'—are the vibrational directions of the faster and slower wave respectively in plate II. These directions form a second set of mutually perpendicular reference axes.

ψ—is the angle which denotes the orientation of plate I with respect to plate II and is measured from α_2' to α_1' or from γ_2' to α_1'. ψ is always positive and is limited to the first quadrant, i.e., $0° \leq \psi \leq 90°$.

[a] The definitions given in this table correspond to those given by Goranson and Adams [3] with some additions. It may be noted that the definitions given here pertain to the special case where the principal directions of plates I and II have been chosen as the reference axes, and as a result are simpler than those of Goranson and Adams in many instances.

* Retired.

[1] Figures in brackets indicate the literature references on page 113.

ψ_c—is the value of ψ when plates I and II are superposed to produce a compound quarter-wave plate.

ϵ', ϵ''—are the ellipticities of the light emergent from plates I and II respectively. The ellipticity is defined as the angle whose tangent is $\pm(b/a)$ where a and b denote the major and minor axes respectively. By convention, with the light coming toward one, ϵ is positive when the light vector along the emergent elliptical vibration moves counterclockwise and negative when clockwise. $-45°\leq\epsilon\leq 45°$.

$\underline{\mathbf{i}}_1$—is the angle between the vibrational direction of the entering plane-polarized light and α_1'. $-90°\leq\underline{\mathbf{i}}_1\leq+90°$.

$\underline{\mathbf{i}}_2$—is the angle between the vibrational direction of the entering plane polarized light and α_2' or γ_2'. $-90°\leq\underline{\mathbf{i}}_2\leq+90°$.

$\underline{\mathbf{i}}_{p_1}$, $\underline{\mathbf{i}}_{p_2}$—are the values of $\underline{\mathbf{i}}_1$ and $\underline{\mathbf{i}}_2$ respectively when the emergent beam of light is linearly polarized after passing through plates I and II.

$\underline{\mathbf{i}}_{c_1}$, $\underline{\mathbf{i}}_{c_2}$—are the values of $\underline{\mathbf{i}}_1$ and $\underline{\mathbf{i}}_2$ respectively when the emergent beam of light is circularly polarized after passing through plates I and II. [It is understood in this case that plates I and II have been superimposed at the angle, ψ_c, to make a compound quarter-wave plate.]

θ_1'—refers to light which has passed through plate I and is the angle between the major axis of the elliptical vibration and α_1'. $-90°\leq\theta_1'\leq+90°$.

θ_2'—refers to light which has passed through plate I and is the angle between the major axis of the elliptical vibration and α_2' or γ_2'. $-90°\leq\theta_2'\leq+90°$.

θ_2''—refers to light which has passed through plates I and II and is the angle between the major axis of the elliptical vibration and α_2' or γ_2'. $-90°\leq\theta_2''\leq+90°$.

Φ—is the phase lag of a given birefractory plate. $0°\leq\Phi\leq 360°$.

Φ_{I}, Φ_{II}—are the phase lags for plates I and II respectively.

When a beam of elliptically polarized light such as the one emerging from plate I passes through a second birefractory plate, the emerging beam, as is well known, is also elliptically polarized. For an elliptically polarized beam of a given ellipticity and azimuth entering a birefractory plate of a given phase lag and azimuth the ellipticity and azimuth of the emerging beam are uniquely determined. Goranson and Adams have shown that the following relations hold:

$$\sin 2\epsilon''=\sin 2\epsilon'\cos\Phi_{\mathrm{II}}\pm\cos 2\epsilon'\sin 2\theta_2'\sin\Phi_{\mathrm{II}} \qquad (5)$$

and

$$\tan 2\theta_2''=\tan 2\theta_2'\cos\Phi_{\mathrm{II}}\mp\frac{\tan 2\epsilon'}{\cos 2\theta_2'}\sin\Phi_{\mathrm{II}}, \qquad (6)$$

where again the parameters are defined in table 1.

These equations in conjunction with tables 1 and 2 and with proper regard to the significance of the $\pm$ and $\mp$ signs, are adequate to determine the correct and unique values of ϵ'' and θ_2'' as follows: Where an alternative in sign occurs (and where it occurs in other equations throughout the paper) the sign chosen depends on which of the two principal directions of plate II is chosen as the reference axis. If ψ is measured from α_2' the upper sign is used, if ψ is measured from γ_2' the lower sign is used.

When either the upper or lower sign is used in (5), it will be found that the solution for ϵ'' is unique because, by definition, $-45°\leq\epsilon''\leq+45°$. When either the upper or lower sign is used in eq (6) it will be found that there are still two solutions for θ_2'', but physical considerations show that only one solution is valid. The correct solution for θ_2'' may be found by referring to table 2.

TABLE 2. *Limits on the correct values of $2\theta_2''$ which may be used to find a unique solution for eq (6)*

$$\cot\Phi_a\equiv\cot 2\epsilon'\sin 2\theta_2'$$

where $0°<\Phi_a<180°$ if $\epsilon'>0°$ and $180°<\Phi_a<360°$ if $\epsilon'<0°$

$(\Phi_{\mathrm{II}}+\Phi_a)$	$2\theta_2''$
$0°<(\Phi_{\mathrm{II}}\pm\Phi_a)<90°$	$0°<2\theta_2''<180°$
$90°<(\Phi_{\mathrm{II}}\pm\Phi_a)<270°$	$-180°<2\theta_2''<0°$
$270°<(\Phi_{\mathrm{II}}\pm\Phi_a)<360°$	$0°<2\theta_2''<180°$

3. Restoration to Plane-Polarization

First we determine what are the relations between the variables in order for the beam, originally plane-polarized, to pass through the two plates and to emerge plane-polarized.

Here $\epsilon''=0$. Hence by (5)

$$\tan 2\epsilon'=\mp\tan\Phi_{\mathrm{II}}\sin 2\theta_2'. \qquad (7)$$

Applying (3) we have

$$\tan\Phi_{\mathrm{I}}\sin 2\theta_1'=\mp\tan\Phi_{\mathrm{II}}\sin 2\theta_2'. \qquad (8)$$

Since the angle between the corresponding principal directions of the two plates is ψ, it follows that

$$\theta_2'=\theta_1'+\psi. \qquad (9)$$

Substituting (9) and (4) in (8), we have (noting that ϵ'' is zero, and that $\underline{\mathbf{i}}$ becomes $\underline{\mathbf{i}}_{p_1}$ the inclination that will result in emerging plane-polarized light)

$$\cot 2\underline{\mathbf{i}}_{p_1} = \left(\mp \frac{\tan \Phi_{\mathrm{I}}}{\tan \Phi_{\mathrm{II}}} - \cos 2\psi\right) \frac{\cos \Phi_{\mathrm{I}}}{\sin 2\psi}. \qquad (10)$$

After the correct minus or plus value has been determined for use in eq (10), there will be two values of $\underline{\mathbf{i}}_{p_1}$, differing by 90°, which will satisfy the equation. These two solutions are both valid and can be associated generally with the principal axes of a single birefractory plate. For each of these solutions there will be separate values of θ_1', θ_2', and θ_2''. These two sets of values will be separated from each other by 90° just as the $\underline{\mathbf{i}}_{p_1}$ values are. The following relations are useful: From (6) and (7) we find

$$\tan 2\theta_2'' = \tan 2\theta_2' \, (\cos \Phi_{\mathrm{II}} + \tan \Phi_{\mathrm{II}} \sin \Phi_{\mathrm{II}}).$$

Multiplying and dividing by $\cos \Phi_{\mathrm{II}}$ we obtain:

$$\tan 2\theta_2'' = \tan 2\theta_2' / \cos \Phi_{\mathrm{II}}. \qquad (11)$$

From (4) and (10) we obtain

$$\cot 2\theta_1' = \left(\mp \frac{\tan \Phi_{\mathrm{I}}}{\tan \Phi_{\mathrm{II}}} - \cos 2\psi\right) \frac{1}{\sin 2\psi}. \qquad (12)$$

We are now in a position, having two plates with Φ_{I} and Φ_{II}, and placed at angle ψ to determine what inclination $\underline{\mathbf{i}}_{p_1}$ of the entering plane-polarized beam will cause the emerging beam to be plane-polarized; and also to calculate θ_2'', the azimuth of the emerging beam. Formally stated the problem is: *Given* Φ_{I}, Φ_{II}, and ψ to calculate $\underline{\mathbf{i}}_{p_1}$ so that the beam will emerge plane-polarized; and also to calculate θ_2''. *Solution*, calculate $\underline{\mathbf{i}}_{p_1}$ from (10), θ_1' from (12), θ_2' from (9) and θ_2'' from (11).

