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A Variant of the Two-Dimensional Riemann Integral

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For a variant of the two-dimensional Riemann integral suggested by S. Marcus, it is shown that the only integrable functions which are continuous (or merely continuous separately in one of the variables) are the constant functions. The integrable discontinuous functions are proven to be constant except possibly on a set which is "small" in a sense made precise in the paper.

In this note the term *rectangle* means a nondegenerate closed rectangular region in the (x, y)-plane, with sides parallel to the coordinate axes. The area and maximum side length of a rectangle R are denoted $|R|_A$ and $|R|_S$ respectively.

A *partition* σ of rectangle R is a finite collection of nonoverlapping rectangles with R as union. If f is a real-valued function defined on R, then to each partition σ there corresponds the class of *Riemann sums*

$$\Sigma\{f(P_J)\cdot|J|_A:J\epsilon\sigma\}$$

resulting from various choices of points $P_J\epsilon J$ for each $J\epsilon\sigma$. The ordinary Riemann integral of f over R can be defined as the limit (if it exists) of Riemann sums for partitions σ with

$$|\sigma|_S=\max\{|J|_S:J\epsilon\sigma\}\rightarrow 0. \tag{1}$$

Marcus [1] proposed studying the analogous integral—which we will call the *(A)-integral* . . . which arises when (1) is replaced by

$$|\sigma|_A=\max\{|J|_A:J\epsilon\sigma\}\rightarrow 0. \tag{2}$$

He observed that the continuous function $f(x, y)=x$ fails to be (A)-integrable.

We shall show in the present note that this is all too typical. It follows readily from our first theorem that the only (A)-integrable functions which are continuous (or merely continuous in at least one variable at each point of R) are the *constants.* The following results and arguments extend in a natural way to higher dimensions.

[1] S. Marcus, On the Riemann Integral in Two Dimensions, Amer. Math. Monthly **71** (1964), pp. 544–545.

THEOREM 1. *Let* f *be bounded on the rectangle*

$$\mathrm{R}=[\mathrm{a},\mathrm{b}]\times[\mathrm{c},\mathrm{d}].$$

Then f *is* (A)-*integrable on* R *if and only if there exists a constant* C *with the property that for each* $\mathrm{n}>0$, *the projections* X_n *on* [a, b] *and* Y_n *on* [c, d] *of the set*

$$\mathrm{S}_\mathrm{n}=\{\mathrm{P}\epsilon\mathrm{R}:|\mathrm{f(P)}-\mathrm{C}|\geqslant\mathrm{n}^{-1}\}$$

have closures $\overline{\mathrm{X}}_\mathrm{n}$ *and* $\overline{\mathrm{Y}}_\mathrm{n}$ *of measure zero.*

PROOF: (a) *Necessity.* Let f be (A)-integrable over R. For $x\epsilon[a, b]$, set

$$f^+(x)=\sup\{f(x, y):y\epsilon[c, d]\},$$

$$f^-(x)=\inf\{f(x, y):y\epsilon[c, d]\},$$

and let $h^+(x)$ and $h^-(x)$ be members of $[c, d]$ such that

$$f(x, h^+(x))\geqslant f^+(x)-1/3\,[f^+(x)-f^-(x)],$$

$$f(x, h^-(x))\leqslant f^-(x)+1/3\,[f^+(x)-f^-(x)].$$

The value I of the (A)-integral of f over R can be obtained as the limit of Riemann sums involving partitions σ of R into vertical strips J. We may on the one hand choose all the points P_J to have the form $(x_J, h^+(j))$, or on the other hand to have the form $(x_J, h^-(x_J))$. This shows that

$$I=(d-c)\int_a^b f(x, h^+(x))dx=(d-c)\int_a^b f(x, h^-(x))dx$$

(ordinary 1-dimensional Riemann integrals), and thus that

$$\int_a^b [f(x, h^+(x))-f(x, h^-(x))]=0. \tag{3}$$

From this and the inequality

$$F(x)=f^+(x, h^+(x))-f^-(x, h^-(x)) \geqslant 1/3[f^+(x)-f^-(x)] \geqslant 0 \tag{4}$$

