JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 69B, No. 3, July–September 1965

Transversals and Matroid Partition*

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(June 9, 1965)

In section 1, *transversal matroids* are associated with "systems of distinct representatives" (i.e., transversals) and, more generally, *matching matroids* are associated with matchings in graphs. The transversal matroids and a theorem of P. J. Higgins on disjoint transversals of a family of sets, along with the well-known graphic matroids and some theorems on decomposition of graphs into forests, motivate some theorems on partitions of general matroids into independent sets. In section 2, the relationship between transversal result and matroid result is illustrated for a special case of later theorems. In section 3, theorems on transversals are proved using network flows. In sections 4 and 5, theorems on matroids are presented which imply various results on decomposition into transversals or into forests. In section 6, the matching matroids are shown to be simply the transversal matroids. For the most part, sections 2, 3, 4–5, and 6 can be read separately.

1. Transversal Matroids

A *matriod* $M=(E, F)$ is a finite set E of elements and a family F of subsets of E, called *independent* sets, such that (1) every subset of an independent set is independent; and (2) for every set $A \subset E$, all maximal independent subsets of A have the same cardinality, called the *rank* $r(A)$ of A.

Sometimes no explicit distinction is made between a matroid and its set of elements, in the same way that no explicit distinction is made between groups, spaces, or graphs and their sets of members. For example, one normally uses the same symbol to denote a space and the set of points in a space. On the other hand, it is often desirable to consider various matroids that have the same set of elements.

The primary example of a matroid is obtained by letting E be the set of columns in a matrix over some field and F the family of linearly independent subsets of columns. In particular, E may be the set of edges in a graph and F the family of edge-sets that comprise "forests" in the graph. A matroid that is abstractly isomorphic to one of the latter kind is called *graphic*.

Our motivation here will be another source of matroids, which is an extensive theory in its own right. It is well known in various contexts, including systems of distinct representatives, (0, 1)-matrices, network flows, matchings in graphs, marriages, and so forth (see [3]).[1] Here we will refer to it very broadly as transversal theory.

Let $Q=\{q_i;\ i=1, \ldots, m\}$ be a family of (not necessarily distinct) subsets of a set $E=\{e_j;\ j=1, \ldots, n\}$. The set $T=\{e_{j(1)}, \ldots, e_{j(t)}\}$, $0 \leq t \leq n$, is called a *partial transversal* (of size t) of Q if T consists of distinct elements in E and if there are distinct integers $i(1), \ldots, i(t)$ such that $e_{j(k)} \in q_{i(k)}$ for $k=1, \ldots, t$. The set T is called a *transversal* or a *system of distinct representatives* of Q if $t=m$.

THEOREM. *Let* Q *be any finite family of (not necessarily distinct) subsets of a finite set* E. (a) *If* F *is the family of partial transversals of* Q, *then* $\mathrm{M_a}=(\mathrm{E, F})$ *is a matroid.* (b) *If* F *is the collection of subfamilies of* Q *that have transversals, then* $\mathrm{M_b}=(\mathrm{Q, F})$ *is a matroid.*

The statements (a) and (b) are equivalent and refer to the same abstract class of matroids because the roles of Q and E are actually symmetric. The situation is easily visualized in the form of the "incidence graph" of (E, Q): a "bipartite" graph, $G=G(E, Q)$, where the nodes in one part are members of Q and the nodes in the other part are the members of E. The edges of G, which all go from one part to the other, are the incidences between Q and E.

A *transversal* matroid is one that is abstractly isomorphic to an M_a (or an M_b). Matroid theory and transversal theory enhance each other via transversal matroids, as do matroid theory and graph theory via graphic matroids.

Let E be any fixed subset of nodes in any given graph G. We assume throughout this paper that each edge of a graph meets two distinct nodes. Let subset $T \subset E$ be a member of F when T meets (is contained in the set of endpoints of) some matching in G. (A *matching* in a graph is a set of its edges such that no two members of the set meet the same node.) We shall show that $M_{G,E}=(E, F)$ is a matroid by verifying axiom (2). In general, where G is not necessarily bipartite and where E is any subset of nodes, we call $M_{G,E}$ a *matching matroid.* For any $A \subset E$, let T_1 and T_2 be maximal subsets of A which meet matchings, say N_1 and N_2, respectively. Consider the subgraph

*This paper is the third in a series [1, 2]. It is, however, self-contained.

Work of the first author is supported by the Army Research Office Durham through the NBS Combinatorial Mathematics Project. The second author is at the RAND Corporation, Santa Monica, Calif. His work is sponsored by the U.S. Air Force Project RAND.

This paper was presented at the Advanced Study Institute on Integer Programming and Network Flow, Tahoe City, Calif., August 21 to July 1, 1965.

C. St. J. A. Nash-Williams, prompted like ourselves by the same earlier papers, developed Theorems 1c, 2c, 1d, and 2d in another way. We are grateful for his correspondence, which has benefited our own work.

We are indebted to Gian-Carlo Rota for his NBS Matroid Seminar lectures, which inspired the discovery of transversal matroids as well as a number of other ideas not yet set forth.