3.1. Special Case for $\Phi_{\mathrm{I}} = \Phi_{\mathrm{II}}$

In many instances it would be convenient to superimpose two identical strips or slabs. Here we would have $\Phi_{\mathrm{I}} = \Phi_{\mathrm{II}}$, which we may call simply Φ. For this case eq (12) becomes

$$\cot 2\theta_1' = -\cot \psi \text{ or } \cot (90 - \psi)$$

from which

$$2\theta_1' = -\psi,\ -\psi + 180° \text{ or } 90° - \psi,\ 90° - \psi + 180°. \qquad (12a)$$

Similarly, from (10)

$$\cot 2\underline{\mathbf{i}}_{p_1} = -\cot \psi \cos \Phi \text{ or } \tan \psi \cos \Phi, \qquad (10a)$$

and, from (9) and (12a)

$$2\theta_2' = \psi,\ \psi + 180° \text{ or } 90° + \psi,\ 90° + \psi + 180°. \qquad (9a)$$

Also (incidentally) from (8) with $\Phi_{\mathrm{I}} = \Phi_{\mathrm{II}}$

$$\sin 2\theta_2' / \sin 2\theta_1' = \mp 1.$$

From (11) and (9a) it follows that

$$\tan 2\theta_2'' = \tan \psi / \cos \Phi \text{ or } -\cot \psi / \cos \Phi, \qquad (11a)$$

and, from (11a) and (10a) when $\Phi_{\mathrm{I}} = \Phi_{\mathrm{II}} = \Phi$

$$2\theta_2'' = -2\underline{\mathbf{i}}_{p_1} \text{ or } -2\underline{\mathbf{i}}_{p_1} + 180°. \qquad (13a)$$

For this simplified case (when Φ and ψ are given), the procedure for determining $\underline{\mathbf{i}}_{p_1}$ and θ_2'' is to calculate $\underline{\mathbf{i}}_{p_1}$ from (10a) and θ_2'' from (13a). If desired, we may obtain θ_1' from (12a) and θ_2' from (9a).

The values of the parameters of interest are presented in two illustrative examples in table 3. In both cases $\Phi_{\mathrm{I}} = \Phi_{\mathrm{II}} = \Phi = 70°$; in one example ψ has been set arbitrarily at 30°, and the second example refers to the situation where $\psi_c = 41.20°$ which is the angle needed to construct a composite quarter-wave plate. (It will be shown in section 4 how the value of ψ_c is found.) Figure 1 shows the relationships of the azimuths for transmitting plane-polarized light with two superimposed plates when $\Phi_{\mathrm{I}} = \Phi_{\mathrm{II}} = \Phi = 70°$ and $\psi_c = 41.20°$.

TABLE 3. *Values of the parameters of interest in producing plane-polarized light and circularly polarized light when it is given that* $\Phi_{\mathrm{I}} = \Phi_{\mathrm{II}} = \Phi = 70°$

Two examples are shown: (1) for an arbitrary $\psi = 30°$, and, (2) for the required $\psi_c = 41.20°$. (The values are given for the situation where the α'-direction of plate II is chosen as the reference axis.)

	$\psi = 30°$	$\psi_c = 41.20°$
Plane-polarized light	$\underline{\mathbf{i}}_{p_1} = -29.68°, 60.32°$	$\underline{\mathbf{i}}_{p_1} = -34.33°, 55.67°$
	$\underline{\mathbf{i}}_{p_2} = 0.32°, -89.68°$	$\underline{\mathbf{i}}_{p_2} = 6.87°, -83.13$
	$\theta_2'' = 29.68, -60.32°$	$\theta_2'' = 34.33°, -55.67°$
	$\theta_1' = -15°, 75°$	$\theta_1' = -20.60°, 69.40°$
	$\theta_2' = 15,° -75°$	$\theta_2' = 20.60°, -69.40°$
Circularly polarized light	Note:	$\underline{\mathbf{i}}_{c_1} = 10.67°, -79.33°$
		$\underline{\mathbf{i}}_{c_2} = 51.87°, -38.13°$
	For	θ_2'' is undefined
	Circularly-polarized light $\psi (= \psi_c) =$ 41.20°	$\theta_1' = 3.80°, -86.20°$
		$\theta_2' = 45.00°, -45.00°$

The parameters are obtained from the following equations (particularly applicable in this simple form since $\Phi_{\mathrm{I}} = \Phi_{\mathrm{II}}$):

Plane-polarized light		Circularly polarized light	
(1) $\underline{\mathbf{i}}_{p_1}$,	eq (10a)	(1) ψ_c	eq (19a)
(2) $\underline{\mathbf{i}}_{p_2}$,	$\underline{\mathbf{i}}_{p_2} = \underline{\mathbf{i}}_{p_1} + \psi$	(2) $\underline{\mathbf{i}}_{c_1}$	eq (17a)
(3) θ_2'',	eq (13a)	(3) $\underline{\mathbf{i}}_{c_2}$	$\underline{\mathbf{i}}_{c_2} = \underline{\mathbf{i}}_{c_1} + \psi$
(4) θ_1',	eq (12a)	(4) θ_2''	is undefined
(5) θ_2',	eq (9a)	(5) θ_1'	eq (18a)
		(6) θ_2'	is given as $\theta_2' = 45°$