it follows that $f^+=f^-$ except on a subset of $[a, b]$ of measure 0. Thus there is a set $X \subset [a, b]$ of measure $b-a$, and a function $g(x)$ defined on X, such that

$$f(x, y)=g(x) \qquad \text{for } (x, y)\epsilon X \times [c, d].$$

We can alternately obtain I as the limit of Riemann sums involving partitions of R into horizontal strips J, with all the points P_J having the same $x\epsilon X$ as abscissa. This shows that

$$I=(b-a)\,(d-c)g(x) \qquad \text{for } x\epsilon X$$

and thus that g is constant, i.e.,

$$f(x, y)=C \qquad \text{for } (x, y)\epsilon X \times [c, d].$$

Similarly, there is a constant C' and a set $Y \subset [c, d]$ of measure $d-c$ such that $f=C'$ on $[a, b]\times Y$. Since $X \times Y$ is nonempty, we infer that $C=C'$ and thus that

$$f=C \qquad \text{on } (X\times[c, d])\cup([a, b]\times Y). \tag{5}$$

For the nonnegative integrand F of eq (3), set

$$Z_n=\{x\epsilon[a, b]: F(x) \geqslant 1/3n\}.$$

We shall prove that $X_n \subset \bar{Z}_n$ and that $\mu(\bar{Z}_n)=0$; it follows (as desired) that $\mu(\bar{X}_n)=0$ and similarly that $\mu(\bar{Y}_n)=0$.

Consider any $x\epsilon X_n$. By (5), $f(x, y)=C$ for some $y\epsilon[c, d]$. Hence $x\epsilon X_n$ implies that $f^+(x)-f^-(x) \geqslant n^{-1}$ and thus, by (4), that $x\epsilon Z_n$, i.e., $X_n \subset Z_n$. Now consider any partition π of $[a,b]$.

$$\begin{aligned}\mu(\bar{Z}_n)&=\Sigma\,\{\mu(\bar{Z}_n \cap K): K\epsilon\pi\}\\ &=\Sigma\,\{\mu(\bar{Z}_n \cap K): K\epsilon\pi, \mu(\bar{Z}_n \cap K)>0\}\\ &\leqslant \Sigma\,\{\mu(\bar{Z}_n \cap K): K\epsilon\pi, Z_n \cap K \neq \phi\}\\ &\leqslant \Sigma\,\{\,|K|: K\epsilon\pi, Z_n \cap K \neq \phi\},\end{aligned}$$

where we have used the fact that $\mu(\bar{Z}_n \cap K)>0$ only if $\bar{Z}_n$, and hence Z_n, meets the interior of K. For each $K\epsilon\pi$ choose $x_K\epsilon K$, with the proviso that $x_K\epsilon Z_n$ if $Z_n \cap K \neq \phi$. Then the preceding chain of inequalities can be continued to

$$\begin{aligned}\mu(\bar{Z}_n) &\leqslant 3n\,\Sigma\,\{F(x_K)\cdot|K|: K\epsilon\pi, Z_n \cap K \neq \phi\}\\ &\leqslant 3n\,\Sigma\,\{F(x_K)\cdot|K|: K\epsilon\pi\}\cdot\end{aligned}$$

For fixed n, it follows from (3) that the last line (a Riemann sum) can be made arbitrarily close to 0 for suitable π. Thus $\mu(\bar{Z}_n)=0$, completing the necessity proof.

(b) *Sufficiency.* To clarify the situation, suppose initially only that $f=C$ on a dense subset of R. Then the oscillation of f at each point of $\bar{S}_n$ is $\geqslant 1/n$. On the other hand any point of $\underline{R}$, at which the oscillation of f is $> 2/n$, must lie in $\bar{S}_n$. From these two easily proved remarks, plus well-known material on Riemann integrability, it follows that f is Riemann integrable if and only if each $\bar{S}_n$ has measure zero, a requirement weaker than the condition for (A)-integrability given in the theorem.

