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Lehman's Switching Game and a Theorem of Tutte and Nash-Williams[1]

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The results cited in the title are unified by the following theorem: For any matroid M and any subsets N and K of elements in M, there exist as many as k disjoint subsets of N which span K and which span each other if and only if there is no contraction matroid $M \times A$ where $N \cap A$ partitions into as few as k sets such that each is independent in $M \times A$ and such that at least one of them does not span $K \cap A$ in $M \times A$.

2.1. The Problem

A. Lehman [3][2] posed the following game to be played between two players on any given matroid M with a distinguished element e. The players are called the cut player and the short player. They take turns and (to be explicit) the cut player goes first. Each player in his turn tags an element of M, other than e, not already tagged. The short player wins if he tags a set of elements which span e. The cut player wins otherwise—that is, the cut player wins if the elements, other than e, which he has not tagged do not span e.

The game, determined by M and e, is called a *short game* if the short player can win against any strategy of the cut player. We will call the game *nonshort* if the cut player can win against any strategy of the short player. Clearly a game is one or the other. For any M and e, Lehman characterizes short games and describes a winning strategy for the short player.

Recall from section 1.4 that a set T in a matroid M is said to span a set A in M if for every $e \epsilon A$, either $e \epsilon T$ or there is a circuit C of M such that $C - e \subset T$. Recall that a base B of M is a set which spans M (e.g., a spanning set of M) and which also is independent.

Where the game is played on a graph G, it is not necessary to have an edge corresponding to e but sufficient to have two distinguished "terminal" nodes, v_1 and v_2, which would be the ends of e. Here, the goal of the short player is to tag a set of edges which contains a path of edges joining v_1 to v_2. The goal of the cut player is to tag a set of edges which separates v_1 from v_2.

A theorem due independently to Tutte [6] and Nash-Williams [4] characterizes for any graph G the maximum number of edge-wise disjoint subgraphs, each connected and containing all nodes of G, into which the edges of G can be partitioned. For a connected graph G, the edges of a connected subgraph which contains every node of G correspond to the elements of a spanning set of the matroid of G, and conversely.

The purpose of the present note is to unify these two theories. Theorem 2 states the straightforward generalization to matroids of the Tutte and Nash-Williams theorem. Theorem 3 is Lehman's main theorem characterizing short games. Theorem 4 is an analogous theorem characterizing nonshort games. (Lehman characterizes nonshort games indirectly by using "dual matroids" which we avoid.) Theorem 5, for the case where $K = N = M$, yields theorem 2. For the case where $k = 2$, it yields the "only if" parts of theorems 3 and 4. The "if" parts of theorems 3 and 4 are proved by describing the winning strategies when the respective conditions hold (in one case this follows Lehman, [3]).

Theorem 1 in section 1.3 and theorem 2 are in a sense dual to each other but not in the usual matroid sense. Each can be proved from the other. We use theorem 1 to prove theorem 5.

Theorem 5 appears interesting in itself. We call it the "cospanning-set theorem" after a main idea of Lehman's theory. For a graph G with a prescribed subset of nodes called terminals, it gives a "good" characterization for the nonexistence of k edge-wise disjoint connected subgraphs (e.g., trees), all with precisely the same set of nodes which includes the terminals.

If the matroid M of the cospanning-set theorem is a finite set of vectors in a space L, then for given subsets N and K of M, the theorem provides a "good" characterization for the nonexistence of as many as k disjoint subsets N_i of N and a subspace L' of L such that each N_i exactly spans L' and such that L' contains K.

2.2. Contractions

We use the following inportant concept on matroids due to Tutte (ch. II of [7]). For any set A of elements in a matroid M, define the circuits of $M \times A$ to be the minimal nonempty intersections of A with circuits of M.

PROPOSITION 6: *The set of elements* A *and the circuits of* M × A *are a matroid* (*denoted by* M × A), *called the contraction of* M *to* A.

PROOF: Axioms 1_c and 2_c for $M \times A$ follow immediately from prop. 3 for M.

