JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 69B, Nos. 1 and 2, January–June 1965

On Measurable Sets and Functions

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(October 6, 1964)

The relation $L=f^{-1}(B)$, where L and f are Lebesgue measurable and B is a Borel set, is studied. Any one of L, B, f can be specified and the relation is solvable; one can also specify two of the three possible pairs. The relation characterizes (in a sense made precise in the text) the classes of Lebesgue measurable functions and sets; that it does so for the class of Borel sets as well is left as a conjecture, whose truth would imply that the functions which preserve Lebesgue measurability as second composition factors [i.e., g in $g(h(x))$] are precisely the Borel measurable functions.

The questions treated in this note, though mainly elementary, occur so naturally in connection with the basic concepts of measure and integration theory as to warrant unified presentation. For concreteness we deal exclusively with subsets of the real line R, and with real-valued functions defined on R. The symbols (BS), (ZS), and (LS) denote the respective classes of Borel sets, sets of zero measure, and Lebesgue-measurable sets; we recall that $L\epsilon(LS)$ if and only if L has a representation of the form

$$L=(B-Z)\cup(Z-B) \qquad B\epsilon(BS), Z\epsilon(ZS). \tag{1}$$

The symbols (BF) and (LF) denote the respective classes of Borel-measurable functions and Lebesgue-measurable functions; we recall that $f\epsilon(LF)$ if and only if

$$L=f^{-1}(B)\epsilon(LS) \qquad \text{for all } B\epsilon(BS). \tag{2}$$

Our first theme is the *solvability* of relation (2) when various subsets of its "variables" (L, B, f) are specified. For example, given $B\epsilon(BS)$ we can trivially find $L\epsilon(LS)$ and $f\epsilon(LF)$ to satisfy (2) by choosing $L=B$ and $f=$(identity); given $f\epsilon(LF)$ we can trivially find $B\epsilon(BS)$ and $L\epsilon(LS)$ to satisfy (2) by choosing *any* $B\epsilon(BS)$ and setting $L=f^{-1}(B)$. A third case (in which L is specified) is treated in the following theorem.

THEOREM 1: *For any* Lϵ(LS), *there is a* Bϵ(BS) *differing from* L *by at most a set of measure zero, and an* fϵ(LF) *differing from the identity on at most a set of measure zero, such that* L$=$f^{-1}(B).

PROOF: If $L=\phi$ or $L=R$, take $B=L$ and $f=$(identity). If $L\epsilon(ZS)$ but $L\neq\phi$, there exists $B\subset L$ with $B\epsilon(BS)$ and $B\neq\phi$; let f be the identity on $(R-L)\cup B$ and[1] let $f(L-B)=\{x\}$ for some $x\epsilon B$. If $R-L\epsilon(ZS)$ and $R-L\neq\phi$, there exists $B'\subset R-L$ with $B'\epsilon(BS)$ and $B'\neq\phi$; let f be the identity on $L\cup B'$, and[2] let $f(R-L-B')=\{x\}$ for some $x\epsilon B'$ (here $B=R-B'$).

Finally, suppose none of the above situations holds. Consider a representation (1) of L, and let f be the identity on $R-Z$. Since it is not true that $B=\phi$, we can define f on $Z-B$ so that $f(Z-B)\subset B$. Since it is not true that $R-B=\phi$, we can define f on $Z\cap B$ so that $f(Z\cap B)\subset R-B$. This completes the proof.

It is natural next to consider the solvability of (2) when *two* of (L, B, f) are specified. Given $B\epsilon(BS)$ and $f\epsilon(LF)$, (2) serves to define an $L\epsilon(LS)$ which satisfies (2). The case in which B and L form the specified pair is treated in the following theorem.

THEOREM 2: *For any* Bϵ(BS) *and* Lϵ(LS), *with sole exceptions* (B$=\phi$, L$\neq\phi$) *and* (B$=$R, L$=\phi$), *there is an* fϵ(LF) *such that* L$=$f^{-1}(B).

PROOF: First suppose $B=\phi$; then if $L=\phi$ any $f\epsilon(LF)$ will do, while if $L\neq\phi$ no f will do. Next suppose $B=R$; if $L\neq\phi$ we can choose $f\epsilon(LF)$ so that $f(R)\subset L$, while if $L=\phi$ no f will do. Finally, if $B\neq\phi$ and $B\neq R$, then we can define f on L so that $f(L)\subset B$, and on $R-L$ so that $f(R-L)\subset R-B$.

The remaining case is that in which $L\epsilon(LS)$ and $f\epsilon(LF)$ are specified. One cannot always find $B\epsilon(BS)$ to satisfy (2) (suppose e.g., that $L\epsilon(LS)-(BS)$ and $f=$(identity)), so that the question must be modified. One might ask for which $f\epsilon(LF)$ it is true that to *each* $L\epsilon(LS)$ there corresponds a $B\epsilon(BS)$ obeying (2); the answer is "no f" even without the measurablility requirement on f, since the cardinality 2^c of (LS) exceeds the cardinality c of (BS). A second modified version is also uninteresting, as the next theorem shows:

THEOREM 3: L$=\phi$ *and* L$=$R *are the only* Lϵ(LS) *such that for each* fϵ(LF), *there exists* Bϵ(BS) *satisfying relation* (2).

