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Lattice Points in a Sphere

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Let $R_3(x)$ be the remainder in the classical lattice point problem for a 3-sphere of radius $\sqrt{x}$ and center (0, 0, 0). We prove that as $x \to +\infty$,

$$R_3(x) = O(x^{3/4} \log x)$$

and

$$R_3(x) = \Omega(x^{1/2} \log \log x).$$

Key Words: Additive analytic number theory, lattice points, remainder formulas.

1. Introduction

In this paper we consider the classical lattice point problem for the three-dimensional sphere. The problem can be described as follows. Let x be a positive real number and let k be a positive integer. Consider a k-dimensional sphere of radius $\sqrt{x}$ and center (0, . . ., 0). Following the notation of Walfisz [4],[2] we let $A_k(x)$ be the number of integer lattice points in this sphere. A simple geometric argument shows that as $x \to +\infty$, $A_k(x) \sim V_k(x)$, where $V_k(x)$ is the volume of the sphere in question. The problem then is to get an asymptotic estimate of the difference $R_k(x) = A_k(x) - V_k(x)$.

Here we are considering only $R_3(x) = A_3(x) - \frac{4}{3}\pi x^{3/2}$. We obtain the following results:

$$R_3(x) = O(x^{3/4} \log x),\ x \to +\infty \tag{1}$$

$$R_3(x) = \Omega(x^{1/2} \log \log x),\ x \to +\infty. \tag{2}$$

Of course (1) is not new. Vinogradov [3] has in fact shown that $R_3(x) = O(x^{\frac{19}{28}+\epsilon})$, $\epsilon > 0$, an upper estimate better than (1). However this result depends upon his difficult theory of exponential sums. Our estimate (1), on the other hand, is better than the elementary result $A_3(x) = O(x)$ and depends only upon a fairly standard application of the circle method.

As far as we can ascertain (2) is new. It is based upon the Ω-estimate for $R_4(x)$ [4, p. 95]

$$R_4(x) = \Omega(x \log \log x),\ x \to +\infty. \tag{3}$$

Walfisz [4, p. 94] gives only $R_3(x) = \Omega(x^{1/2})$, $x \to +\infty$. In [1] it is shown that $\lim\limits_{x\to\infty} |R_3(x)x^{-1/2}| = +\infty$, but this is of course weaker than (3).

2. Preliminaries

Landau's formula for $A_k(x) \cdot (k \geq 4)$ is [4, p. 29]

$$A_k(x) = \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2}\right)} \sum_{1 \leq q \leq x^{1/2}} \sum_{h (\mathrm{mod}\ q)}{}' \left(\frac{S(h,q)}{q}\right)^k \cdot \sum_{1 \leq n \leq x} n^{k/2-1} e^{-2\pi i n h/q} + O(x^{k/4} \log x),\ x \to +\infty. \tag{4}$$

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[2] Figures in brackets indicate the literature references at the end of this paper.

Here $S(h, q) = \sum_{a(\mathrm{mod}\ q)} e^{2\pi iha^2/q}$ is the famous Gaussian sum about which we need only the fact that

$$|S(h, q)| \leqslant Kq^{1/2}, \tag{5}$$

where K is independent of h and q [4, p. 10]. The notation Σ' indicates that we are to sum over only those h such that $(h, q)=1$.

If (4) held for $k=3$ we could apply it to derive (1) without much difficulty. However since the proof of (4) given in [4] fails for $k<4$, we replace it for $k=3$ with the following formula obtainable by the same general method.

$$A_3(x) = 2\pi \sum_{n\leqslant x} n^{1/2} + O(x^{3/4}\log x),\ x\to+\infty. \tag{6}$$

Once we have (6), (1) is easily obtainable.

We will also need the following standard result [4, p. 25].

LEMMA *1* (*Euler Summation Formula*). *Let* $\Psi(t)=t-[t]-\frac{1}{2}$. *If* $f(t)$ *has a continuous derivative in the interval* $a\leqslant t\leqslant b(a<b)$, *then*

$$\sum_{a<m\leqslant b} f(m) = \int_a^b f(t)dt + \Psi(a)f(a) - \Psi(b)f(b) + \int_a^b \Psi(t)f'(t)dt. \tag{7}$$

This is proved by integrating $\int_a^b \Psi(t)f'(t)dt$ by parts.