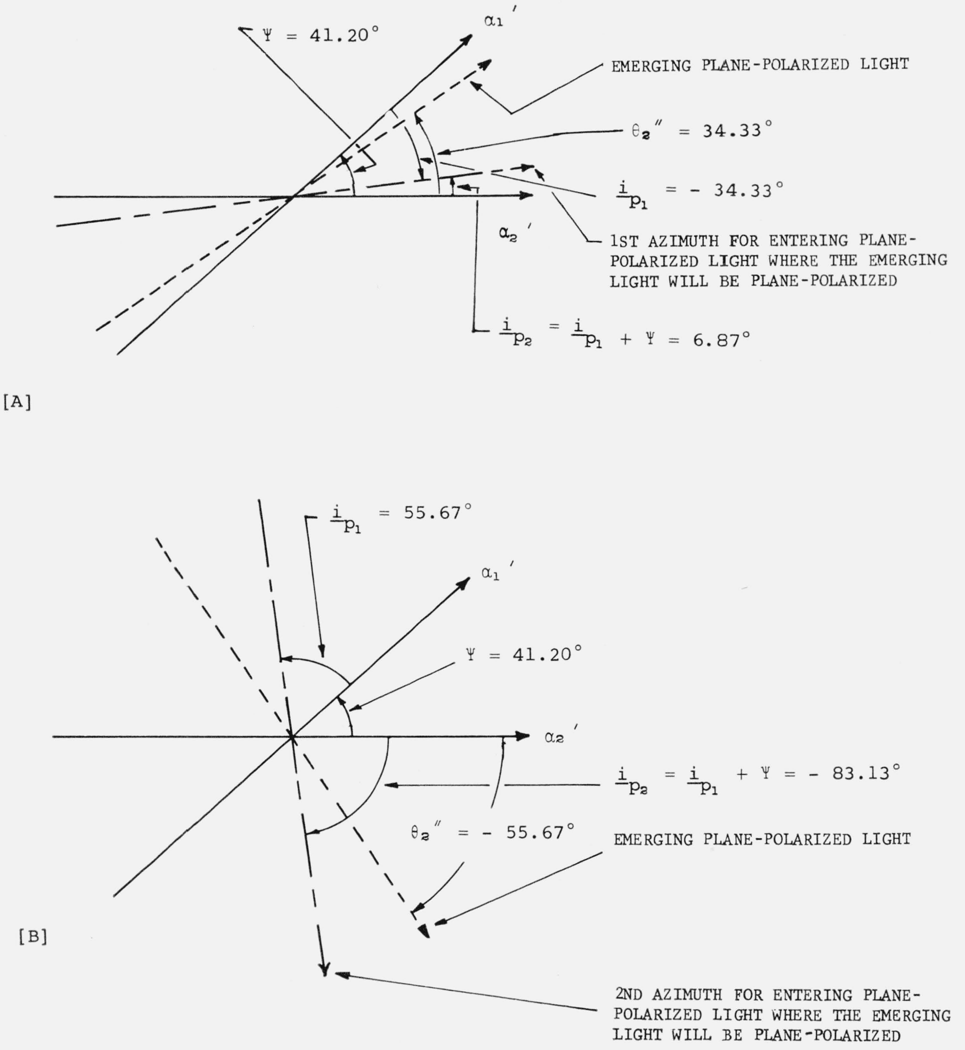

FIGURE 1. *Illustration of the azimuthal relationships in the transmission of plane-polarized light by two superimposed doubly refracting plates when $\Phi=70°$ and $\Psi=\Psi_c=41.20°$.*

It is shown in [A] and [B] that there are two mutually perpendicular entering azimuths for which plane-polarized light will emerge plane-polarized and that the vibration direction will be rotated through the angle $(\theta_2''-\underline{\mathbf{i}}_{p_2})$ which is the same for both.

For any combination of birefractory plates, therefore, there can be found two mutually perpendicular azimuths for incident plane-polarized light such that the emergent light is also plane-polarized. In this respect such a combination is like a single birefractory plate. The effect of the combination differs (in general), however, from that of a single plate in that the combination rotates the vibration direction through the angle $(\theta_2''-\underline{\mathbf{i}}_{p_2})$. It turns out that the sense of this rotation will be either positive (counterclockwise) or negative (clockwise) depending on whether the plates are combined so that ψ is measured from the α_2' or γ_2' direction. In table 4 are presented values of the parameters for emerging plane-polarized light when the two plates have different $\Phi_{\mathrm{I}}=\Phi_{\mathrm{II}}=\Phi$, but with the arbitrary parameter, ψ, set at 30° for all cases.

TABLE 4. *Values of $\underline{\mathbf{i}}_{p_1}$ and other parameters of interest in producing plane-polarized from two plates having different $\Phi_I = \Phi_{II} = \Phi$ but with the arbitrary parameter ψ set at 30° for all cases*

(The values are given for the situation where the α'-direction of plate II is chosen as the reference axis.)

Φ	0	15	30	45	60	75	90	105	120	135	150	165	180
Ψ	30	30	30	30	30	30	30	30	30	30	30	30	30
$2\underline{\mathbf{i}}_{p_1}$	−30.0	−30.9	−33.7	−39.2	−49.2	−65.8	−90.0	65.8	49.2	39.2	33.7	30.9	30.0
	150.0	149.1	146.3	140.8	130.8	114.2	90.0	−114.2	−130.8	−140.8	−146.3	−149.1	−150.0
$2\Theta_1'$	−30.0	−30.0	−30.0	−30.0	−30.0	−30.0	−30.0	−30.0	−30.0	−30.0	−30.0	−30.0	−30.0
	150.0	150.0	150.0	150.0	150.0	150.0	150.0	150.0	150.0	150.0	150.0	150.0	150.0
$2\Theta_2''$	30.0	30.9	33.7	39.2	49.2	65.8	90.0	−65.8	−49.2	−39.2	−33.7	−30.9	−30.0
	−150.0	−149.1	−146.3	−140.8	−130.8	−114.2	−90.0	114.2	130.8	140.8	146.3	149.1	150.0
$2\Theta_2'$	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0	30.0
	−150.0	−150.0	−150.0	−150.0	−150.0	−150.0	−150.0	−150.0	−150.0	−150.0	−150.0	−150.0	−150.0

The parameters are obtained from the following equations (particularly applicable in this simple form since $\Phi_I = \Phi_{II}$):

(1) $2\underline{\mathbf{i}}_{p_1}'$ -------------------- eq (10a)
(2) $2\Theta_1'$ -------------------- eq (12a)
(3) $2\Theta_2''$ -------------------- eq (13a)
(4) $2\Theta_2'$ -------------------- eq (9a)

4. Simple Method for Producing Circularly Polarized Light

A simple method for getting ψ_c and $\underline{\mathbf{i}}_{c_1}$ for given Φ_I and Φ_{II} begins as follows: Suppose we let θ'_2 equal 45°. Then, from (5) with $\epsilon''=45°$, sin $(2\epsilon'+\Phi_{II})=1$, and therefore

$$\epsilon' = 45 \mp \Phi_{II}/2. \tag{14}$$

If $\underline{\mathbf{i}}_1$ is chosen to give ϵ' given by eq (14), then θ_1' can be calculated and $\psi = \psi_c$ can be calculated by eq (9) to give $\theta_2' = 45°$. This combination of $\underline{\mathbf{i}}_1$ and ψ_c will (by the conditions leading up to eq (14)) yield $\epsilon''=45°$, i.e., emergent laevorotary circularly polarized light.

By (14)—for this case of emerging circularly polarized light—

$$\sin 2\epsilon' = \pm \cos \Phi_{II} \tag{15}$$

and

$$\tan 2\epsilon' = \pm \cot \Phi_{II}. \tag{16}$$

From (1) and (15) we have

$$\sin 2\underline{\mathbf{i}}_{c_1} = \pm \cos \Phi_{II} / \sin \Phi_I, \tag{17}$$

and from (3) and (16)

$$\sin 2\theta_1' = \pm \cot \Phi_{II} / \tan \Phi_I. \tag{18}$$

and from (9) and the above assumption that $\theta_2' = 45°$

$$\psi_c = 45° - \theta_1'. \tag{9b}$$

It is important to note the following points in regard to eqs (17), (18), and (9b) and the "combination" plate considered here: (1) The equations admit solutions only if $90° \leq \Phi_I + \Phi_{II} \leq 270°$. (2) Careful examination shows that only one pair of solutions is valid in each equation, although there are additional apparent solutions. (3) The two solutions for ψ_c are complementary. One solution for ψ_c is obtained if α_2' is chosen as the reference axis, the second solution is obtained if γ_2' is chosen as the reference axis. (4) As noted in section 3.1 there will be a rotation of the principal directions of vibration for this "combination" plate, the sense of this rotation depends upon the choice of ψ_c. (5) The equations were developed to produce left-handed circularly polarized light, i.e., $\epsilon''=45°$. However, for a given "combination" plate, light incident at azimuth, $\underline{\mathbf{i}} = \underline{\mathbf{i}}_c + 90°$ will emerge as right-handed circularly polarized light, i.e., $\epsilon'' = -45°$.