[1] Figures in brackets indicate the literature references at the end of this paper.

$N \subset G$ formed by the edge-set

$$N_1 + N_2 = (N_1 - N_2) \cup (N_2 - N_1)$$

and the endpoints of its members. The connected components of N are simple open and closed paths because each node of N meets either one or two edges of N. Set

$$T_1 + T_2 = (T_1 - T_2) \cup (T_2 - T_1)$$

consists precisely of the path-ends of N that are in A; $(T_1 - T_2)$ are the nodes of A that meet N_1 but not N_2, and $(T_2 - T_1)$ are the nodes of A that meet N_2 but not N_1. Suppose T_2 is larger than T_1; then $T_2 - T_1$ is larger than $T_1 - T_2$. In this case, some component of N must be an open path, say P, which has one end v in $T_2 - T_1$ and the other end not in $T_1 - T_2$. Regarding path P as its edge-set, $N_1 + P = (N_1 - P) \cup (P - N_1)$ is a matching. This matching meets T_1 in A and in addition it meets v in A. Thus, we contradict the hypothesis that T_1 is a maximal subset of A which meets a matching. Therefore, T_1 and T_2 have the same cardinality and it follows that $M_{G,E} = (E, F)$ is a matroid.

General matching matroids are discussed in section 6.

2. Introduction

P. J. Higgins [4] gives conditions for a family Q of sets to have k mutually disjoint partial transversals of prescribed sizes $n_1, n_2, \ldots, n_k$. In section 4 we present conditions for a matroid M to have k mutually disjoint independent sets of prescribed sizes $n_1, n_2, \ldots, n_k$. Where the matroid is graphic, for example, this result is new.

The following two closely related matroid theorems are presented in [1] and [2] as generalizations of theorems by Nash-Williams and Tutte on graphs. Theorem 2, below, for the case of transversals, handles a special case of the Higgins problem; it will be generalized to cover the Higgins problem. Theorem 1 is new for the case of transversals; it will be generalized analogously.

THEOREM 1. *The elements* E *of a matroid* M *can be partitioned into as few as* k *sets, each independent in* M, *if and only if* $|A| \leq k \cdot r(A)$ *for all* $A \subset E$.

THEOREM 2. *The elements* E *of a matroid* M *can be partitioned into as many as* k *sets, each a spanning set of* M, *if and only if* $|A| \geq k(r(E) - r(\bar{A}))$ *for all* $A \subset E$.

As usual $|A|$ denotes cardinality of set A, and $\bar{A}$ denotes the complement of A (with respect to E). A *spanning set* of a matroid M is a subset of E which contains a maximal independent set.

A *base* of a matroid M is a maximal independent set, i.e., a minimal spanning set. Each base has cardinality equal to $r(E)$, the *rank* of the matroid.

For any family B of subsets of a set E, a *covering* in B is a subfamily whose union is E, and a *packing* in B is a subfamily whose members are disjoint. Where B is the family of bases of matroid M, theorem 1 describes the minimum cardinality of a covering in B, and theorem 2 describes the maximum cardinality of a packing in B.

Applied to a transversal matroid M_b, where the members of a family Q are the matroid elements and where the subfamilies that have transversals are the independent sets of elements, Theorem 1 says that *a family* Q *of sets can be partitioned into as few as* k *subfamilies, each having a transversal, if and only if* $|A| \leq k \cdot \rho(A)$ *for every subfamily* $A \subset Q$. Here $\rho(A)$ denotes the maximum cardinality of a subfamily of A which has a transversal, i.e., the maximum cardinality of a partial transversal of A. The statement is not interesting when $k=1$; for abstract matroids there is nothing interesting to say in this case.

Where A is a family of subsets of a set E, where $N(E, A)$ is the (0, 1)-incidence matrix of members of E (rows) versus members of A (columns), and where $G(E, A)$ is the bipartite incidence graph of (E, A), the value $\rho(A)$ is called the *term rank* of A, $N(E, A)$, and $G(E, A)$, respectively. One of the two fundamental forms of the fundamental theorem of transversal theory is due to P. Hall. It describes when a family A (or Q) itself has a transversal. The other fundamental form of the fundamental theorem is König's formula for term rank: $\rho(A)$, the maximum cardinality of a partial transversal of A or of a matching in $G(E, A)$ (i.e., a set of 1's which might be called a matching in $N(E, A)$) is equal to the minimum cardinality of a set of nodes that meets all edges in $G(E, A)$ (i.e., a set of rows and columns that together contain all 1's of $N(E, A)$).

Let $\sigma(A)$, for $A \subset Q$, denote the cardinality of the union of the members of A. It is a consequence of the König formula for term rank that the inequalities $|A| \leq k \cdot \rho(A)$ for all $A \subset Q$ are equivalent to the inequalities $|A| \leq k \cdot \sigma(A)$ for all $A \subset Q$. Thus the latter are also necessary and sufficient for Q to have a partition into k subfamilies, each with a transversal. When $k=1$, this is P. Hall's theorem on transversals.