3. Proof of (6) and (1)

Many of the calculations done in the proof of (4) [4, pp. 29–35] are valid for $k=3$. In particular we have [4, p. 33, formula (21)]

$$A_3(x) = \sum_{q\leqslant x^{1/2}} \sum_{h(\mathrm{mod}\ q)}{}' \left(\frac{S(h, q)}{q}\right)^3 \int_{\theta(h, q)} w^{-3/2}$$

$$\sum_{n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi in\left(y+\frac{h}{q}\right)\right\} dy + O(x^{3/4}\log x),\ x\to+\infty. \tag{8}$$

In (8), $w=x^{-1}-2yi$, and $\theta(h, q)$ is an interval described as follows. Let h'/q' and h''/q'' be the two Farey fractions of order $x^{1/2}$ closest to h/q with say $h'/q'<h/q<h''/q''$, and consider the interval $\left[\frac{h'+h}{q'+q}, \frac{h+h''}{q+q''}\right]$. Then $\theta(h, q)$ is obtained from this interval by translating h/q to the origin, that is,

$$\theta(h, q) = \left[\frac{h'+h}{q'+q} - \frac{h}{q}, \frac{h+h''}{q+q''} - \frac{h}{q}\right].$$

For our purpose here the essential fact about $\theta(h, q)$ is [4, p. 30]

$$\begin{cases} |y| \leqslant q^{-1}x^{-1/2}, \text{ for } y\epsilon\theta(h, q) \\ |y| \geqslant 2^{-1}q^{-1}x^{-1/2}, \text{ for } y\notin\theta(h, q). \end{cases} \tag{9}$$

for any Farey fraction h/q or order $x^{1/2}$.

By (8) we have

$$A_3(x)=\int_{\theta(0,1)} w^{-3/2}\sum_{n\le x}\exp\left\{\frac{\pi n}{x}-2\pi iny\right\}dy$$

$$+\sum_{2\le q\le x^{1/2}}\ \sideset{}{'}\sum_{h(\mathrm{mod}\ q)}\left(\frac{S(h,q)}{q}\right)^3\int_{\theta(h,q)} w^{-3/2}\sum_{n\le x}\exp\left\{\frac{\pi n}{x}-2\pi in\left(y+\frac{h}{q}\right)\right\}dy$$

$$+O(x^{3/4}\log x),\ x\to+\infty. \tag{10}$$

Again we observe that the calculations of [4, pp. 33–34] are valid for $k=3$. These yield

$$\int_{\theta(0,1)} w^{-3/2}\sum_{n\le x}\exp\left\{\frac{\pi n}{x}-2\pi iny\right\}dy=\int_{-\infty}^{\infty} w^{-3/2}\sum_{n\le x}\exp\left\{\frac{\pi n}{x}-2\pi iny\right\}dy+O(x^{3/4}),\ x\to+\infty.$$

Now

$$\int_{-\infty}^{\infty} w^{-3/2}\sum_{n\le x}\exp\left\{\frac{\pi n}{x}-2\pi iny\right\}dy=\sum_{n\le x}e^{\pi n/x}\int_{-\infty}^{\infty} w^{-3/2}e^{-2\pi iny}dy,$$

and by [4, p. 35] (again valid for $k=3$),

$$\int_{-\infty}^{\infty} w^{-3/2}e^{-2\pi iny}dy=\frac{\pi^{3/2}}{\Gamma(3/2)}e^{-\pi n/x}n^{1/2}=2\pi e^{-\pi n/x}n^{1/2}.$$

Thus, we have

$$\int_{-\infty}^{\infty} w^{-3/2}\sum_{n\le x}\exp\left\{\frac{\pi n}{x}-2\pi iny\right\}dy=2\pi\sum_{n\le x}n^{1/2},$$

and (10) becomes

$$A_3(x)=2\pi\sum_{n\le x}n^{1/2}+\sum_{2\le q\le x^{1/2}}\ \sideset{}{'}\sum_{h(\mathrm{mod}\ q)}\left(\frac{S(h,q)}{q}\right)^3$$

$$\int_{\theta(h,q)} w^{-3/2}\sum_{n\le x}\exp\left\{\frac{\pi n}{x}-2\pi in\left(y+\frac{h}{q}\right)\right\}dy+O(x^{3/4}\log x),\ x\to+\infty. \tag{11}$$

Let Σ denote the multiple sum on the right-hand side of (11); to prove (6) it is sufficient to show that $\Sigma=O(x^{3/4}\log x)$, as $x\to+\infty$.

By (5) and (9),

$$|\Sigma|\le K\sum_{2\le q\le x^{1/2}}q^{-3/2}\ \sideset{}{'}\sum_{h(\mathrm{mod}\ q)}\int_{|y|\le q^{-1}x^{-1/2}}|w|^{-3/2}\left|\sum_{n\le x}\exp\left\{\frac{\pi n}{x}-2\pi in\left(y+\frac{h}{q}\right)\right\}\right|dy. \tag{12}$$

We apply the familiar method of partial summation to estimate the inner sum. Let $T(n)=\sum_{1\le k\le n}e^{-2\pi ik(y+h/q)}$. Then since $T(n)$ is a geometric series

$$|T(n)|\le 2|e^{\pi i(y+h/q)}-e^{-\pi i(y+h/q)}|^{-1}=\left|\sin\pi\left(y+\frac{h}{q}\right)\right|^{-1}.$$