[1] This paper is a sequel to the preceding one, "Minimum Partition of a Matroid Into Independent Subsets." The numbering system there, including references, is continued here. This work was supported by the Army Research Office (Durham) and the Defense Communications Agency.

[2] Figures in brackets indicate the literature references on page 72.

COROLLARY: *Where* A *and* $\overline{A}$ *are complementary subsets of matroid* M, $\overline{A}$ *is closed (a span) in* M *if and only if matroid* $M \times A$ *contains no "loops," that is elements of rank zero.*

PROPOSITION 7: *Where* K *and* A *are subsets of matroid* M, *subset* T′ *of* A *spans* $K \cap A$ *in* $M \times A$ *if and only if there is a subset* T *of* M *such that* $T' = T \cap A$ *and such that* T *spans* K *in* M.

COROLLARY: *The spanning sets of matroid* $M \times A$ *are precisely the intersections of* A *with spanning sets of* M.

PROOF OF PROP. 7: Suppose $T' = T \cap A$ where T spans K in M. Since T spans K, for any element e in $K \cap A$, either $e \in T$ or there is a circuit C in M such that $C - e \subset T$. If $e \in T$ then $e \in T'$ and hence T' spans e in $M \times A$. If there is a C then, by definition of $M \times A$, there is a circuit C' of $M \times A$ such that $e \in C' \subset C$. It follows that $C' - e \subset T'$ and hence T' spans e in $M \times A$. Thus, the "if" part is proved.

Suppose subset T' of A spans $K \cap A$ in $M \times A$. Let $T = T' \cup \overline{A}$ where $\overline{A}$ is the complement of A in M. Then $T' = T \cap A$. Let e be any element of K. If $e \in T$, then T spans e in M. Otherwise, $e \in K \cap A$, *and* $e \notin T'$. Since T' spans e in $M \times A$, there is a circuit C' of $M \times A$ such that $e \in C'$ and $C' - e \subset T'$. By definition of $M \times A$, there is a circuit C of M such that $C' = C \cap A$. Therefore, T spans e, since $e \in C$ and $C - e \subset T$. Thus, the "only if" part is proved.

Tutte uses $M \cdot A$ to denote what we mean by the submatroid A of M; he does not follow Whitney's informality of letting A mean both a matroid and its set of elements. We will use Tutte's notation and also, where convenient, we will depart from it again by referring to $M \times A$ simply as the contraction matroid, A, of M just as we refer to $M \cdot A$ as the submatroid, A, of M. Also, A denotes the elements of either.

Where $M(G)$ is the matroid of graph G, the matroid of a subgraph H of a graph G is the submatroid of $M(G)$ which contains the elements corresponding to the edges of H, and conversely. The matroid of a "contraction graph" H of G is the contraction of $M(G)$ which contains the elements corresponding to the edges of H, and conversely.

The most instructive way to describe the meaning of contraction graph is visually. The contraction graph H of G whose edges are the set H of edges in G is the graph obtained from G by contracting to a point each edge of G not in H.

It should be pointed out that in order for there to be a contraction H of G for every subset H of edges in G, we must extend our meaning of graph (in sec. 1.1) to graphs which include edges which "meet the same node at both ends." These "loop" edges are circuits by themselves; they correspond to matroid elements which are not contained in any independent set of the matroid. This sort of matroid element corresponds in a matrix to a column of all zeros. In a matrix $N(G)$, a loop of graph G can be represented by a column of $N(G)$ which contains a 2 in the row corresponding to the node met and which contains zeros elsewhere. Relative to the matroid structure, the column is all zeros, mod. 2.

We have pointed out how any contraction of the matroid of a graph can be represented as the matroid of a graph. It is also possible to represent any contraction of the matroid of a matrix as the matroid of a matrix.