To see this equivalence, suppose that $|A| > k \cdot \rho(A)$ for some $A \subset Q$. In the incidence graph $G(E, A)$, let $E_1 \cup A_1$, $E_1 \subset E$ and $A_1 \subset A$, be a minimum cardinality set of nodes that meets all of the edges. By the König theorem, $\rho(A) = |E_1| + |A_1|$. Let $A_2 = A - A_1$. The set-union of members of A_2, that is, the other ends of all the edges that meet A_2, is E_1, so $\sigma(A_2) = |E_1|$. Combining, we have

$$|A_2| = |A| - |A_1| > k(|E_1| + |A_1|) - |A_1|$$

$$= k \cdot |E_1| + (k-1) \cdot |A_1| \geq k \cdot \sigma(A_2).$$

On the other hand, clearly $\rho(A) \leq \sigma(A)$ for all $A \subset Q$. *Therefore,* $|A| \leq k \cdot \rho(A)$ *for all* $A \subset Q$ *is equivalent to* $|A| \leq k \cdot \sigma(A)$ *for all* $A \subset Q$. *Thus,* Q *can be partitioned into as few as* k *subfamilies, each with a transversal, if and only if the latter holds.*

We do not recommend this matroid approach as the way to derive the transversal result. Theorem 1 in general is not easy, and, even after it is established, using it with the König theorem to get the transversal result is no easier than deriving the transversal result directly from P. Hall's theorem as follows. Let each element $e \in E$ be replicated k times to obtain $e_1, \ldots, e_k \in E'$. To obtain Q', let $q' \in Q'$ consist of all the replications of the elements in $q \in Q$. Then $|A| \leq k \cdot \sigma(A)$ for all $A \subset Q$ is equivalent to $|A'| \leq \sigma(A')$ for all $A' \subset Q'$. By P. Hall's theorem the latter is equivalent to the existence of a transversal for Q'. That, in turn, is equivalent to there being a partition of Q into as few as k subfamilies, each having a transversal.

Section 3 presents a derivation of transversal theorems using network flows. Section 4 presents a different derivation of the corresponding matroid theorems. Both derivations suggest computationally good algorithms. Section 5 presents another application of section 4, and section 6 relates general matching matroids to transversal matroids.

3. Transversal Covers and Packings

In this section we focus attention on the transversal matroid $M_a = (E, F)$, F being the family of partial transversals of Q. We shall use network flows to derive results on covers and packings in F. For background material on network flows, we refer to [3]. In particular, the max-flow min-cut theorem and integrity theorem will be applied.[3]

Consider the directed network shown in figure 1. In figure 1 we have, in addition to a source-node u and a sink-node v, three tiers of nodes: $e_1, e_2, \ldots, e_n$ (elements of E); $q_1, q_2, \ldots, q_m$ (subsets of E that comprise the family Q); and $p_1, p_2, \ldots, p_k$ (partial transversals). The directed edges of this network and their flow capacities are listed below:

	Edges	Capacities
(u, e_j)	$j=1, \ldots, n,$	$c(u, e_j)=1,$
$(e_j, q_i),$	corresponding to $e_j \in q_i,$	$c(e_j, q_i)=\infty,$
$(q_i, p_r),$	$i=1, \ldots, m; \quad r=1, \ldots, k,$	$c(q_i, p_r)=1,$
$(p_r, v),$	$r=1, \ldots, k,$	$c(p_r, v)=n_r.$

An integral flow from source to sink in this network produces k mutually disjoint partial transversals of respective sizes $s_1 \leq n_1, s_2 \leq n_2, \ldots, s_k \leq n_k$ in the following manner. Take a chain decomposition of the flow and put e_j in p_r if, for some $i = 1, 2, \ldots, m$, the edges (e_j, q_i) and (q_i, p_r) occur in a chain of this decomposition. Conversely, k mutually disjoint partial transversals of sizes $s_1 \leq n_1, s_2 \leq n_2, \ldots, s_k \leq n_k$ yield an integral flow from source to sink. Using the

[3] In a graph where the edges e_i are directed and have positive integer capacities c_i, the maximum number of chains (directed paths, not necessarily distinct) from a node u to a node v, such that each e_i is contained in at most c_i of these chains, equals the minimum of the total capacity of the edges directed from U to $\bar{U}$ where $(U, \bar{U})$ is any partition of all the nodes into two parts such that $u \in U$ and $v \in \bar{U}$. The family of chains is called a chain decomposition of a maximum flow from u to v. The set of edges directed from a U to $\bar{U}$ is called a cut separating source u from sink v.

