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Character Subgroups of *F*-Groups[1]

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A necessary and sufficient condition is given that a subgroup of an F-group G be definable by the vanishing of an additive character on G.

1. An *F-group* is an abstract group given by a presentation of the form

$$A_1, B_1, \ldots, A_g, B_g, E_1, \ldots, E_s, P_1, \ldots, P_t;$$

$$E_1^{m_1} = \ldots = E_s^{m_s} = (\Pi\gamma_i)(\Pi E_j)(\Pi P_k) = 1,$$

where $\gamma_i = A_i B_i A_i^{-1} B_i^{-1}$. Here m_i is a rational integer $\geqslant 2$, $s \geqslant 0$, $t \geqslant 0$, and $s+t+g>0$. The consideration of F-groups arises naturally in the study of discontinuous groups, since it is known that except for certain trivial exceptions every Fuchsian group is an F-group when considered as an abstract group and that, conversely, every F-group has a faithful representation as a Fuchsian group.

If G is a group, an *additive character* on G is any homomorphism χ of G into the additive group of the complex numbers. That is, $\chi(XY) = \chi(X) + \chi(Y)$, for X and Y in G. By a *character subgroup* of G we mean any subgroup H of G such that H is the kernel of some nontrivial additive character on G. Otherwise stated, there exists a nontrivial character χ on G such that $H = \{X \epsilon G \mid \chi(X) = 0\}$.

Henceforth G will denote an F-group, Δ the normal closure in G of $E_1, \ldots, E_s$, and G' the commutator subgroup of G.

The purpose of this note is to prove the following theorems.

THEOREM 1: *A subgroup* H *of* G *is a character subgroup of* G *if and only if* H *is normal in* G, $H \supset G'\Delta$, *and* G/H *has no elements of finite order.*

THEOREM 2: *A subgroup* H *of* G *is contained in a character subgroup of* G *if and only if* $H \cdot G'\Delta$ *is of infinite index in* G.

2. The essence of the proof of these theorems is contained in the two lemmas of this section.

LEMMA 1: $G/\Delta G'$ *is a free abelian group on finitely many generators.*

PROOF: A simple calculation shows that $G/\Delta G'$ is isomorphic to $(G/\Delta)/(G/\Delta)'$. (This holds for any normal subgroup Δ of G.) Suppose $t \geqslant 1$. Then eliminating the generator P_t from the presentation has the concomitant effect of removing the relation $(\Pi\gamma_i)E_1 \ldots E_s P_1 \ldots P_t = 1$. Thus G/Δ is free of rank $2g+t-1$ and we can conclude that $(G/\Delta)/(G/\Delta)'$ is free abelian of rank $2g+t-1$.

If $t=0$, then G/Δ is a fundamental group on $2g$ generators with the sole relation $\prod_{i=1}^{g} \bar{\gamma}_i = 1$, where $\bar{\gamma}_i = \gamma_i \Delta$. Hence G/Δ is isomorphic to G^*/N where G^* is a free group of rank $2g$ on the generators $A_i\Delta$, $B_i\Delta$, $1 \leqslant i \leqslant g$, and N is the normal closure in G^* of $\prod_{i=1}^{g} \bar{\gamma}_i$. Thus, $(G/\Delta)/(G/\Delta)'$ is isomorphic to $(G^*/N)/(G^*/N)'$, which in turn is isomorphic to $G^*/NG^{*\prime}$. But N is contained in $G^{*\prime}$ and therefore $G^*/NG^{*\prime} = G^*/G^{*\prime}$, a free abelian group of rank $2g$.

LEMMA 2: *Let* G *be a free abelian group of finite rank* r *and let* H *be a subgroup of* G *such that* G/H *has no elements of finite order. Then* H *is a character subgroup of* G.