By *deleting* (or *cutting*) set of elements A in matroid M, we mean replacing matroid M by its submatroid on the set $M - A$. By *contracting* (or *shorting*) set of elements A in matroid M, we mean replacing matroid M by its contraction to the set $M - A$. Clearly, from the definition of submatroid, we can get a submatroid M' of M by deleting the elements of M not in M' one after another in any order. Clearly, from the corollary to prop. 7, we can get a contraction matroid M' of M by contracting the elements of M not in M' one after another in any order. It can be proved that for any elements a and b in a matroid M, deleting a and then contracting b is the same as contracting b and then deleting a. The proof is omitted. These results can be summarized by the following:

PROPOSITION 8: *The operations of deleting certain elements together with the operations of contracting certain other elements in a matroid are associative and commutative.*

The above proposition is equivalent to Tutte's identities 3.33 in [7]. Tutte defines a *minor* of a matroid M to be any matroid obtained from M by deleting certain elements and contracting certain other elements in M.

The following theorem is presented by Tutte (theorem 3.53 of [7]) in terms of "dendroids."

PROPOSITION 9: *If* A *and* $\overline{A}$ *are complementary sets of elements in matroid* M, *then the elements in a base of* $M \times A$ *together with the elements in a base of* $M \cdot \overline{A}$ *are the elements in a base of* M.

Proof omitted.

COROLLARY: $r(M \cdot A) + r(M \times \overline{A}) = r(M)$.

We have been calling $r(M \cdot A)$ the rank $r(A)$ of set A in matroid M. We denote $r(M \times A)$ as function $t(A)$ of sets A in matroid M.

The following theorem, which for the case of connected graphs is the one due to Tutte and Nash-Williams, completely parallels theorem 1. The "if" part of theorem 2 follows immediately from the "if" part of theorem 5 (where $M = N = K$).

THEOREM 2: *The elements of a matroid* M *can be partitioned into as many as* k *sets, each a spanning set of* M, *if and only if there is no subset* A *of elements of* M *for which*

$$|A| < k \cdot t(A).$$

Any contraction graph of a connected graph is connected. Using the last paragraph of 1.7, observe that where M is the matroid of a connected graph G, $t(A)$ is the number of nodes minus one of a contraction graph of G, and $|A|$ is the number of edges in that contraction graph.

Notice that, since $t(A) = r(M) - r(\overline{A})$, theorem 2 is easily stated without the notion of contraction.

To prove the "only if" part of theorem 2, assume that M partitions into k sets, each spanning M. By taking a subset of each of them, we get disjoint bases

$B_i(i=1, \ldots, k)$. Let A be any subset of M and let $\bar{A}$ be its complement. Since B_i is independent, $r(\bar{A}) \geq |\bar{A} \cap B_i|$. Since B_i is a base, $|A \cap B_i| + |\bar{A} \cap B_i| = |B_i| = r(M)$. Combining the two gives $|A \cap B_i| \geq r(M) - r(\bar{A}) = t(A)$. Therefore $|A| \geq \Sigma_i |A \cap B_i| \geq k \cdot t(A)$.

2.3. Short Games

It turns out to be just as easy to analyze games where, for the graph case, any subset of nodes of G are distinguished as terminals and the goal of the short player is to tag a set of edges in G which contains the edges of a connected subgraph containing all the terminals. To interpret this game in matroid terms, adjoin to G a set of new edges which form a connected graph K containing precisely the terminals as nodes. Then relative to the matroid of graph $G \cup K$, the goal of the short player is to tag a set of elements corresponding to edges in G which spans the set of elements corresponding to edges in K.

For any matroid M and nonempty subsets N and K, consider the game $L(M, N, K)$ where, as before, the cut player and short player take turns tagging different elements of N, the cut player going first. The short player wins if he tags a set of elements which span K. Otherwise, the cut player wins. Call $L(M, N, K)$ a short game if the short player can win against any strategy of the cut player.

Lehman's main theorem (explicitly for the case where K is a single element) is

THEOREM 3: *$L(M, N, K)$ is a short game if and only if N contains two disjoint sets, A_0 and B_0, of elements which span each other and which span K.*

Notice that in the two-terminal graph case, the short player wants to get a path joining the terminals. The structure characterizing when he can is two edgewise disjoint trees each containing the terminals and each containing *precisely* the same nodes as the other.