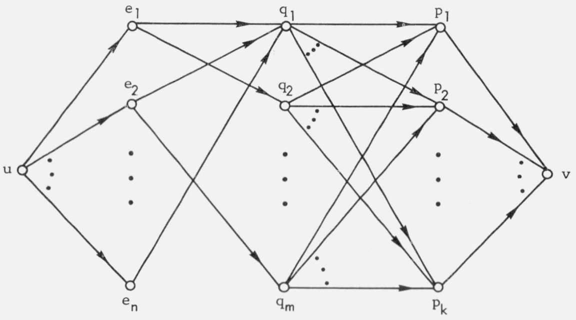

FIGURE 1.

integrity theorem and max-flow min-cut theorem for network flows, it follows that the maximum number of elements contained in a union of k (mutually disjoint) partial transversals of respective sizes $s_1 \leq n_1, s_2 \leq n_2, \ldots, s_k \leq n_k$ is equal to the capacity of a minimum cut separating source and sink in this network. We proceed to calculate this.

Let A, B, C be arbitrary subsets of $E = \{e_1, e_2, \ldots, e_n\}$, $Q = \{q_1, q_2, \ldots, q_m\}$, and $P = \{p_1, p_2, \ldots, p_k\}$, respectively, and denote their complements in these sets by $\bar{A}, \bar{B}, \bar{C}$. The capacity of an arbitrary cut separating u and v is then represented by the sum

$$\sum_{e_j \in \bar{A}} c(u, e_j) + \sum_{\substack{e_j \in A \\ q_i \in \bar{B}}} c(e_j, q_i) + \sum_{\substack{q_i \in B \\ p_r \in \bar{C}}} c(q_i, p_r) + \sum_{p_r \in C} c(p_r, v).$$

We wish to minimize this over $A \subset E, B \subset Q, C \subset P$. Using the table of edge capacities, this reduces to computing the minimum of

$$|\bar{A}| + |B| \cdot |\bar{C}| + \sum_{p_r \in C} n_r$$

over $A \subset E, B \subset Q, C \subset P$ such that the set of edges leading from A to $\bar{B}$ is empty. Thus, for given A and C, we may take B to consist solely of those nodes of Q which are joined by edges to some node of A. In the language of set representatives, B consists of those sets represented by elements of A. Moreover, for $\bar{C}$ of fixed cardinality $|\bar{C}| = k - s$, we may take C to correspond to the s smallest n's. Thus, choosing the notation so that $0 < n_1 \leq n_2 \leq \ldots \leq n_k$, and letting $\sigma(A)$ denote the cardinality of the subfamily of Q represented by elements of A, we are led to minimizing

$$|\bar{A}| + (k-s)\,\sigma(A) + \sum_{r=1}^{s} n_r$$

over $A \subset E$ and $s = 0, 1, \ldots, k$. For fixed A, the minimization over s can be carried out explicitly. Indeed, let n_j^* be the number of integers among the n_r, $r = 1, 2, \ldots, k$, such that $n_r \geq j$, $j = 1, 2, \ldots$. Thus $[n_j^*]$ and $[n_r]$ are conjugate partitions of the integer $\sum_{r=1}^{k} n_r$. It is not hard to see, especially in terms of a partition diagram, that

$$\min_{0 \le s \le k} \left[(k-s)\sigma(A) + \sum_{r=1}^{s} n_r \right] = \sum_{j=1}^{\sigma(A)} n_j^*.$$

This proves that the maximum number of elements of E *contained in a union of* k *(mutually disjoint) partial transversals of* Q, *having respective sizes* $s_1 \le n_1$, $s_2 \le n_2, \ldots, s_k \le n_k$, *is equal to*

$$(*) \qquad \min_{A \subset E} \left[|\bar{A}| + \sum_{j=1}^{\sigma(A)} n_j^* \right].$$

Here $\sigma(A)$ *denotes the number of sets in the family* Q *that are represented by elements of* A.

The following two theorems, which give necessary and sufficient conditions for the existence of covers and packings composed of partial transversals of prescribed sizes, are consequences of this result. (Nash-Williams originated a similar viewpoint for related theorems on matroids.)

THEOREM 1a. *Let* Q *be a finite family of subsets of a finite set* E. *The family* Q *has* k *partial transversals of respective sizes* $n_1, n_2, \ldots, n_k$ *whose union is* E *if and only if* (i) $n_i \le \rho(E)$, $i=1, 2, \ldots, k$, *and* (ii) *for every* $A \subset E$, *the inequality*

$$|A| \le \sum_{j=1}^{\sigma(A)} n_j^*$$

holds.

Here $\rho(E)$ denotes the term rank of the bipartite incidence graph (or matrix) of elements of E versus sets of the family Q, that is, $\rho(E)$ is the rank of the matroid $M_a=(E, F)$. The proof of sufficiency of (i) and (ii) makes use of the fact that M_a is a matroid in extending the k partial transversals of sizes s_i to partial transversals of sizes n_i, $i=1, 2, \ldots, k$.

THEOREM 2a. *Let* Q *be a finite family of subsets of a finite set* E. *The family* Q *has* k *mutually disjoint partial transversals of respective sizes* $n_1, n_2, \ldots, n_k$ *if and only if, for every* $A \subset E$, *the inequality*

$$|A| \ge \sum_{j=\sigma(\bar{A})+1}^{\infty} n_j^*$$

holds.

Using the König theorem in an argument similar to that in section 2 shows that the rank function $\rho(A)$ of matroid M_a can be used in place of $\sigma(A)$ in (*), hence also in theorems 1a and 2a.