Suppose now that f satisfies the condition. Replace f by $f-C$, which is (A)-integrable if and only if f is. Thus we can assume $C=0$. Next introduce

$$f^{(+)}=\max\,(f, 0) \qquad f^{(-)}=\max\,(-f, 0);$$

then $f=f^{(+)}-f^{(-)}$ is (A)-integrable over R if $f^{(+)}$ and $f^{(-)}$ are. Thus we can assume at the outset that $f \geqslant 0$, and that X_n and Y_n are the projections of

$$S_n=\{P\epsilon R\,; f(P) \geqslant n^{-1}\}\cdot$$

Consider any $\delta>0$. First choose n so large that

$$(b-a)\,(d-c)n^{-1}<\delta/5. \tag{6}$$

Since $\mu(\bar{X}_n)=0$, we can cover $\bar{X}_n$ by the union of the interiors of a finite set $\{I_i: 1 \leqslant i \leqslant m\}$ of closed subintervals of $[a, b]$, such that

$$\sum_1^m |I_i| < \{\delta/5B\}^{1/2}, \tag{7}$$

$$\sum_1^m |I_i| < \delta/5B(d-c), \tag{8}$$

where B is an upper bound for f on R. Similarly, we can cover $\bar{Y}_n$ by the union for the interiors of a finite set $\{I_j': 1 \leqslant j \leqslant n\}$ of closed subintervals of $[c, d]$, such that

$$\sum_1^n |I_j'| < \{\delta/5B\}^{1/2}, \tag{9}$$

$$\sum_1^n |I_j'| < \delta/5B(b-a). \tag{10}$$

In terms of the $N=mn$ rectangles $R_{ij}=I_i\times I_j'$, we have

$$S_n \subset \cup_{\,i,j}R_{ij}. \tag{11}$$

Now consider any partition σ of R such that

$$|\sigma|_A < \delta/20BN, \tag{12}$$

and any associated Riemann sum

$$\Sigma\{f(P_J)\cdot|J|_A: J\epsilon\sigma\}\cdot \tag{13}$$

We partition the members of σ, and thus the summands of (13), into five classes. First are the members J of σ which do not meet any R_{ij}, and hence by (11) do not meet S_n. Then $f(P_j) < 1/n$ for each such J, and so by (6) these J's contribute $< \delta/5$ to (13). Second come the members of σ which lie within some R_{ij}; the contribution of these to (13) is $\leqslant B\sum_{i,j}|R_{ij}|_A$, which by (7) and (9) is $< \delta/5$. Third are the members of σ, not considered so far, which contain a vertex of some R_{ij}; there are at most $4N$ of these, each of area $\leqslant |\sigma|_A$, and so by (12) they add $< \delta/5$ to (13). Fourth come those members of σ, not previously considered, which meet a horizontal side of some R_{ij}. Among these, consider for example those which for fixed i meet at least one horizontal side of any

$$R_{ij}=[a_i, b_i]\times[c_j, d_j]=I_i\times I'_j.$$

Because these rectangles contain no vertex of any R_{ij}, they all lie in the vertical strip $(a_i, b_i)\times[c, d]$, so that their contribution to (13) is $< B(d-c)|I_i|$. Summing over i, we find that the contribution of the fourth class to (13) is $\leqslant B(d-c)\sum_1^m |I_i|$, which by (8) is $< \delta/5$. The fifth class is analogous to the fourth except for referring to the vertical sides of the rectangles R_{ij}; like the fourth class it contributes $< \delta/5$ to (13).

We have shown that the Riemann sum (13) is $< \delta$ whenever (12) holds. Since $f \geqslant 0$, it follows that the (A)-integral of f over R exists (and is zero), completing the sufficiency proof.