Lehman calls two (or more) sets which span each other *cospanning*. Let us verify that two disjoint cospanning sets A_0 and B_0 in N which span K provide a winning strategy in the game $L(M, N, K)$ for the short player. All that we need consider is the span $M_0 = S(A_0) = S(B_0)$ in M. Clearly, we can take A_0 and B_0 to be bases of submatroid M_0; assume that they are. If the cut player tags an element not in $A_0 \cup B_0$, we can pretend that at the same time he also tags some element of $A_0 \cup B_0$. Clearly, the short player would not be taking an illegal advantage by pretending this. Therefore, suppose the cut player in his first turn tags element a_0 in A_0.

By axiom 2 (in 1.1) there is an element b_0 of B_0 such that $(A_0 - a_0) \cup b_0$ is a base of M_0. The short player should tag an element b_0. It follows from prop. 7 that disjoint sets $A_1 = A_0 - a_0$ and $B_1 = B_0 - b_0$ are spanning sets of the contraction matroid $M_1 = M_0 - b_0$ of M_0.

Since it is the cut player's turn again, the situation of A_1 and B_1 relative to M_1 is as it was for A_0 and B_0 relative to M_0 except that M_1 is smaller. Assuming there is a strategy for the succeeding turns whereby the short player can tag a set of elements which contains a base T of reduced matroid M_1, then by prop. 9 the set $T \cup b_0$ of elements, which the short player will have tagged, is a base of matroid M_0 and hence spans set K in matroid M.

When B_0 contains only one element b_0, then b_0 itself spans M_0 and K. Hence, by induction on the number of elements, we have a winning strategy for the short player. This proves the "if" part of theorem 3. The harder "only if" part will follow from theorem 4 and theorem 5.

2.4. Nonshort Games

The notion of contraction can always be used in place of the more familiar notion of "matroid duality," and conversely, because of a theorem (3.27 of [7]) relating the contraction matroids of an M to the submatroids of the "dual to M." Sometimes one notion is convenient, sometimes the other. We do not use duality here. Lehman in treating the same topic uses mainly duality.

Lehman's interpretation of his dual results characterizing when the cut player can win for the case of graphs does not directly provide a "good" characterization in the sense of the absolute supervisor. Clearly his characterization of a short game is good in the case of graphs. However, he does not give the following analogous characterization for nonshort games. (Compare Lehman's theorem (26) and its graph interpretation with our theorem 4 and its contraction graph interpretation. See also the comment on his theorem (26) which follows his theorem (29).)

THEOREM 4: *$L(M, N, K)$ is a non-short game if and only if there is a contraction matroid M' of matroid M where set $N' = N \cap M'$ can be partitioned into two sets I_1 and I_2 such that I_1 and I_2 are both independent in M' and such that I_2 does not span the set $K' = K \cap M'$ in M'.*

Let us verify that an M', I_1, and I_2 provide a winning strategy for the cut player in game $L(M, N, K)$. If I_1 does not span K' in M' then the cut player can tag anything on his first turn. Otherwise, he should tag an element e_1 in I_1 such that $I_1 - e_1$ does not span K' in M'.

Since I_2 does not span K', there is an element $e \epsilon K'$ such that $r(e) \neq 0$. If $e \epsilon I_1$, then e is an element e_1. Otherwise, by axiom 2′ there is a *unique* circuit C in $I_1 \cup e$ and so any element of $C - e$ is an element e_1. ($C - e$ is not empty since $r(e) \neq 0$.) Now neither the untagged elements $I_1' = I_1 - e_1$ of I_1 nor the untagged elements $I_2' = I_2$ of I_2 span K' in contraction matroid M'.

Even if the short player tags an element not in M', clearly the cut player is not taking an illegal advantage by pretending the short player also tags an untagged element in M' if there are any. Therefore, assume the short player does tag one, say e_2 in I_2'. Consider the contraction matroid $M'' = M' - e_2$ of M. (By prop. 8, the contraction of matroid M' to set M'' is the same as the contraction of matroid M to set M''.) By the circuit definition of contraction matroid, set $I_2'' = I_2' - e_2$ will be independent in M'' and will not span $K'' = K' \cap M''$ in M''.