The situation of theorem 2a is the problem studied by Higgins. His conditions are not the same as those of theorem 2a, but are instead stated in terms of subfamilies B of Q rather than subsets A of E. They may be derived from theorem 2a by use of the König theorem (and vice versa), or can be obtained directly by eliminating A and C, rather than B and C, in the minimization argument leading to (*).

4. Matroid Partition

THEOREM 1b. *The set* E *of elements of a matroid* M *can be covered by a family of independent subsets* I_i $(i=1, \ldots, k)$ *of prescribed sizes* $n_i \le r(E)$ *if and only if, for every* $A \subset E$,

$$|A| \le \sum_{j=1}^{r(A)} n_j^* = \sum_i \min(n_i, r(A)).$$

THEOREM 2b. *The set* E *of elements of a matroid* M *contains mutually disjoint independent subsets* $I_i (i=1, \ldots, k)$ *of prescribed sizes* $n_i \le r(E)$ *if and only if, for every* $A \subset E$,

$$|A| \ge \sum_{j=r(\bar{A})+1}^{r(E)} n_j^* = \sum_i [n_i - \min(n_i, r(\bar{A}))].$$

Here $r(A)$ denotes rank relative to matroid M. The equations in theorems 1b and 2b are obvious.

Using lemma 1, theorems 1b and 2b follow immediately from theorems 1c and 2c below.

LEMMA 1. *For any matroid* $M=(E, F)$ *and any nonnegative integer* n, *let* $F_{(n)}$ *denote the members of* F *which have cardinality at most* n. *Then* $M_{(n)}=(E, F_{(n)})$ *is a matroid. Where* r(A) *is the rank function for* M, *the rank function for* $M_{(n)}$ *is*

$$r_{(n)}(A) = \min(n, r(A)).$$

We call $M_{(n)}$ the *truncation* of M at n.

The proof of lemma 1 is obvious.

Let $r_i(A)$ be the rank functions for any family of matroids $M_i=(E, F_i)$, $i=1, \ldots, k$, on the set E of elements.

THEOREM 1c. *Set* E *can be partitioned into a family of subsets* $I_i (i=1, \ldots, k)$, *where* $I_i \in F_i$, *if and only if for every* $A \subset E$,

$$|A| \le \sum_i r_i(A).$$

THEOREM 2c. *There is a family of mutually disjoint sets* $I_i (i=1, \ldots, k)$, *where* I_i *is a maximal member (base) in* F_i, *if and only if for all* $A \subset E$,

$$|A| \ge \sum_i r_i(E) - \sum_i r_i(\bar{A}).$$

Where each M_i is a graph, theorem 2c is equivalent to a theorem of Tutte [5].

Since it can be shown that a truncation of a graphic or a transversal matroid is not necessarily graphic or transversal, theorems 1b and 2b for these cases do not follow from theorems 1c and 2c for these cases as in general. A similar remark applies to the way 2c will be derived from 1c. Thus we observe that the general matroid concept is useful even where primary interest is more special. The proof of 1c, on the other hand, is arranged so that the only matroids it will mention are those of the theorem. Hence, the proof applies directly to any special class of matroids (including classes of one). Everything in references [1] and [2] applies directly to the case of only graphs. The proofs in [2] do not apply directly to the case of only transversals because, as will be shown at another time, a "contraction" of a transversal matroid is not necessarily transversal.

LEMMA 2. *Let* A *be any subset of the elements of a matroid* M. *Let* I *be any independent subset of* A. *A maximal set* S, *such that* I ⊂ S ⊂ A *and* r(S) = r(I) = |I|, *is the unique set consisting of* I *and elements* e ∈ A *such that* e ∪ I *is dependent.*

Set S is called the *span of I in A.*

PROOF. Consider $e \epsilon A - I$. By the definition of rank, I is a maximal independent subset of any S. Thus, if $e \cup I$ is independent, then $e \notin S$. And thus, on the other hand, if $e \cup I$ is dependent, then I is a maximal independent subset of $e \cup S$. Hence by axiom 2 for matroids, $r(e \cup S) = |I|$, and so $e \epsilon S$.

LEMMA 3. *The union of any independent set* I *and any element* e *of a matroid* M *contains at most one minimal dependent set.*

A minimal dependent set is called a *circuit* of M.

PROOF. Suppose $I \cup e$ contains two distinct circuits C_1 and C_2. Assume I is minimal for this possibility. We have $e \epsilon C_1 \cap C_2$. There is an element $e_1 \epsilon C_1 - C_2$ and an element $e_2 \epsilon C_2 - C_1$. Set $(I \cup e) - (e_1 \cup e_2)$ is independent since otherwise $I - e_1$ is a smaller independent set than I for which $(I - e_1) \cup e$ contains more than one circuit. Set I and set $(I \cup e) - (e_1 \cup e_2)$ are maximal independent subsets of set $I \cup e$. This contradicts axiom 2.