4.1. Special Case for $\Phi_I = \Phi_{II}$

For two similar plates we put $\Phi_I = \Phi_{II} = \Phi$, and (17) becomes

$$\sin 2\underline{\mathbf{i}}_{c_1} = \pm \cot \Phi \tag{17a}$$

and (18) becomes

$$\sin 2\theta_1' = \pm \cot^2 \Phi. \tag{18a}$$

Now from (9) sin $2\theta_1' = \cos 2\psi$; therefore from (18a)

$$\cos 2\psi_c = \pm \cot^2 \Phi. \tag{19a}$$

Then, from (17a) and (19a) we may readily calculate the impinging azimuth and the requisite separation, ψ_c, to give emerging circularly polarized light. Incidentally, it should be noted that a solution for ψ_c is obtained when Φ ranges from 45 to 135° but not from 0 to 45° or from 135 to 180°.

As a supplement to the example shown in figure 1, figure 2 shows the relationships of the azimuths for producing circularly polarized light from two superimposed plates when $\Phi_I = \Phi_{II} = \Phi = 70°$. Figure 3 shows the rotation of the principal planes of vibration in a "combination" quarter-wave plate when $\Phi = 70°$. As shown, the rotation may be either positive or negative depending on whether the upper or lower sign is chosen in eq (19a) when solving for ψ_c.

Table 5 gives values of ψ_c, $\underline{\mathbf{i}}_{c_1}$, and $\underline{\mathbf{i}}_{p_1}$ for various values of Φ when two similar plates are combined to make a quarter-wave plate. Here it is evident that

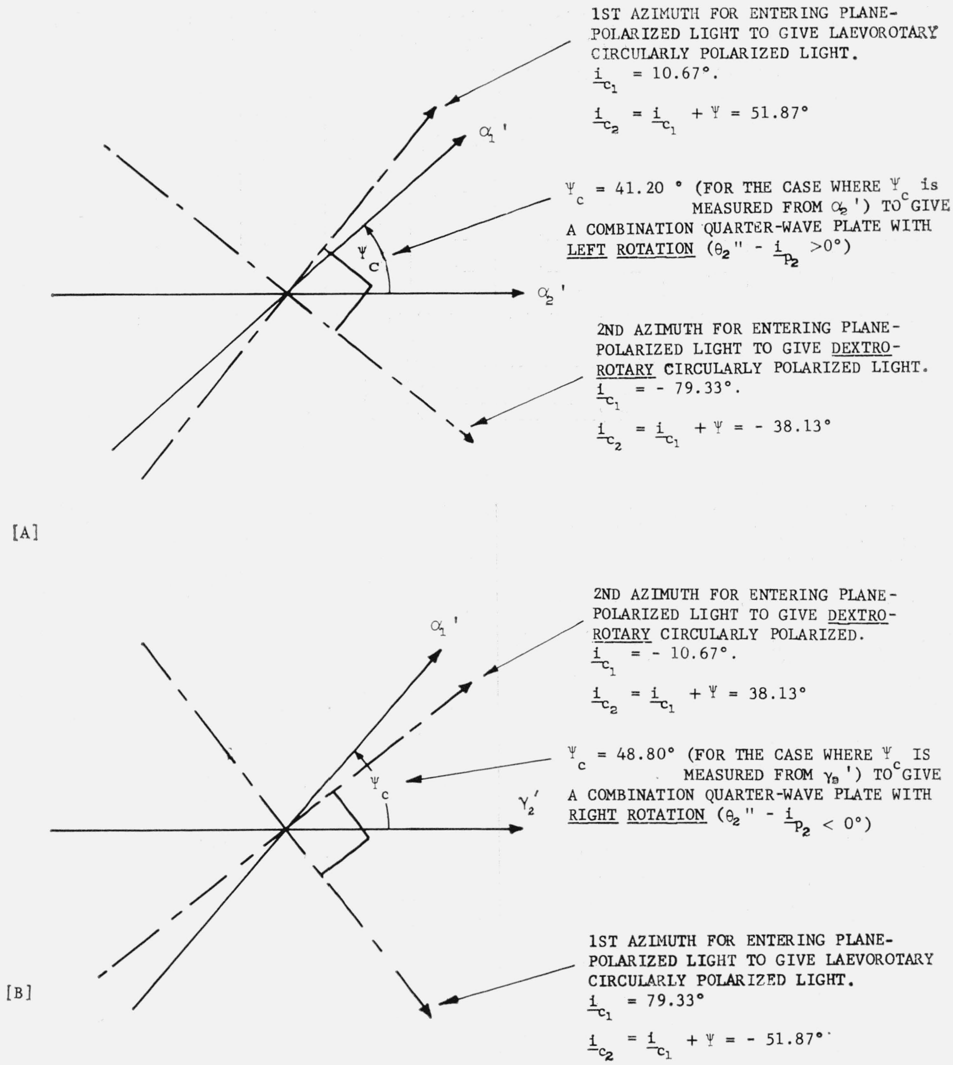

FIGURE 2. *Illustration of the azimuthal relationships in the production of circularly polarized light by two superimposed doubly refracting plates when $\Phi = 70°$.*

In case [A], $\Psi = \Psi_c$ is measured from α_2' and rotates the principal vibration directions to the left. In case [B], $\Psi = \Psi_c = 48.80°$ is measured from γ_2' and rotates the principal vibration directions to the right by an equal amount (see fig. 3). It is shown in each case that there exist mutually perpendicular azimuths of incidence which yield laevo- and dextrorotary circularly polarized light respectively, which are at 45° to the azimuths for the production of plane-polarized light (fig. 3).

the choice of ψ_c is fixed by the phase lags of the plates. A striking feature of the relation of $\underline{\mathbf{i}}_{c_1}$ to $\underline{\mathbf{i}}_{p1}$ is brought out by the table, namely that $(\underline{\mathbf{i}}_{c_1} - \underline{\mathbf{i}}_{p_1})$ appears to be always $\pm 45°$. As a matter of fact in a pair of plates adjusted to transform plane-polarized light into circularly polarized light the difference between the two $\underline{\mathbf{i}}$'s is exactly 45°. That is, it may be readily shown that

$$\sin^2 2\underline{\mathbf{i}}_{c_1} = \cos^2 2\underline{\mathbf{i}}_{p_1} \tag{20}$$

The proof of relation (20) may be found in appendix 1. In this respect the "combination" quarter-wave plate behaves like an ordinary single plate with 90° retardation.