Again by the definition of contraction matroid, if e_2 is not in the span of I_1' in M' then I_1' is independent in M''. In this case, the cut player should tag some element e_1' such that $I_1''=I_1'-e_1'$ does not span K'' in M''.

By the definition of contraction matroid and prop. 3, if e_2 is in the span of I_1' in matroid M' then set I_1' contains just one circuit of contraction M'' and does not span K'' in M''. In this case, the cut player should tag some element e_1' in the one circuit of I_1' in M'', so that $I_1''=I_1'-e_1'$ is independent in M''.

Thus in either case, after the cut player takes his second turn, the untagged elements of M'' partition into sets I_1'' and I_2'' where, in contraction M'', both are independent and neither spans K''. There are no elements tagged by the short player in M''. The situation is identical to the one in M' right after the cut player took his first turn except that M'' has fewer untagged elements.

Hence, by induction on the number of untagged elements in the contraction matroid, if the cut player tags as described, he eventually reaches a contraction matroid $M^{(h)}$ in which all the elements are tagged by him, and yet for which there is an $e\epsilon K^{(h)}=M^{(h)}\cap K$ such that $r(e)\neq 0$ in $M^{(h)}$ (since $K^{(h)}$ is not spanned by the empty $I_1^{(h)}$ or the empty $I_2^{(h)}$). The cut player will then have won the game, because for the short player to win he must tag a set, say T, which spans K in matroid M. By prop. 7, for any such T and any set $M^{(h)}$ in M, $T\cap M^{(h)}$ must span $K^{(h)}=K\cap M^{(h)}$ in matroid $M^{(h)}$, which is impossible. This proves the "if" part of theorem 4.

2.5. Cospanning-Sets Theorem

We still have to prove the "only if" parts of theorems 3 and 4. They follow immediately from theorem 5 (for the case $k=2$). We proved the part of theorem 3 which says "$P \Rightarrow$ (L is a short game)". We proved the part of theorem 4 which says "$Q \Rightarrow$ (L is not a short game)". Theorem 5 says "$P \Leftrightarrow$ not Q". Logic yields that "(L is a short game)$\Rightarrow P$" and "(L is not a short game)$\Rightarrow Q$".

THEOREM 5: *For any matroid* M *and any subsets* N *and* K *of elements in* M, *there exist as many as* k *disjoint subsets of* N *which span each other and which span* K, *if any only if there is no contraction matroid* M′ *of* M *where* N ∩ M′ *partitions into as few as* k *sets such that each is independent in* M′ *and such that at least one of them does not span* K ∩ M′

PROOF: The "only if" part of theorem 5 follows from the "if" part of prop. 7. Suppose in matroid M there exist k disjoint subsets T_i of $N\subset M$, which span each other and which span $K\subset M$. Let M' be any contraction of M. Where a set T_i is the T of prop. 7; where set M' is the A and matroid M' is the $M\times A$ of prop. 7; and where S, the span (closure) in M of each T_i, is the K of prop. 7; prop. 7 says that $T_i'=T_i\cap M'$ spans $S'=S\cap M'$ in matroid M'. Since each T_i' spans S', each T_i' contains at least $r(S')$ elements where $r(S')$ is the rank of set S' in matroid M'. Since all the sets T_i' are mutually disjoint, $N\cap S'$ contains at least $k\cdot r(S')$ elements.

On the other hand, suppose $N\cap M'$ partitions into as few as k independent sets I_i of M' where one of them I_1 does not span $K\cap M'$ and hence does not span S'. Since each $I_i'=I_i\cap S'$ is independent, each I_i' contains at most $r(S')$ elements. Since I_1' does not span S', it contains fewer than $r(S')$ elements. Therefore $N\cap S'=\cup I_i'$ contains fewer than $k\cdot r(S')$ elements. Thus, the "only if" part of theorem 5 is proved.