PROOF OF 1c. Suppose that $\{I_i\}$ $(i = 1, \ldots, k)$ is a partition of E, where $I_i \epsilon F_i$. Then for arbitrary $A \subset E$,

$$|A| = \sum_i |A \cap I_i| = \sum_i r_i(A \cap I_i) \leq \sum_i r_i(A).$$

Conversely, suppose that for every $A \subset E$, the inequality holds. Let $\{I_i\}$ $(i = 1, \ldots, k)$ be a family of disjoint sets such that I_i is independent in M_i. Any number of these may be empty. Suppose there is an

$$e \epsilon E - \bigcup_i I_i.$$

We shall show how to rearrange elements among the sets I_i to make room for e in one of them while preserving the mutual disjointness and the independence of I_i in M_i. This will prove the theorem.

If $e \epsilon S$ for any $S \subset E$, then for some i, $|I_i \cap S| < r_i(S)$. Otherwise,

$$|S| \geq |\bigcup_i (I_i \cap S) \cup e|$$

$$= 1 + \sum_i |I_i \cap S| > \sum_i r_i(S)$$

would contradict the hypothesis.

Let $S_0 = E$. Inductively, starting with $j - 1 = 0$, if $e \epsilon S_{j-1}$ then for some $I_{i(j)}$ such that

$$|I_{i(j)} \cap S_{j-1}| < r_{i(j)}(S_{j-1}),$$

we define S_j to be the span in S_{j-1}, with respect to matroid $M_{i(j)}$, of $I_{i(j)} \cap S_{j-1}$. Since

$$r_{i(j)}(S_j) < r_{i(j)}(S_{j-1}),$$

S_j is a proper subset of S_{j-1}. Therefore we must eventually reach an S_h such that $e \notin S_h$ and $e \epsilon S_j$ for $0 \leq j < h$.

(Where the matroids M_i are identical, the construction above is the same as the corresponding part of the proof of theorem 1 in [1]. The rest of references [1] and [2] goes through essentially unchanged for a version, concerning possibly distinct matroids, which includes theorems 1c and 2c. However, we continue here with a substantially trimmed version.)

If $e \cup I_{i(h)}$ is independent in $M_{i(h)}$, the present proof is finished. Otherwise $e \cup I_{i(h)}$ contains a circuit C of $M_{i(h)}$. Set $(e \cup I_{i(h)}) \cap S_{h-1}$ is not dependent in $M_{i(h)}$, because then, by lemma 2 and by the definition of S_h, since $e \epsilon S_{h-1}$, we would have $e \epsilon S_h$. Thus let m be the smallest integer, $0 < m < h$, such that $(e \cup I_{i(h)}) \cap S_m$ is independent in $M_{i(h)}$. There is an $e' \epsilon C - S_m$. By lemma 3, $e \cup I_{i(h)} - e'$ is independent in $M_{i(h)}$.

Replacing $I_{i(h)}$ by $e \cup I_{i(h)} - e'$, we now need to dispose of e' instead of e. However, we can show that sequence $(I_{i(1)}, S_1), \ldots, (I_{i(m)}, S_m)$, with the roles of e and e' interchanged, is of the same construction as $(I_{i(1)}, S_1), \ldots, (I_{i(h)}, S_h)$, only shorter. Since the original $e \cup (I_{i(h)} \cap S_{j-1})$ is dependent in $M_{i(h)}$, for all j, $1 \leq j \leq m$, by lemma 3 we have $e' \epsilon C \subset S_{j-1}$. Consider the terms $(I_{i(j)}, S_j)$, $1 \leq j \leq m$, one after another in order. Assume there is no change in S_{j-1}. If originally $I_{i(j)} \neq I_{i(h)}$, then there is no change at all in $(I_{i(j)}, S_j)$. If originally $I_{i(j)} = I_{i(h)}$, then, even though e and e' are interchanged in $I_{i(j)}$, by lemma 2 and the definition of S_j, since $e \cup e' \subset C \subset S_{j-1}$, there is no change in S_j. Thus the theorem is proved.

PROOF OF 2c. For any family of matroids, $M_i = (E, F_i)$ $(i = 1, \ldots, k)$, with rank functions $r_i(A)$, consider the additional matroid $M_0 = (E, F_0)$ where the members of F_0 are the subsets of E that have cardinality at most $|E| - \sum_i r_i(E)$. Matroid M_0 is a truncation of the matroid in which all subsets of E are independent. The existence of mutually disjoint sets $I_i (i = 1, \ldots, k)$, where I_i is a maximal member of F_i, is equivalent to the existence of a partition of E into a family of sets, I_0 and I_i $(i = 1, \ldots, k)$, such that $I_0 \epsilon F_0$ and $I_i \epsilon F_i$. By theorem 1c, the existence of that partition is equivalent to the condition that

$$|A| \leq \min\left(|E| - \sum_i r_i(E), |A|\right) + \sum_i r_i(A)$$

for all $A \subset E$.

That condition in turn is equivalent to

$$|A| \leq |E| - \sum_i r_i(E) + \sum_i r_i(A)$$

for all $A \subset E$, which is equivalent to

$$|\bar{A}| \geq \sum_i r_i(E) - \sum_i r_i(A)$$

for all $A \subset E$. Thus theorem 2c is proved.