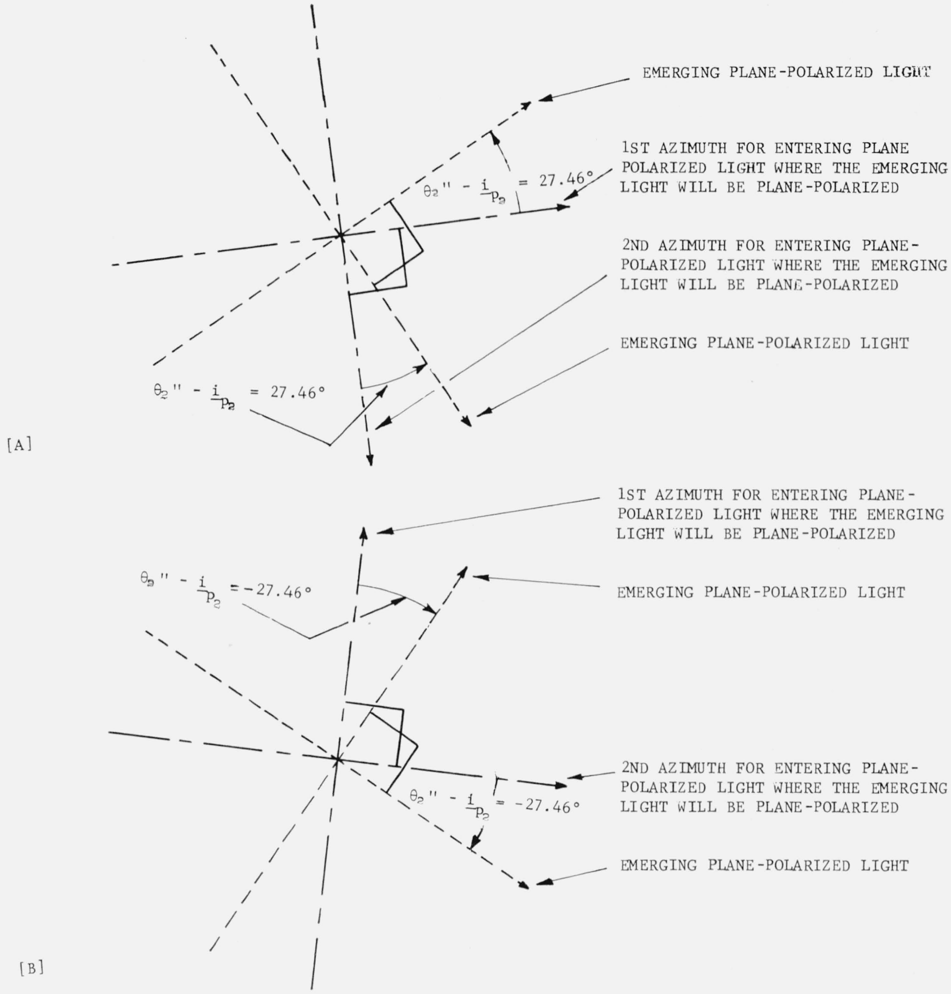

FIGURE 3. *Illustration of the rotation of the principal vibration directions in combination quarter-wave plates when $\Phi=70°$.*

Two situations corresponding to the two solutions of eq (19A) are shown: In [A], $\Psi=\Psi_c=41.20°$ and is measured from α_2': In [B], $\Psi=\Psi_c=48.80°$ and is measured from γ_2'. The magnitude of the rotation is the same in both cases. Case [A] is the same as that in figure 1 where the azimuthal relations of the principal vibration directions of the plates are shown. Note that the azimuths in each case are mutually perpendicular and at 45° to those for producing circularly polarized light (fig. 2)

TABLE 5.—*Values of ψ_c, $\underline{\mathbf{i}}_{c_1}$, $\underline{\mathbf{i}}_{p_1}$, and related parameters in producing circularly polarized light from two plates having different $\Phi_I=\Phi_{II}=\Phi$*

(The values are given for the situation where the α'—direction of Plate II is chosen as the reference axis.)

Φ	45	50	60	70	80	90	100	110	120	130	135
cot Φ	1.000	0.840	0.577	0.3640	0.1762	0	−0.1762	−0.364	−0.577	−0.840	−1.000
cot² Φ	1.000	.704	.333	.1326	.0312	0	.0312	.1326	.333	.706	1.000
$2\underline{\mathbf{i}}_{c_1}$	90.00	57.2	35.2	21.34	10.2	0	−10.2	−21.34	−35.2	−57.2	−90.0
	−90.0	−122.8	−144.8	−158.66	−169.8	180	169.8	158.66	144.8	122.8	90.0
$2\psi_c$	0	45.4	70.6	82.40	88.4	90	88.4	82.40	70.6	45.4	0
ψ_c	0	22.7	35.3	41.20	44.2	45.0	44.2	41.20	35.3	22.7	0
cot ψ_c	∞	2.42	1.412	1.142	1.028	1.000	1.028	1.142	1.412	2.42	∞
cos Φ	0.707	0.642	0.500	0.342	0.1736	0	−0.1736	−0.342	−0.500	−.642	−0.707
−cot Ψ_c cos Φ	−∞	−1.55	−.706	−.3918	−.1789	0	.1789	.3918	.706	1.55	∞
$2\underline{\mathbf{i}}_{p_1}$	0	−32.8	−54.8	−68.66	−79.8	−90.0	79.8	68.66	54.8	32.8	0
	180.0	147.2	125.2	111.34	100.2	90.0	−100.2	−111.34	−125.2	−147.2	180.0

The parameters are obtained from the following equations (particularly applicable in this simple form since $\Phi_I=\Phi_{II}$):

(1) ψ_c ---------------------- eq (19a)
(2) $\underline{\mathbf{i}}_{c_1}$ ---------------------- eq (17a)
(3) $\underline{\mathbf{i}}_{p_1}$ ---------------------- eq (10a)

In another respect also the "combination" plate resembles a simple quarter-wave plate, namely that when a beam of plane-polarized light is passed through, the azimuth of the ellipse of the emerging beam is independent of the inclination $\underline{\mathbf{i}}_1$ of the entering beam. For proof of this interesting relation see appendix 2.

There is a third way in which the "combination" plate resembles an ordinary quarter-wave plate. To show this, we first recall that by (1) a beam of plane-polarized light passing through a plate with Φ equal to 90° emerges with ϵ equal to $\underline{\mathbf{i}}$. For our "combination" plate—let us put

$$\Delta = \underline{\mathbf{i}}_1 - \underline{\mathbf{i}}_{p_1}. \tag{21}$$

It turns out, as shown in appendix 3, that whereas with the ordinary quarter-wave plate $\epsilon' = \underline{\mathbf{i}}$, for the combination quarter-wave plate

$$\epsilon'' = \underline{\mathbf{i}}_1 - \underline{\mathbf{i}}_{p_1} \equiv \Delta. \tag{22}$$

Thus, there is the same relation, between ϵ'' and Δ as there is between ϵ' and $\underline{\mathbf{i}}$ for the simple quarter-wave plate.

5. Design of Half-Shade Systems

Precision measurements in polarimetry usually depend upon making a brightness match between two parts of a field of view. Half-shade devices have been constructed which permit the observer to ascertain very precisely the azimuth of plane-polarized light, and the azimuth and ellipticity of elliptically polarized light [6, 9, 10]. As examples, the Lippich half-nicol and the Cornu-Jellet split nicol are widely known as azimuth half-shades, and the Brace half-shade has been used extensively in the measurement of ellipticity. The superimposition of doubly refracting plates makes it possible to construct several novel half-shade devices. These half-shades, particularly adapted to monochromatic analysis, are interesting in themselves, and they serve to illustrate some interesting features of the "combination" plates discussed thus far.

5.1. Azimuth Half-Shades

The inherent rotation of the principal vibration directions of a "combination" quarter-wave plate which was discussed in sections 3.1 and 4.1 may be used to construct an azimuth half-shade. It is convenient to denote by α_L' and γ_L' the principal directions (after rotation) of a laevorotary compound plate, and by α_D' and γ_D' the principal directions (after rotation) of a dextrorotary compound plate.

(a) In figure 4 it is shown that two compound plates may be cut along γ_L' and γ_D' and these two directions placed in adjacent positions along the vibration direction, P, of a nicol prism. The combination of nicol and two plates in fixed position then acts effectively as an azimuth half-shade with half-shadow angle of $2|\theta_2'' - \underline{\mathbf{i}}_{p2}|$. The characteristics of this half-shade are analogous to those of the Lippich half-nicol or the Cornu-Jellet split nicol. As an equivalent arrangement to the one shown in figure 4, α_L' may be interchanged for γ_L', when, at the same time, α_D' is interchanged for γ_D'.

