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On Convex Metrics

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It is shown that a metric on a linear space, if convex in each variable, must also be invariant under translation, and so must arise from a norm. The question occurs in connection with determining the optimal location of a central facility.

Introduction

Given a finite number of points $p_1, \ldots, p_k$ in the plane, consider the problem of finding a point x that minimizes the sum of Euclidean distances $\Sigma d(p_i, x)$. More general versions of this problem arise in spatial economics, concerning optimal locations for a central office, plant, or warehouse (compare [3]). Most of these will be based on metrics d more general than the Euclidean metric. Among them, the class of metrics that are convex functions in each variable command particular interest: in this case, local minima are automatically global minima, facilitating minimization decisively. We shall show in this paper that convex metrics are invariant under translation, and therefore arise from a norm. For the concepts of topologies, metrics, and norms in linear spaces see, for instance, [1] [2] and [2].

1. *Metrics and norms.* Let L be a linear space over the field R of real numbers. A function $f: L \to R$ is *convex* if

$$f(\lambda x + \mu y) \leqslant \lambda f(x) + \mu f(y)$$

holds for all $x, y \in L$ and all $\lambda, \mu \geqslant 0$ with $\lambda + \mu = 1$.

A *weak metric* in L is a function $d: L \times L \to R$ satisfying the axioms:

(M1) $d(x, y) \geqslant 0$ *for all* $x, y \in L$,

(M2) $d(x, x) = 0$ *for all* $x \in L$,

(M3) $d(x, z) \leqslant d(x, y) + d(y, z)$ (*triangle inequality*).

d is not necessarily a full-fledged metric in that it need not satisfy the axioms of definiteness

(M4) $d(x, y) = 0$ *implies* $x = y$

and symmetry

(M5) $d(x, y) = d(y, x)$.

[1] Supported by U.S. Post Office Department, Office of Research and Engineering. This does not imply official endorsement of the views expressed.
[2] Figures in brackets indicate the literature references at the end of this paper.

If d is a weak metric in L, then so is d^*, which is defined by $d^*(x, y) := d(y, x)$. Both metrics coincide if d is symmetric.

The purpose of this paper is to examine weak metrics that are convex functions. Such metrics are, for instance, generated by "weak norms." A *weak norm* is a function $n: L \to R$ which satisfies the following three axioms (for all $x, y \in L$):

(N1) $n(x) \geqslant 0$

(N2) $n(\alpha x) = \alpha n(x)$ *for* $\alpha \geqslant 0$ (*homogeneity*)

(N3) $n(x+y) \leqslant n(x) + n(y)$.

As a consequence of (N2) we have $n(0) = 0$. If n fulfills also

(N4) $n(x) = 0$ *implies* $x = 0$,

then n is a proper *norm*.

A weak norm may be employed to define a metric by putting

$$d(x, y) := n(x - y).$$

The axioms (M1), (M2), (M3) are readily verified. Note that n need not satisfy (N2) in order to define a weak metric. For instance $n(x) := |x|^{1/2}$ defines a weak metric on R without being homogeneous. We call a weak metric which is generated by a weak (homogeneous) norm *normal*.

2. *Homogeneous weak metrics.* Every normal weak metric is invariant under translation:

$$d(x+u, y+u) = d(x, y) \text{ for all } x, y, u.$$

However, as the example $d(x, y) := |x - y|^{1/2}$ illustrates, invariance under translation does not guarantee normality. We need additional hypotheses, and we therefore introduce the following concept of homogeneity. A weak metric is called *homogeneous in the first argument* if for all $x, u \in L$ and $\alpha \geqslant 0$ one has

$$d(x + \alpha u, x) = \alpha d(x + u, x).$$

A weak metric d is normal if and only if it is both invariant under translation and homogeneous in either the first or the second argument. Indeed, if d is translation invariant, then it can be written as $d(x, y) = n(x-y)$, where n is defined by $n(v) := d(v, 0)$. If d is homogeneous, then so is n, and the triangle inequality implies (N3).

Now it turns out that homogeneity implies invariance under translation. The following theorem was suggested by A. J. Goldman.

THEOREM 1: *If a weak metric* d *in a linear space is homogeneous in its first argument, then it is also invariant under translation and therefore normal.*

PROOF: The theorem is a consequence of the following relation [3]

$$\text{(T)} \qquad d(x, y) \leq \frac{1}{\alpha} d(x, x+u) + d(x+u, y+u) + \frac{1}{\alpha} d(y+u, y)$$

which holds for all x, y, u in L and all $\alpha > 0$. Indeed, letting $\alpha \to +\infty$ we obtain $d(x, y) \leq d(x+u, y+u)$ for all x, y, u in L. Replacing x, y, u by $x+u, y+u, -u$ gives $d(x+u, y+u) \leq d(x, y)$.