5. Another Application

Let J_i ($i=1, \ldots, k$) be mutually disjoint independent sets in a matroid $M=(E, F)$. Let $E' = E - (\cup J_i)$.

THEOREM 1d. *Set* E *can be partitioned into a family of independent sets* $\mathrm{I_i \epsilon F}$ ($\mathrm{i}=1, \ldots, \mathrm{k}$) *such that* $\mathrm{J_i \subset I_i}$ *if and only if, for every* $\mathrm{A \subset E'}$,

$$|\mathrm{A}| \leq \sum_{\mathrm{i}} [\mathrm{r(A \cup J_i) - r(J_i)}].$$

THEOREM 2d. *There is a family of mutually disjoint bases* $\mathrm{I_i}$ ($\mathrm{i}=1, \ldots, \mathrm{k}$) *of* M *such that* $\mathrm{J_i \subset I_i}$ *if and only if, for every* $\mathrm{A \subset E'}$,

$$|\mathrm{A}| \geq \sum_{\mathrm{i}} [\mathrm{r(E) - r((E' - A) \cup J_i)}].$$

For any matroid $M=(E, F)$ and any $E_0 \subset E$, let F_0 consist of sets $I \epsilon F$ such that $I \subset E_0$. Then $M \cdot E_0 = (E_0, F_0)$, obviously a matroid, is called a *submatroid* of M (obtained from M by *deleting* the elements of $\bar{E}_0 = E - E_0$). The rank of a subset of E_0 is the same in $M \cdot E_0$ as in M.

For any matroid $M=(E, F)$ and any $E_0 \subset E$, let J be any maximal subset of $\bar{E}_0 = E - E_0$ which is a member of F. In other words, let J be any base of submatroid $M \cdot \bar{E}_0$. Let F_0 consist of sets $I \epsilon F$ such that $I \subset E_0$ and such that $J \cup I \epsilon F$. It follows easily from the definition of matroid that $M \times E_0 = (E_0, F_0)$ is a unique matroid, called the *contraction* of M to E_0 (obtained from M by *contracting* the elements of $\bar{E}_0$). Where r and r_0 denote the rank functions for matriods M and $M \times E_0$, respectively, we have for every $A \subset E_0$,

$$r_0(A) = r(A \cup \bar{E}_0) - r(\bar{E}_0).$$

Theorem 1d follows immediately from theorem 1c by letting the M_i of 1c (for $i=1, \ldots, k$) be the matroid obtained from matroid M of 1d by contracting the elements of J_i and then deleting all the other elements of $E - E'$.

To prove 2d from 2c, we obtain each M_i of 2c from M of 2d in the same way as above. If, for some i, $r(E' \cup J_i) < r(E)$, then no base of M is contained in $E' \cup J_i$ and so there is no family of bases I_i as described in 2d. In this case the inequality in 2d does not hold where A is the empty set. Otherwise, $r(E' \cup J_i) = r(E)$ for each i. In this case, if J'_i is a base of M_i, then $J_i \cup J'_i$ is a base of M. Thus, in this case, 2d follows from 2c.

6. Addendum on Matchings

An element of a matroid M is called *isolated* if it is contained in every base of M, i.e., if it is contained in no circuits of M. Clearly, any number of isolated elements can be "added" to any transversal matroid M_a, thereby obtaining another transversal matroid. With respect to the graph representation $G(E, Q)$ of M_a, for every isolated element e' added to M_a, simply add a node e' to E and join it to a new node q' added to Q.

Several elements of a matroid M are said to be *in series* with each other either when they are all isolated, or else when none of them is isolated and each base of M contains all but possibly one of them.

A set of elements is in series in matroid M *if and only if the elements are contained in exactly the same circuits of* M.

Suppose some base I of M contains neither of elements e_1 and e_2 of M. Then $I \cup e_1$ contains a circuit of M that contains e_1 but not e_2.

Suppose an element e_1 is contained in a circuit C of M that does not contain nonisolated element e_2 of M. Let I be a base of M which does not contain e_2. The rank of $(I \cup C) - e_1$ is as large as the rank of I; otherwise every maximal independent subset of $I \cup C$ would contain e_1, but then e_1 would be contained in no circuit in $I \cup C$. Therefore $(I \cup C) - e_1$ contains a base of M; this base contains neither e_1 nor e_2. Thus the theorem is proved.

"Replacing an element e_i in a matroid M by a set $E_i^k = \{e_i^1, \ldots, e_i^k\}$ of new elements in series" yields a matroid $M^{(i,k)}$. The circuits of $M^{(i,k)}$ and the elements of $M^{(i,k)}$ are identical with those of M except that e_i is replaced by the members of E_i^k. Each base B of M which contains e_i corresponds to a base $(B - e_i) \cup E_i^k$ of $M^{(i,k)}$. Each base B of M which does not contain e_i corresponds to k bases of $M^{(i,k)}$ of the form $B \cup E_i^k - e_i^j$, $j=1, \ldots, k$. We omit proof that $M^{(i,k)}$ is a matroid, which is not difficult using the description of the bases.