(b) The half-shade described in section (a) above depends upon the rotation of the optic axes which is a unique feature of "Combination" plates, and the two parts of the biplate may have an arbitrary path difference. On the other hand, half-wave plates have long been known for their ability to rotate the plane of polarization of incident light [11]. The Laurent plate is a half-wave plate placed partly over a polarizing prism and rotated in fixed combination with the prism [11]. Chauvin has described an analyzer consisting of the same combination [9]. The advantage of these half-shade devices lies in the fact that the sensitivity may be varied by varying the angle between the vibration direction of the polarizing prism and one of the principal directions of the half-wave plate.

A half-wave plate may be simply constructed by superposing two of the "combination" quarter-wave plates described in this paper. However one of the quarter-wave plates must be dextrorotary and the other laevorotary and they must have equal amounts of rotation of the optic axes. The two plates must be combined so as to compensate for the inherent rotation of each one.

(c) It is known that when a doubly refracting material is placed with its principal vibration direction in a diagonal position between two crossed quarter-wave plates, that the combination acts as as an optically rotary material [3]. Two specimens may be cut from the same doubly refracting sheet and positioned so that the fast direction of one specimen adjoins the slow direction of the other.

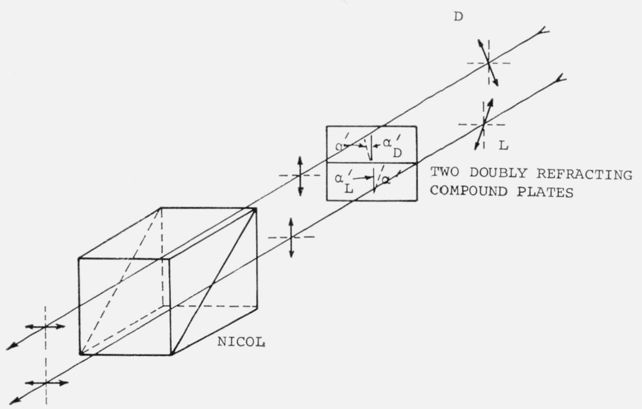

Figure 4. *An azimuth half-shade consisting of two compound plates in fixed combination with an analyzing nicol prism.*

Incident plane-polarized light with azimuth shown at D causes the top half of the field of view to show total extinction. Incident plane-polarized light with azimuth shown at L causes the bottom half of the field of view to show total extinction.

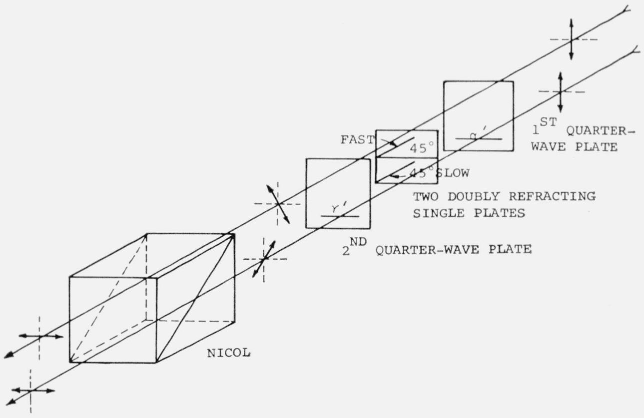

FIGURE 5. *An azimuth half-shade consisting of a divided doubly refracting material between two crossed quarter-wave plates.*

The analyzing nicol rotates independently and is shown in a position where the two halves of the field of view have equal brightness.

If these two plates are placed so that their principal vibration directions are in a diagonal position between two crossed quarter-wave plates, the field of view being evenly divided, this fixed combination may be employed as an azimuth half-shade whose half-shadow angle is equal to Φ, the phase lag of the divided plate. One possible arrangement is shown in figure 5.

This half-shade acts then in a manner analogous to two plates of quartz cut perpendicularly to the optic axis, one plate being laevorotary and the other being dextrorotary. The combination described above acts effectively as a Soliel or Nakamura half-shade [6].

The arrangement shown in figure 5 is correct when using quarter-wave plates with no unusual features. However, in constructing the half-shade from quarter-wave plates described in this paper, the inherent rotations of the plates themselves must be recognized and accounted for. For example, in the arrangement shown in figure 5, it is possible to use a compound laevorotary plate for the first quarter-wave plate, and to set the axes so that α'_L is fixed at the azimuth of α' shown in the figure. In this case a dextrorotary compound plate must be used for the second quarter-wave plate, γ' (not γ'_D) of this second plate will still have the azimuth of γ' shown in the figure.

5.2. An Elliptic Half-Shade

The Bravais biplate is well known as an elliptic half-shade [10]. It consists of two quarter-wave plates with their edges in juxtaposition and the fast ray of one plate parallel to the slow ray of the other. This biplate serves to show when elliptically polarized light has been completely restored to plane-polarized light, because the two halves of the field match at all azimuths, the brightness level

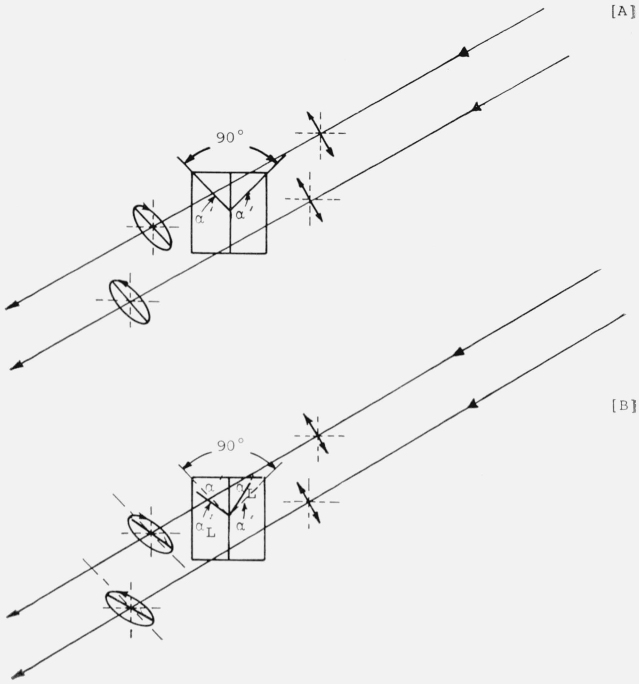

FIGURE 6. *Bravais biplates.*

A An elliptic half-shade in which the two parts are single plates of equal and opposite order (quarter-wave).
B An elliptic half-shade in which the two parts are *compound* plates of equal and opposite order (quarter-wave).

alone varying, but the match is destroyed as soon as the incident light becomes elliptically polarized. A Bravais biplate is shown in figure 6A.

Because the azimuth of this biplate may be varied with respect to the analyzing nicol, it is perfectly all right to use compound quarter-wave plates with their associated rotation of the principal axes. It is important, however, that both halves of the biplate have the same amount and sign of rotation. Figure 6B shows a Bravais biplate constructed from two laevorotary quarter-wave plates. The biplate might as easily be constructed from two dextrorotary quarter-wave plates.