PROOF: Since H is a subgroup of G, H is a free abelian group of rank $s \leqslant r$. Suppose that

$$G = \{g_1, \ldots, g_r\}, H = \{h_1, \ldots, h_s\} \cdot$$

Then there is an $s \times r$ rational integral matrix A such that $h = Ag$, where

$$g = \begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix}, \qquad h = \begin{bmatrix} h_1 \\ \vdots \\ h_s \end{bmatrix}.$$

Let U be any $s \times s$ unimodular rational integral matrix and V any $r \times r$ unimodular rational integral matrix. Then the elements of the vector Uh are generators of H, the elements of the vector Vg are generators of G, and

$$Uh = UAV^{-1} \cdot Vg \cdot$$

Choose U and V so that UAV^{-1} is in Smith Normal Form. Then we can write

$$Vg = \begin{bmatrix} g_1^* \\ \vdots \\ g_r^* \end{bmatrix}, \qquad Uh = \begin{bmatrix} h_1^* \\ \vdots \\ h_s^* \end{bmatrix},$$

with $h_i^* = m_i g_i^*$, for some positive integer $m_i (1 \leqslant i \leqslant s)$. Since $m_i g_i^* \epsilon H$ and G/H contains no elements of finite order, it follows that $g_i^* \epsilon H$ and hence that $m_i = 1 (1 \leqslant i \leqslant s)$. That is, $h_i^* = g_i^*$, for $1 \leqslant i \leqslant s$.

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The desired character χ may now be defined as follows. Choose $\chi(g_i^*)=0$ for $1 \leqslant i \leqslant s$ and $\chi(g_{s+1}^*), \ldots, \chi(g_r^*)$ linearly independent over the rational numbers. It is clear that $\chi(X)=0$ for $X \epsilon H$. Furthermore, suppose that $X \epsilon G$ and that $\chi(X)=0$. Writing

$$X=c_1 g_1^* + \ldots + c_r g_r^*,$$

with rational integral c_i, we find that

$$\chi(X)=c_1\chi(g_1^*)+\ldots+c_r\chi(g_r^*)$$

$$=c_{s+1}\chi(g_{s+1}^*)+\ldots+c_r\chi(g_r^*)=0.$$

Since $\chi(g_{s+1}^*), \ldots, \chi(g_r^*)$ are linearly independent over the rationals, $c_{s+1}=\ldots=c_r=0$ and $X \epsilon H$. Thus H is a character subgroup of G.

3. We are now in a position to prove Theorems 1 and 2.

Proof of Theorem 1. Suppose H is a normal subgroup of G such that $H \supset G'\Delta$ and G/H has no elements of finite order. Then $(G/G'\Delta)/(H/G'\Delta) \cong G/H$ has no elements of finite order. By Lemma 1, $G/G'\Delta$ is a free abelian group, and therefore, by Lemma 2, $H/G'\Delta$ is a character subgroup of $G/G'\Delta$. The character in question on $G/G'\Delta$ can be extended to a character on G in an obvious way, and it is clear that H is a character subgroup of G.

Conversely, suppose that H is the character subgroup of G corresponding to the nontrivial character χ. Clearly H is normal in G and $H \supset G'\Delta$. Suppose $X \epsilon G$ and $X^n \epsilon H$ for some positive integer n. Then $\chi(X^n)=n\chi(X)=0$ so that $\chi(X)=0$, and $X \epsilon H$. Thus G/H has no elements of finite order, and the proof of Theorem 1 is complete.

Proof of Theorem 2. Suppose H is a subgroup of G such that $HG'\Delta$ is of infinite index in G. Since $[G/G'\Delta\colon HG'\Delta/G'\Delta]=[G\colon HG'\Delta]$, we conclude that $HG'\Delta/G'\Delta$ is of infinite index in the free abelian group $G/G'\Delta$. Hence $HG'\Delta/G'\Delta$ is a free abelian group on *strictly fewer* generators than the number of generators of $G/G'\Delta$. A simple calculation involving linear equations shows that there is a nontrivial character on $G/G'\Delta$ which vanishes on $HG'\Delta/G'\Delta$. Extending to a character on G, we obtain a nontrivial character on G which vanishes on $HG'\Delta$ and *a fortiori* on H.

Conversely, if H is a subgroup of G such that a nontrivial character χ vanishes on H, then $HG'\Delta$ is of infinite index in G. For choose $X \epsilon G$ such that $\chi(X) \neq 0$. Then $\chi(X^n)=n\chi(X) \neq 0$, $(n=1, 2, 3, \ldots)$, so that $X^n \notin HG'\Delta$. This completes the proof.

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