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The Condition of Certain Matrices*

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By reversing the usual direction of application, a common procedure for solving integral equations numerically is used to obtain the asymptotic P-condition numbers of two well-known test matrices.

Key Words: Asymptotic P-condition, integral equations, test matrices.

Todd [1][1] has recently suggested the matrix $A_{17} = (|i-j|)$, $1 \leq i, j \leq n$, as an addition to the well-known collection of test matrices [2]. The P-condition numbers of the matrices are used as a measure of their difficulty for numerical purposes. Where the condition numbers are not explicitly known, the asymptotic behavior in n is given. Lehmer's matrix A_7

$$\begin{aligned} a_{ij} &= i/j \qquad i \leq j \\ &= j/i \qquad j < i \end{aligned}$$

is exceptional in that the correct order in n is not known. A simple idea will allow us to obtain the asymptotic condition number of A_{17} and the correct order for A_7.

Hilbert's first method for integral equations [3] approximates the eigenvalues of the kernel $K(x, y)$, $0 \leq x, y \leq 1$, by those of the matrix

$$\frac{1}{n}\left(K\left(\frac{i}{n}, \frac{j}{n}\right)\right).$$

If K is bounded and Riemann integrable, then the eigenvalues of the matrices tend to those of the integral equation as n tends to ∞. We reverse this procedure. We wish to estimate the behavior of the eigenvalues of a set of matrices as n tends to ∞. If we can regard them as arising from the application of Hilbert's first method to a fixed kernel, then we may hope for an asymptotic result.

To estimate the largest eigenvalue of A_{17}, let us form

$$\frac{1}{n^2}(|i-j|) = \frac{1}{n}\left(\left|\frac{i}{n} - \frac{j}{n}\right|\right).$$

We regard them as arising from the approximation of the kernel

$$K(x, y) = |x-y|.$$

A simple computation gives the largest eigenvalue of K as

$$\lambda_1 = \tfrac{1}{2} z_0^{-2}$$

where z_0 is the unique real root of coth $z = z$. This equation has been studied [4] and z_0 is approximately $z_0 = 1.9967864$. We conclude that the largest eigenvalue of A_{17} is asymptotically

$$\lambda_1 \sim \tfrac{1}{2} z_0^{-2} n^2.$$

It is easy to estimate the accuracy of the approximation using the bounds of [5] but we shall not go into this.

We need the reciprocal of the eigenvalue of A_{17} smallest in modulus. It is obtained as the largest eigenvalue of the inverse matrix.

Let us introduce the matrices

$$M = -2A_{17}^{-1},$$

$$N = \begin{Bmatrix} 1, -1 & & & \\ -1, 2, -1 & & & \\ & \ddots & \ddots & \ddots \\ & -1, 2, -1 & & \\ & & -1, 1 & \end{Bmatrix}$$

and Q where $\begin{cases} q_{11} = q_{1n} = q_{n1} = q_{nn} = \dfrac{1}{n-1} \\ q_{ij} = 0 \text{ otherwise.} \end{cases}$

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[1] Figures in brackets indicate the literature references at the end of this paper.

It is readily verified that

$$A_{17}^{-1} = \tfrac{1}{2}(Q - N)$$

or

$$M + Q = N.$$

Q is obviously a positive semidefinite matrix. By a well-known inequality [6, p. 73],

$$\lambda_1(N) = \lambda_1(M + Q) \geqslant \lambda_1(M).$$

Rutherford [7] has shown that the eigenvalues of N are

$$4 \cos^2 \frac{i\pi}{2n} \qquad i = 1, 2, \ldots, n.$$

and the eigenvector associated with $\lambda_1(N)$ is

$$x_j = \sin \frac{j\pi}{n} - \sin \frac{(j-1)\pi}{n} \qquad j = 1, \ldots, n.$$

The Rayleigh quotient furnishes a lower bound to $\lambda_1(M)$. A convenient approximate eigenvector is that associated with $\lambda_1(N)$. For this vector

$$\lambda_R(M) = \frac{(x, Mx)}{(x, x)} = \lambda_1(N) - \frac{(x_1 + x_n)^2}{(n-1)(x, x)}.$$

A little algebraic manipulation gives

$$\lambda_R(M) = \lambda_1(N) - \frac{8 \sin^2 \dfrac{\pi}{n}}{n(n-1)\lambda_1(N)}.$$

Thus

$$\lambda_R(M) \leqslant \lambda_1(M) \leqslant \lambda_1(N)$$

and we see

$$\lambda_1(M) = \lambda_1(N) + 0(n^{-4}).$$

From this

$$\left| \frac{1}{\lambda_n(A_{17})} \right| = 2 \cos^2 \frac{\pi}{2n} + 0(n^{-4})$$

and finally

$$P(A_{17}) \sim z_0^{-2} n^2 \cos^2 \frac{\pi}{2n}.$$

The same methods applied to the kernel

$$K(x, y) = \begin{cases} x/y & x \leqslant y \\ y/x & y < x \end{cases}$$

show that for A_7, $\lambda_1 = 0(1)$. The inverse is explicitly

$$\begin{array}{ll} \dfrac{4i^3}{4i^2 - 1} & i = j,\ i < n \\[2ex] \dfrac{n^2}{2n-1} & i = j = n \\[2ex] -\dfrac{i(i+1)}{2i+1} & |i - j| = 1 \\[2ex] 0 & |i - j| > 1. \end{array}$$

Gerschgorin's theorem immediately gives $\left|\dfrac{1}{\lambda_n}\right| = 0(n)$. Todd and Newman comment that $P(A_7) \geqslant n$. From this we conclude that $P(A_7) \sim cn$ where c is a constant.

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