5.3. Half-Shades Uniquely Adapted to the Friedel Method

The third Friedel method has been employed by Goranson and Adams for the precise determination of optical path difference [3]. According to this method, plane-polarized light is made to pass through a doubly refracting specimen with one of the principal vibration directions of the specimen set at 45 deg with respect to the vibration direction of the polarizer. The light is then made to pass through a quarter-wave plate with one of its principal directions parallel to the vibration direction of the original plane-polarized light. The light emergent

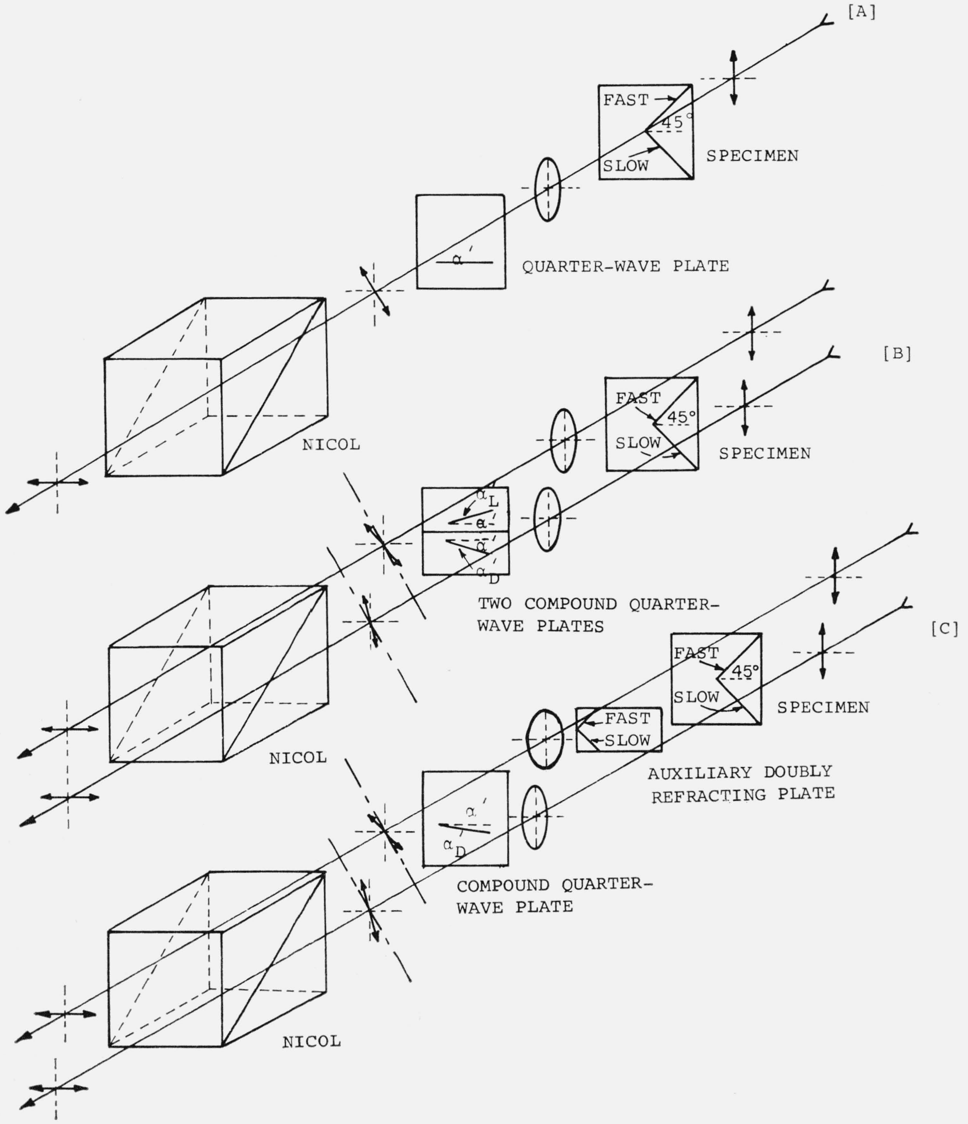

FIGURE 7. *Friedel method III.*

A There is no divided field. The analyzing nicol must be rotated to the left for complete extinction.
B Two compound quarter-wave plates produce a divided field. The analyzing nicol must be rotated to the left to make a brightness match between the two halves of the field.
C A divided field is produced by an auxiliary doubly refracting plate. The analyzing nicol must be rotated to the left to make a brightness match between the two halves of the field.

from the quarter-wave plate is plane-polarized, and an extinction position may be found by rotating the analyzer. The measured angle between the azimuths of the incident and emergent plane-polarized beams of light is equal to one-half the phase angle of the specimen. The arrangement of the optical elements for this method is shown in figure 7A.[3]

One may employ a compound quarter-wave plate described in this paper instead of the more familiar type of quarter-wave plate which is usually obtained by splitting sheets of mica to the proper thickness. In this case, however, it is necessary to know the sign and amount of the inherent rotation of the quarter-wave plate itself and to account for it in setting the zero position of the analyzer.

(a) An effective half-shade may be obtained if two compound quarter-wave plates (one dextrorotary, and one laevorotary) are placed with their

[3] Although no half-shade device is shown in figure 7A the original instrument designed by Goranson and Adams [3] was equipped with a Wright biquartz wedge plate [12].

edges in juxtaposition so as to divide the field and with the α' direction of each alined with the principal vibration direction of the polarizer. This arrangement is shown is figure 7B. In this case there is no correction to be made to the zero position of the analyzer, and, in measuring, the analyzer is rotated until the two halves of the divided field have the same intensity.

(b) Another half-shade device may be obtained if a second doubly refracting plate is placed after the specimen so that it covers half the field and has its principal vibration directions alined with those of the specimen. The auxiliary plate may be selected so that the rotation incurred by its phase lag just balances the inherent rotation of a compound quarter-wave plate. Actually, Φ of this auxiliary plate must have the magnitude $4|\theta_2'' - \underline{i}_{p_2}|$, where θ_2'' and $\underline{i}_{p_2}$ refer to the "combination" quarter-wave plate. A typical arrangement is shown in figure 7C for the case of a dextrorotary compound quarter-wave plate. In making a measurement, as with the half-shade device discussed in part (a), the analyzer is rotated until the two halves of the field have the same intensity.

6. References

[1] E. G. Coker and L. N. G. Filon, A Treatise on Photoelasticity (Cambridge University Press, Cambridge, 1957).

[2] H. T. Jessop and F. C. Harris, Photoelasticity (Dover Publications, Inc., New York, 1949).

[3] R. W. Goranson and L. H. Adams, J. Frank. Inst. **216,** 475 (1933).

[4] G. Friedel, Bull. Soc. Min. France **16,** 19 (1893).

[5] R. M. Waxler and L. H. Adams, J. Res. NBS **65A** (Phys. and Chem.) No. 4, 283 (1961).

[6] H. G. Jerrard, J. Opt. Soc. Am. **44,** 289 (1954).

[7] F. H. Smith, Research (London) **8,** 385 (1955).

[8] Z. Tuzi and H. Oosima, Sci. Papers Inst. Phys. Chem. Res. Tokyo **36,** 72 (1939).

[9] H. G. Jerrard, J. Opt. Soc. Am. **38,** 35 (1948).

[10] C. A. Skinner, J. Opt. Soc. Am. and Rev. Sci. Instr. **10,** 491 (1925).

[11] A. C. Hardy and F. H. Perrin, The Principles of Optics (McGraw-Hill Book Company, Inc., New York, 1932).

[12] F. E. Wright, Am. J. Sci. **26,** 377, 391 (1908).