THEOREM 2. *Let* X* *and* Y* *be subsets of* [a, b] *and* [c, d] *respectively. A necessary and sufficient condition for the existence of an* (A)-*integrable function on* R, *constant except on a set whose projections are* X* *and* Y*, *is that*

$$X^*=\bigcup_n X^*_n, \qquad Y^*=\bigcup_n Y^*_n$$

where each

$$\mu(\overline{X}^*_n)=\mu(\overline{Y}^*_n)=0.$$

PROOF: Necessity follows from theorem 1. To prove sufficiency, assume X^* and Y^* admit representations of the indicated type. Set

$$X_n=\bigcup_1^n X^*_m, \qquad Y_n=\bigcup_1^n Y^*_m;$$

then we have

$$X^*=\bigcup_n X_n, \qquad Y^*=\bigcup_n Y_n,$$

$$\mu(\overline{X}_n)=0, \qquad \mu(\overline{Y}_n)=0.$$

Let F_n and G_n be the characteristic functions of X_n and Y_n respectively, and set

$$F(x)=\sup\{n^{-1}F_n(x)\},$$

$$G(y)=\sup\{n^{-1}G_n(y)\},$$

$$f(x,y)=\min\{F(x), G(y)\}.$$

Then $f \geqslant 0$, and furthermore

$$X^*\times Y^*=\{P\epsilon R: f(P)>0\},$$

$$X_n\times Y_n=\{P\epsilon R: f(P)\geqslant n^{-1}\},$$

so that f is (A)-integrable by theorem 1.

THEOREM 3. *Let* X* *and* Y* *be subsets of* [a, b] *and* [c, d] *respectively. A necessary and sufficient condition for the* (A)-*integrability of every bounded function which is constant except on a set whose projections are* X* *and* Y*, *is that the closures of* X* *and* Y* *have measure zero.*

PROOF: Sufficiency follows directly from theorem 1. To prove necessity, let f be the characteristic function of $X^*\times Y^*$; then in the notation of theorem 1 we have $X_1=X^*$ and $Y_1=Y^*$, so that by theorem 1 the (A)-integrability of f requires that the closures of X^* and Y^* be of zero measure.

By specializing $f(x, y)$, we can obtain results on 1-dimensional integration (which could of course be proved much more directly). Let F be a function defined on $[a, b]$ and X^* a subset of $[a, b]$, and consider the conditions

$$F\geqslant 0, \tag{14}$$

$$X^*=\{x\epsilon[a, b]: F(x)>0\}, \tag{15}$$

$$\int_a^b F(x)dx=0. \tag{16}$$

Note that (16) asserts the *existence* along with the vanishing of the Riemann integral.

THEOREM 4. *A necessary and sufficient condition for the existence of an* F *satisfying* (14), (15), (16) *is that* X* *be an at most countable union of sets, each with closure of measure zero. A necessary and sufficient condition that every bounded* F *obeying* (14) *and* (15) *should also satisfy* (16), *is that the closure of* X* *have measure zero.*

PROOF: (a) To any F satisfying (14) and (15), we can associate a function f defined on $R=[a, b]\times[0, 1]$ by setting

$$f(x, 0)=F(x), f(x, y)=0 \qquad \text{for } y\epsilon(0, 1].$$

Then f is zero except precisely on $X^*\times\{0\}$, and is bounded if and only if F is. It is easy to show that f has a (necessarily vanishing) (A)-integral over R if and only if (16) holds.

If there exists an F satisfying (14), (15), and (16), then the corresponding f is (A)-integrable and so X^* is as in theorem 2. This proves the necessity part of the first assertion.

If the closure of X^* has measure zero, then for each bounded F obeying (14) and (15), it follows from theorem 3 that the corresponding f is (A)-integrable and thus that F satisfies (16). This proves the sufficiency part of the second assertion.

(b) Assume X^* is an at most countable union of sets whose closures have measure zero. By the sufficiency proof of theorem 2, there is an (A)-integrable function f on R such that $f \geqslant 0$ and

$$X^* \times \{0\} = \{P \epsilon R : f(P) > 0\}.$$

This implies that $F(x) = f(x, 0)$ satisfies (14), (15), and (16). Thus the sufficiency part of the first assertion is proved.

(c) Suppose every bounded F which obeys (14) and (15) also satisfies (16). In particular this applies to the characteristic function F_0 of X^*. The function f_0, which is associated to F_0 as in (a), is simply the characteristic function of $X^* \times \{0\}$. Because F_0 obeys (16), f_0 must be (A)-integrable, and as in the necessity proof of theorem 3 it follows that the closure of X^* has measure zero. Thus the necessity part of the second assertion is proved.

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