The "if" part of theorem 5 follows from propositions 8 and 9 and theorem 1. Let M be any matroid, let N and K be any subsets of M, and let k be any positive integer. Suppose A_0 is a maximal subset of N such that $|A_0|=k\cdot r(A_0)$ and $|A|\leqslant k\cdot r(A)$ for all $A\subset A_0$. Set A_0 may be empty. By theorem 1, A_0 partitions into k independent sets, I_i. Since $|A_0|=k\cdot r(A_0)$, each I_i must be a base of submatroid A_0 and of course also a base of $S(A_0)$, the span of A_0 in M.

Let M' be the contraction matroid of M obtained by contracting $S(A_0)$ in M. Suppose A_1 is a subset of $N'=N\cap M'$ such that $|A_1|=k\cdot r(A_1)$ and $|A|\leqslant k\cdot r(A)$ in matroid M' for all $A\subset A_1$. Then like A_0 in M, A_1 partitions into k bases I_i' of submatroid A_1 of M'. By prop. 8, submatroid A_1 of M' is the contraction to A_1 of the submatroid $A_1\cup S(A_0)$ of M. Call it minor A_1. By prop. 9, a base of minor A_1 together with a base of submatroid $S(A_0)$ of M is a base of submatroid $A_1\cup S(A_0)$ of M.

In particular, by pairing the sets I_i' one-to-one with the sets I_i, we get k disjoint bases $I_i''=I_i'\cup I_i$ of submatroid $A_1\cup S(A_0)$ of M. Since $\cup I_i''=A_0\cup A_1\subset N$ and since the sets I_i'' span each other in M, $|A_0\cup A_1|=k\cdot r(A_0\cup A_1)$ and, by the "only if" part of theorem 1, $|A|\leqslant k\cdot r(A)$ in M for all $A\subset A_0\cup A_1$. However, A_0 was taken to be maximal for this property, and hence A_1 is empty. Thus, matroid M' contains no nonempty A_1, as defined.

Since $S(A_0)$ is closed in M, the matroid M', obtained by contracting $S(A_0)$, contains no element of rank zero (corollary to prop. 6). Suppose $N'=N\cap M'$ contains a nonempty set A_2 such that $|A_2|\geqslant k\cdot r(A_2)$ in M'. Take A_2 to be minimal. By the nonexistence in M' of a nonempty A_1 as described above, we have that $|A_2|>k\cdot r(A_2)$. Since there are no elements of rank zero, A_2 contains at least two elements. Deleting an element from A_2 to get a nonempty A_3, we have $|A_3|\geqslant k\cdot r(A_2)\geqslant k\cdot r(A_3)$ in M', which contradicts the minimality of A_2. Therefore, for all nonempty subsets A of N', $|A|<k\cdot r(A)$ in M'.

Suppose some element $g\epsilon K$ is contained in matroid M'. Since g does not have zero rank, there exists a matroid M_h, which contains the elements of M' plus a new auxiliary element h, such that h and g form a circuit in M_h and such that submatroid M_h-h of M_h is the matroid M'. It is easy to verify that M_h is such a matroid where the circuits of M_h are (1) the set consisting of g and h, (2) the circuits of M', and (3) sets $(C-g)\cup h$ where C is a circuit of M' which contains g. Let $N_h=N'\cup h$. It follows from the relation $|A|<k\cdot r(\mathrm{A})$ in matroid M' for all nonempty $A\subset N'$, that $|A_h|\leqslant k\cdot r(A_h)$ in M_h for all A_h in N_h.

Hence by theorem 1, N_h can be partitioned into k independent sets I_i^h of M_h, including the set, say I_1^h, which contains h. In matroid M' the set $I_1^h - h$ is independent and does not span g. All of the other sets I_i^h are independent in M'. These sets I_i^h and $I_1^h - h$ are a partition of N'.

Thus, if there is no such partition of $N' = N \cap M'$ for contraction M' of M then no element of K is in M'. Thus $K \subset S(A_0)$. In this case, the k bases I_i of submatroid $S(A_0)$ of M span each other and span K in M. This completes the proof of theorem 5.

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