7. Appendix 1. Relation Between $\underline{\mathbf{i}}_{c_1}$ and $\underline{\mathbf{i}}_{p_1}$ in an "Adjusted" Combination Plate

Given: the four quantities Φ, ψ_c, $\underline{\mathbf{i}}_{p_1}$ and $\underline{\mathbf{i}}_{c_1}$ and the three equations. (When ψ is measured from α_2')

$$\cos 2\psi_c = +\cot^2 \Phi \tag{19a}$$

$$\tan \psi_c = -\tan 2\underline{\mathbf{i}}_{p_1} \cos \Phi \tag{10a}$$

$$\sin 2\underline{\mathbf{i}}_{c_1} = \cot \Phi. \tag{17a}$$

To prove:

$$\sin^2 2\underline{\mathbf{i}}_{c_1} = \cos^2 2\underline{\mathbf{i}}_{p_1} \tag{20}$$

from (19a),

$$\frac{1-\cot^2 \Phi}{1+\cot^2 \Phi} = \frac{1-\cos 2\psi_c}{1+\cos 2\psi_c}$$

$$\sin^2 \Phi - \cos^2 \Phi = \frac{\sin 2\psi_c}{\cos 2\psi_c} = \tan^2 \psi_c$$

dividing by $\cos^2 \Phi$ gives

$$\frac{1}{\cot^2 \Phi} - 1 = \frac{\tan^2 \psi_c}{\cos^2 \Phi}$$

or

$$\cot^2 \Phi = \frac{1}{1+\dfrac{\tan^2 \psi_c}{\cos^2 \Phi}}. \tag{23}$$

From (23) and (10a) we have,

$$\cot^2 \Phi = \frac{1}{1+\tan^2 2\underline{\mathbf{i}}_{p_1}} = \cos^2 2\underline{\mathbf{i}}_{p_1}. \tag{24}$$

From (24) and (17a) we have $\sin^2 2\underline{\mathbf{i}}_{c_1} = \cos^2 2\underline{\mathbf{i}}_{p_1}$. (The proof when ψ is measured from γ_2' is similar.)

8. Appendix 2

Given eq (6) and other relations pertaining to the "combination" quarter-wave plate, it is required to prove that the resulting azimuth, θ_2'', is independent of the inclination, $\underline{\mathbf{i}}_1$, of the entering plane-polarized beam.

By (3) the eq (6) becomes

$$\tan 2\theta_2'' = \tan 2\theta_2' \cos \Phi - \frac{\tan \Phi \sin \theta_1'}{\cos 2\theta_2'} \sin \Phi. \tag{6a}$$

In order to eliminate θ_2' and θ_1', we proceed as follows: By (9) and (4), we have

$$\tan 2\theta_2' = \frac{\tan 2\underline{\mathbf{i}}_1 \cos \Phi + \tan 2\psi_c}{1-\tan 2\underline{\mathbf{i}}_1 \cos \Phi \tan 2\psi_c}. \tag{25}$$

Also, we obtain from (9)

$$\frac{\sin 2\theta_1'}{\cos 2\theta_2'} = \frac{1}{\cot 2\theta_1' \cos 2\psi_c - \sin 2\psi_c}. \tag{26}$$

From this, with (4) and (19a)

$$\frac{\sin 2\theta_1'}{\cos 2\theta_2'} = \frac{1}{\dfrac{\cot^2 \Phi}{\tan 2\underline{\mathbf{i}}_1 \cos \Phi} - \sqrt{1-\cot^4 \Phi}} \tag{27}$$

and, by (19a)

$$\tan 2\psi_c \left(\equiv \frac{\sin 2\psi_c}{\cos 2\psi_c}\right) = \frac{\sqrt{1-\cot^4 \Phi}}{\cot^2 \Phi}. \tag{28}$$

Now substitute $\tan 2\psi_c$ from (28) in (25), and put the resulting value of $\tan 2\theta_2'$ and also the value of

$\sin 2\theta_1'/\cos 2\theta_2'$ from (27) in (6a), and we obtain an expression equivalent to the second term of (6) or (6a). Multiply the numerator and denominator by $\sin^2 \Phi$ to get a modified expression, which we call A, i.e.,

$$\tan 2\theta_2'' = A.$$

It will be found that $A \cos \Phi/\sqrt{-\cos 2\Phi}$ turns out to be simply unity (the quantities $\underline{\mathbf{i}}_1$ and Φ happily dropping out). Therefore,

$$\tan 2\theta_2'' = \frac{\sqrt{-\cos 2\Phi}}{\cos \Phi} = \tan 2\psi_c \cos \Phi. \tag{29}$$

The azimuth of the elliptically polarized beam is therefore independent of $\underline{\mathbf{i}}_1$.

9. Appendix 3. Ellipticity of a Light Beam Emerging From an "Adjusted" Combination Plate

We have $\Phi_{\mathrm{I}} = \Phi_{\mathrm{II}} \equiv \Phi$ and ψ is measured from α_{II}'. By substituting (1) and (2) in (5) we obtain

$$\sin 2\epsilon'' = \sin 2\underline{\mathbf{i}}_1 \sin \Phi \cos \Phi + \cos 2\underline{\mathbf{i}}_1 \sin \Phi \sin 2\theta_2'/\cos 2\theta_1'. \tag{30}$$

From (9),

$$\sin 2\theta_2' = \sin 2\theta_1' \cos 2\psi + \cos 2\theta_1' \sin 2\psi, \tag{9c}$$

and, introducing (9c) into (30),

$$\sin 2\epsilon'' = \sin 2\underline{\mathbf{i}}_1 \sin \Phi \cos \Phi + \cos 2\underline{\mathbf{i}}_1 \sin \Phi (\tan 2\theta_1' \cos 2\psi + \sin 2\psi). \tag{31}$$

From (19a),

$$\sin 2\psi_c = \sqrt{1 - \cot^4 \Phi}, \tag{19b}$$

and, introducing (4), (19a), and (19b) into (31),

$$\sin 2\epsilon'' = \sin 2\underline{\mathbf{i}}_1 \sin \Phi \cos \Phi + \cos 2\underline{\mathbf{i}}_1 \sin \Phi (\tan 2\underline{\mathbf{i}}_1 \cos \Phi \cot^2 \Phi + \sqrt{1 - \cot^4 \Phi}). \tag{32}$$

Next, noting that, in general,

$$\sin \Phi \cos \Phi \cot^2 \Phi = \cos^3 \Phi/\sin \Phi$$

and

$$\sin \Phi \cos \Phi + \cos^3 \Phi/\sin \Phi = \cot \Phi$$

also

$$\sin \Phi \sqrt{1 - \cot^4 \Phi} = \sqrt{1 - \cot^2 \Phi},$$

$$\sin 2\epsilon'' = \sin 2\underline{\mathbf{i}}_1 \cot \Phi + \cos 2\underline{\mathbf{i}}_1 \sqrt{1 - \cot^2 \Phi}. \tag{33}$$

From (24),

$$\pm \sin 2\underline{\mathbf{i}}_{p_1} = \sqrt{1 - \cot^2 \Phi}. \tag{24a}$$

From the geometry the sign can be either plus or minus; taking the case where the sign is minus—

$$\sin 2\epsilon'' = \sin 2\underline{\mathbf{i}}_1 \cos 2\underline{\mathbf{i}}_{p_1} - \cos 2\underline{\mathbf{i}}_1 \sin 2\underline{\mathbf{i}}_{p_1}. \tag{34}$$

$$\sin 2\epsilon'' = \sin (2\underline{\mathbf{i}}_1 - 2\underline{\mathbf{i}}_{p_1}), \tag{35}$$

or

$$\epsilon'' = \underline{\mathbf{i}}_1 - \underline{\mathbf{i}}_{p_1} \equiv \Delta. \tag{22}$$

(Paper 69C2-190)