PROOF: $L=\phi$ and $L=R$ *are* solutions, since one can take $B=L$ independent of f. For any other $L\epsilon(LS)$, choose f to be the characteristic function of some $L'\epsilon(LS)$ different from both L and $R-L$; then $f^{-1}(B)\neq L$ for *all* $B\subset R$, since $f^{-1}(B)$ will be one of the four sets ϕ, R, L', $R-L'$ according to the membership or nonmembership of 0 and 1 in B.

Our second theme concerns the role of relation (2) in *characterizing* the three classes involved ((LF),

[1] Omit this clause if $B=L$.
[2] Omit this clause if $B'=R-L$.

(LS), (BS)), in the sense of the equations

$$(LF)=\{f:(f:R\rightarrow R), f^{-1}(B)\epsilon(LS) \text{ for all } B\epsilon(BS)\},$$
$$(LS)=\{L:L\subset R, L=f^{-1}(B) \text{ for some } B\epsilon(BS) \text{ and } f\epsilon(LF)\},$$
$$(BS)=\{B:B\subset R, f^{-1}(B)\epsilon(LS) \text{ for all } f\epsilon(LF)\}.$$

The first of these equations holds trivially; i.e., the relation (2) used to define (LF) certainly characterizes (LF). Theorem 1 shows that the right side of the second equation contains (LS); since the inclusion in the opposite direction is trivial, (LS) is also characterized by (2). The right side of the third equation clearly contains (BS), so that what remains to be proved is equivalent to the following statement, which the writer has been unable to settle:

CONJECTURE: *If* S *is not a Borel set then for at least one* fϵ(LF), $f^{-1}(S)$ *fails to be Lebesgue measurable.*

Our final theme is the preservation of measurability under function composition. Composition will be denoted by an asterisk, i.e., $(f*g)(x)=f(g(x))$. We set

$$(LCF)=\{f:(f:R\rightarrow R), f*g\epsilon(LF) \text{ for all } g\epsilon(LF)\}$$

where "*LCF*" is a mnemonic for "left composition factor." Taking g as the identity shows that $(LCF)\subset(LF)$; it is an unpleasant fact that the inclusion is strict. Some texts include a proof that (LCF) contains the continuous functions, while others give the sharper result that the class of Borel measurable functions $(BF)\subset(LCF)$. We shall show why this may be the best possible result:

THEOREM 4: (BF) = (LCF) *if the Conjecture is true.*

PROOF: First assume $f\epsilon(BF)$; then for any $g\epsilon(LF)$ and $B\epsilon(BS)$ we have $f^{-1}(B)\epsilon(BS)$ and therefore

$$\{x:f(g(x))\epsilon B\}=g^{-1}(f^{-1}(B))\epsilon(LS)$$

so that $(f*g)^{-1}(B)\epsilon(LS)$ for all $B\epsilon(BS)$. Thus $f*g\epsilon(LF)$ for all $g\epsilon(LF)$, proving $f\epsilon(LCF)$. Next assume $f\epsilon(LF)-(BF)$. Then $f^{-1}(B)\epsilon(LS)-(BS)$ for some $B\epsilon(BS)$. By the Conjecture, there exists $g\epsilon(LF)$ for which $g^{-1}(f^{-1}(B))=(f*g)^{-1}(B)$ is *not* in (LS); thus $f*g$ is not in (LF), and hence f is not in (LCF), completing the proof.

Similarly, we define

$$(RCF)=\{g:(g:R\rightarrow R), f*g\epsilon(LF) \text{ for all } f\epsilon(LF)\}.$$

Taking f as the identity shows that $(RCF)\subset(LF)$, and it is known[3] that (RCF) does not even contain all continuous strictly monotone functions. For an alternate characterization of (RCF), we set

$$(SLF)=\{g:(g:R\rightarrow R), g^{-1}(L)\epsilon(LS) \text{ for all } L\epsilon(LS)\}$$

where "*SLF*" is a mnemonic for "strongly Lebesgue measurable function."

THEOREM 5: (SLF) = (RCF)

PROOF: First assume $g\epsilon(SLF)$; then for any $f\epsilon(LF)$ and $B\epsilon(BS)$ we have $f^{-1}(B)\epsilon(LS)$ and therefore

$$(f*g)^{-1}(B)=g^{-1}(f^{-1}(B))\epsilon(LS).$$

Therefore $f*g\epsilon(LF)$ for each $f\epsilon(LF)$; i.e., $g\epsilon(RCF)$. Next assume $g\epsilon(LF)-(SLF)$; then there exists $L\epsilon(LS)$ for which $g^{-1}(L)$ is not in (LS), and by Theorem 1 $L=f^{-1}(B)$ for some $f\epsilon(LF)$ and $B\epsilon(BS)$. Thus $(f*g)^{-1}(B)$ is not in (LS), and so $g\epsilon(LF)-(RCF)$, completing the proof.

It would be interesting to explore the class (SLF) more thoroughly.

[3] See p. 83 of Halmos' "Measure Theory," van Nostrand, 1950.

(Paper 69B1–140)