We proceed to prove relation (T). By homogeneity and the triangle inequality (M3), we get

$$d(x, y) = \frac{1}{\alpha} d(y+\alpha(x-y), y)$$
$$\leq \frac{1}{\alpha} d(y+\alpha(x-y), y+u) + \frac{1}{\alpha} d(y+u, y).$$

Again by homogeneity and (M3), it follows that

$$\frac{1}{\alpha} d(y+\alpha(x-y), y+u$$
$$= \frac{1}{\alpha} d\left(y+u+\alpha\left(x-y-\frac{1}{\alpha}u\right), y+u\right)$$
$$= d\left(x+u-\frac{1}{\alpha}u, y+u\right) \leq d\left(x+u-\frac{1}{\alpha}u, x+u\right)$$
$$+ d(x+u, y+u) = \frac{1}{\alpha} d(x, x+u) + d(x+u, y+u),$$

which remained to be shown.

In the presence of invariance under translation, there is no difference between homogeneity in the first and homogeneity in the second argument: $d(x+\alpha u, x) = \alpha d(x+u, x)$ implies $d(x, x+\alpha(-u)) = \alpha d(x, x+(-u))$. Hence theorem 1 admits the following

COROLLARY: *If a weak metric* d *is homogeneous in its first argument, then it is also homogeneous in its second argument.*

3. *Convex weak metrics.* If a weak metric d is *convex* in its first argument then,

$$d(\lambda x + \mu y, z) \leq \lambda d(x, z) + \mu d(y, z)$$

for all $x, y, z \epsilon L$ and all $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. Normal metrics are *convex in each argument.* The converse of this statement will be the main result of this section. As a first step in this direction we prove the

THEOREM 2: *A weak metric* d *which is convex in each argument is also homogeneous in each argument.*

PROOF. The equation $d(x+\alpha u, x) = \alpha d(x+u, x)$ is clearly correct when $\alpha = 0$ or $\alpha = 1$, and the case $\alpha > 1$ can be reduced to the case $\alpha < 1$ by replacing α by $\frac{1}{\alpha}$ and u by αu. Thus we may assume $0 < \alpha < 1$. By convexity in the first argument,

$$d(x+\alpha u, x) = d(\alpha(x+u) + (1-\alpha)x, x) \leq \alpha d(x+u, x)$$
$$+ (1-\alpha) d(x, x) = \alpha d(x+u, x).$$

To prove the inequality in the opposite direction, we use the triangle inequality and convexity in the second argument to write

$$d(x+u, x) - d(x+\alpha u, x) \leq d(x+u, x+\alpha u)$$
$$= d(x+u, \alpha(x+u) + (1-\alpha)x) \leq \alpha d(x+u, x+u)$$
$$+ (1-\alpha) d(x+u, x) = (1-\alpha) d(x+u, x),$$

from which it follows that $\alpha d(x+u, x) \leq d(x+\alpha u, x)$ as desired. Homogeneity in the second argument is proved analogously.

Theorem 2, together with Theorem 1, establishes our main theorem:

THEOREM 3. *A weak metric* d *which is convex in both arguments is normal.*

Let us turn to another question. If L is a linear space, then so is $L \times L$. This brings up the question of how joint convexity of d, that is, convexity of d as a function on $L \times L$, relates to separate convexity in each argument. Here we have

THEOREM 4: *If a weak metric* d *is convex in each argument separately, then* d *is also a convex function on* L × L.

The surprising fact about this theorem is that it is apparently not possible to conduct a proof by simple combination of the inequalities that define convexity in each argument separately. The proof presented here relies on our main theorem 3 in that it uses the fact that d can be expressed by a weak norm n.

[3] The proof of Theorem 1 via relation (T) has been suggested by the referee of the American Mathematical Monthly to which this paper had been submitted previously. The original more geometric version of the proof can be found in an informal communication by this author.

Joint convexity requires

$$d(\lambda(x, y)+\mu(u, v))=d(\lambda x+\mu u, \lambda y+\mu v)$$
$$\leqslant \lambda d(x, y)+\mu d(u, v)$$

to hold for all x, y, u, v and all $\lambda, \mu \geqslant 0$ with $\lambda+\mu=1$. Now we have for a normal weak metric

$$d(\lambda x+\mu u, \lambda y+\mu v)=n(\lambda(x-y)+\mu(u-v)) \leqslant \lambda n(x-y)$$
$$+\mu n(u-v)=\lambda d(x, y)+\mu d(u, v).$$

4. *A counter example.* Convexity in one argument only is not sufficient to establish invariance under translation. This is shown by the following example.

Let L be the line of real numbers, and define $d:L \rightarrow R$ as follows

$$d(x, y):=\begin{cases} 2(y-x) & \text{if } x<y \\ 2(x-y) & \text{if } 0 \leqslant y \leqslant x \\ 2x-y & \text{if } y<0 \leqslant x \\ x-y & \text{if } y \leqslant x<0. \end{cases}$$

It is easily verified that d is a weak metric and convex in the first argument. However, d is not invariant under translation.

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References

[1] Blumenthal, L. M., Theory and Applications of Distance Geometry, (Clarendon Press, London, 1953).
[2] Koethe, G., Topologische Lineare Räume I (Springer, Heidelberg, 1960).
[3] Kuhn, H. W. and Kuenne, R. E., An efficient algorithm for the solution of the generalized Weber problem in spatial economics, Journal of Regional Science (to be published).

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