For any transversal matroid $\mathrm{M_a}$, *containing element* $\mathrm{e_i}$, *the matroid* $\mathrm{M_a^{(i,k)}}$ *is also transversal.*

Let M_a be represented by a bipartite graph $G = G(E, Q)$ as described in section 1; a base of M_a consists of the endpoints in E of a maximum cardinality matching in G. By thinking of bases, it is easy to see that we obtain from G a similar representation $G^{(i,k)}$ for matroid $M_a^{(i,k)}$ as follows. Replace node $e_i \epsilon E$ of G by the set E_i^k of new nodes. Join each $e_i^j \epsilon E_i^k$ to the same nodes in Q to which e_i was joined. Also add to Q a set Q' of $k-1$ new nodes, each joined to precisely the members of E_i^k. We then have $G^{(i,k)}$. A base of matroid $M_a^{(i,k)}$ consists of the endpoints in $(E - e_i) \cup E_i^k$ of a maximum cardinality matching in $G^{(i,k)}$.

Clearly, if $A \subset E$ for matching matroids $M_{G,A}$ and $M_{G,E}$, then $M_{G,A}$ is the submatroid of $M_{G,E}$ whose set of elements is A. Clearly, any submatroid of a transversal matroid is transversal.

Every matching matroid is a transversal matroid. (Thus, the two classes of matroids are abstractly the same.)

In view of the preceding observations on submatroids, it suffices to show that where G is any graph and where V is all of its nodes, $M_{G,V}$ is a transversal matroid. Clearly, B is a base in matroid $M_{G,V}$ if and only if B is the set of endpoints of some maximum (cardinality) matching L in G.

Section 6 of [7] implies the following theorem (which essentially strengthens some other known theorems, a characterization by Tutte of graphs in which no matching meets all the nodes, and a formula by Berge for what we regard here as the rank of $M_{G,V}$).

(*) *From any graph* G, *by deleting the set* J *of nodes which meet every maximum (cardinality) matching and deleting all the edges which meet* J, *the remainder consists of connected components,* 0_i, *containing respectively* $2r_i+1$ *nodes where* r_i *is an integer.* (*If* G *is bipartite, each* 0_i *is a single node.*) *Let* Q *consist of the nodes* u *in* J *which in* G *are joined to at least one node in* $\cup\, 0_i$. *Every maximum matching in* G *contains* r_i *edges in* 0_i, *for each* i, *and contains an edge joining* u *to a node in* $\cup\, 0_i$, *for each* $u \in Q$.

What is actually proved in [7] is theorem (*) where "Every" is replaced by "Some" in the last sentence. However, because each 0_i has an odd number of nodes, because every edge leaving an 0_i goes to a $u \epsilon Q$, and because each edge has two ends, it is easy to see that any matching which is not as described in the theorem meets fewer nodes in $\cup\, 0_i$. Hence, it has smaller cardinality than the matching, described in the theorem, which is proved in [7] to exist.

(Unless some matching in G meets every node, there are more 0_i's than there are u's. The theorem of Tutte says that a graph contains no matching that meets all of the nodes if and only if there exists a subset Q of the nodes such that deleting Q and its incident edges from G leaves more than $|Q|$ components which have odd numbers of nodes.)

For any graph G, whose node set is V, the set $J \subset V$ defined in (*) is the set of isolated elements in matroid $M_{G,V}$. Denoting the set of nodes in 0_i by E_i, theorem (*) says that each maximum matching meets all but possibly one node in E_i; thus, set E_i is in series in matroid $M_{G,V}$. By "contracting" the subgraphs 0_i to single nodes e_i, comprising a set E, and then by deleting $J-Q$ and all edges which do not meet an e_i, we obtain from G a bipartite graph $G(E, Q)$.

Let M_a be the transversal matroid, with set E of elements, associated with $G(E, Q)$. It follows easily from theorem (*) that matroid $M_{G,V}$ is obtained from matroid M_a by replacing each e_i by the set E_i in series and by adding set J of isolated elements.

The structure of transversal matroids and some other related matroids will be further described in a later paper.

7. References

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(Paper 69B3–145)

Note added in proof: Theorem 1, the subject of [1], generalized here, was proved for the case where the matroid is a set of vectors in a vector space by Alfred Horn [A characterization of unions of linearly independent sets, J. London Math. Soc. **30** (1955), 494–496] and by R. Rado [A combinatorial theorem on vector spaces, J. London Math. Soc. **37**, (1962), 351–353]. In the Abstracts of Short Communications, International Congress of Mathematicians, Stockholm 1962, p. 47, Rado remarks that "This theorem is of interest since in contrast to other propositions on vector spaces its proof has not yet been extended to abstract independence relations I (H. Whitney, Amer. J. Math. 1935, R. Rado, Canadian J. Math. 1949). It remains to decide if (i) the theorem is true for all I, or (ii) its validity constitutes a new necessary condition for representability of I in a vector space." Theorem 1 confirms (i).