Since $|y|\le q^{-1}x^{-1/2}$, $q^{-1}(h-x^{-1/2})\le y+\dfrac{h}{q}\le q^{-1}(h+x^{-1/2})$, while $q\ge 2$ implies that $1\le h\le q-1$;

thus if $x \geqslant 1$ (say), $O \leqslant y+\frac{h}{q} \leqslant 1$. Therefore

$$|\sin \pi \left(y+\frac{h}{q}\right)|^{-1} \leqslant \max \left\{\frac{1}{2(y+h/q)}, \frac{1}{2(1-y-h/q)}\right\}.$$

Also,

$qy+h \geqslant h-x^{-1/2} \geqslant h-\frac{1}{2}$, and $q-qy-h \geqslant q-h-x^{-1/2} \geqslant q-h-\frac{1}{2}$, if $x \geqslant 4$. We conclude that

$$|T(n)| \leqslant q\left\{\frac{1}{2h-1}+\frac{1}{2q-2h-1}\right\} \leqslant q\left\{\frac{1}{h}+\frac{1}{q-h}\right\}.$$

Now,

$$\sum_{1\leqslant n\leqslant x} \exp\left\{\frac{\pi n}{x}-2\pi i n\left(y+\frac{h}{q}\right)\right\}=\sum_{1\leqslant n\leqslant x} e^{\pi n/x}\{T(n)-T(n-1)\}$$

$$=\sum_{1\leqslant n\leqslant x} T(n)\{e^{\pi n/x}-e^{\pi(n+1)/x}\}+e^{\pi([x]+1)/x}T([x]),$$

and we have

$$\left|\sum_{1\leqslant n\leqslant x} \exp\left\{\frac{\pi n}{x}-2\pi i n\left(y+\frac{h}{q}\right)\right\}\right| \leqslant q\left\{\frac{1}{h}+\frac{1}{q-h}\right\} \sum_{1\leqslant n\leqslant x}\{e^{\pi(n+1)/x}-e^{\pi n/x}\}$$

$$+q\left\{\frac{1}{h}+\frac{1}{q-h}\right\} e^{\pi([x]+1)/x} \leqslant 2q\left\{\frac{1}{h}+\frac{1}{q-h}\right\} e^{\pi(x+1)/x} \leqslant K'q\left\{\frac{1}{h}+\frac{1}{q-h}\right\},$$

where K' is independent of h, q, and x. This, with (12), leads to

$$\sum=O\left(\sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2} \sum_{h \pmod q}{}' \left\{\frac{1}{h}+\frac{1}{q-h}\right\} \int_0^{q^{-1}x^{-1/2}} |w|^{-3/2}\, dy\right), x \to +\infty.$$

But $|w|^{-3/2}=x^{3/2}(1+4x^2y^2)^{-3/4} \leqslant \min\{x^{-3/2}, (2y)^{-3/2}\}$, so that

$$\sum=O\left(\sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2} \sum_{h \pmod q}{}' \left\{\frac{1}{h}+\frac{1}{q-h}\right\}\left\{\int_0^{x^{-1}} x^{3/2}\, dy+\int_{x^{-1}}^{q^{-1}x^{-1/2}} y^{-3/2}\, dy\right\}\right)$$

$$=O\left(\sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2} \sum_{h \pmod q}{}' \left\{\frac{1}{h}+\frac{1}{q-h}\right\} x^{1/2}\right)$$

$$=O\left(x^{1/2} \sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2} \log q\right)$$

$$=O(x^{3/4}\log x), \text{ as } x \to +\infty,$$

and (6) is proved.

To obtain (1) we simply apply (7) to $\sum_{1\leqslant n\leqslant x} n^{1/2}$. This gives

$$\sum_{1\leqslant n\leqslant x} n^{1/2}=\int_0^x t^{1/2}dt-\Psi(x)x^{1/2}+\frac{1}{2}\int_0^x \Psi(t)\cdot t^{-1/2}dt=\frac{2}{3}x^{3/2}+O(x^{1/2}), \; x \to +\infty.$$

Together with (6), this implies

$$A_3(x)=\frac{4}{3}\pi x^{3/2}+O(x^{3/4}\log x),\ x\to+\infty,$$

and the proof of (1) is complete.

4. Proof (2)

We begin with two lemmas (cf. [4, pp. 49–50]).

Lemma 2. $A_k(x)=\sum_{-\sqrt{x}\leq m\leq\sqrt{x}} A_{k-1}(x-m^2)$, *for* $k\geq 2$.

Proof: Clear.

Lemma 3. $\sum_{-\sqrt{x}\leq m\leq\sqrt{x}}(x-m^2)^{k/2}=\int_{-\sqrt{x}}^{\sqrt{x}}(x-t^2)^{k/2}dt+O(x^{\frac{k-1}{2}}),\ x\to+\infty.$

Proof: By Lemma 1,

$$\sum_{-\sqrt{x}\leq m\leq\sqrt{x}}(x-m^2)^{k/2}=\sum_{-\sqrt{x}<m\leq\sqrt{x}}(x-m^2)^{k/2}=\int_{-\sqrt{x}}^{\sqrt{x}}(x-t^2)^{k/2}dt-k\int_{-\sqrt{x}}^{\sqrt{x}}\Psi(t)(x-t^2)^{\frac{k}{2}-1}t\,dt.$$

But by the second mean value theorem of the integral calculus,

$$\int_{-\sqrt{x}}^{\sqrt{x}}\Psi(t)(x-t^2)^{\frac{k}{2}-1}t\,dt=O(x^{\frac{k}{2}-1+\frac{1}{2}})=O(x^{\frac{k-1}{2}}),\text{ as }x\to+\infty,$$

since $\int_{-\sqrt{x}}^{\sqrt{x}}\Psi(t)dt$ is bounded, independently of x.

To prove (2) we assume

$$R_3(x)=o(x^{1/2}\log\log x),\ x\to+\infty, \tag{13}$$

and show that this leads to a contradiction. By lemma 2, and the definition of $R_3(x)$,

$$A_4(x)=\sum_{-\sqrt{x}\leq m\leq\sqrt{x}}A_3(x-m^2)=\frac{4}{3}\pi\sum_{-\sqrt{x}\leq m\leq\sqrt{x}}(x-m^2)^{3/2}+\sum_{-\sqrt{x}\leq m\leq\sqrt{x}}R_3(x-m^2).$$

By (13), given any $\epsilon>0$ there exists $N>3$ such that if $x>N$, then $|R_3(x)|<\epsilon x^{1/2}\log\log x$. Also (13) implies that for *any* $x>3$, $|R_3(x)|<Kx^{1/2}\log\log x$, where K is independent of x. Therefore, assuming that $x>N$, we have

$$\left|\sum_{-\sqrt{x}\leq m\leq\sqrt{x}}R_3(x-m^2)\right|\leq\sum_{-\sqrt{x-N}<m<\sqrt{x-N}}\left|R_3(x-m^2)\right|+\sum_{\sqrt{x-N}\leq|m|\leq\sqrt{x}}\left|R_3(x-m^2)\right|$$

$$<2\epsilon(x-N)^{1/2}x^{1/2}\log\log x+\frac{KN}{(x-N)^{1/2}}x^{1/2}\log\log x$$

$$+R_3(0)+R_3(1)+R_3(2),$$

where we have used the fact that $x^{1/2} \log\log x$ is monotone and observed that there are at most $N/(x-N)^{1/2}$ integers in the range $\sqrt{x-N} \leq |m| \leq \sqrt{x}$. Now holding N fixed and letting $x \to +\infty$, we have

$$\overline{\lim_{x\to+\infty}} \frac{\left| \sum_{-\sqrt{x}<m<\sqrt{x}} R_3(x-m^2) \right|}{x \log\log x} \leq 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$\sum_{-\sqrt{x}\leq m\leq\sqrt{x}} R_3(x-m^2) = o(x \log\log x), \text{ as } x \to +\infty,$$

so that

$$A_4(x) = \frac{4}{3}\pi \sum_{-\sqrt{x}\leq m\leq\sqrt{x}} (x-m^2)^{3/2} + o(x \log\log x),\ x \to +\infty.$$

Lemma 3, with $k=3$, implies that

$$\sum_{-\sqrt{x}\leq m\leq\sqrt{x}} (x-m^2)^{3/2} = \frac{3}{8}\pi x^2 + O(x),\ x \to +\infty,$$

and we get

$$A_4(x) = \frac{\pi x^2}{2} + o(x \log\log x),\ x \to +\infty,$$

in contradiction to (3). Thus (13) is impossible, and the proof of (2) is complete.

5. Remarks

1. The method used here is the derivation of a o-estimate for $R_4(x)$ from an assumed o-estimate for $R_3(x)$. Thus an improved Ω-estimate for $R_4(x)$ would immediately give an improvement on (2), by the same method.

2. This process can be applied to give an O-estimate for $R_3(x)$, given an O-estimate for $R_2(x)$. If we start with Vinogradov's result [2]

$$R_2(x) = O(x^{\frac{17}{53}+\epsilon}),\ \epsilon > 0,\ x \to +\infty,$$

we get

$$R_3(x) = O(x^{\frac{87}{106}+\epsilon}),\ \epsilon > 0,\ x \to +\infty,$$

an estimate which is, however, weaker than (1).

6. References

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