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On the Surface Duality of Linear Graphs*

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THEOREM: A 1-1 correspondence between the edges of two connected graphs is a duality with respect to some polyhedral surface embedding if and only if for each vertex v of each graph, the edges which meet v correspond in the other graph to the edges of a subgraph G_v which is connected and which has an even number of its edge-ends to each of its vertices (where if an edge meets v at both ends its image in G_v is counted twice). Using the Euler formula, the characteristic of the surface is determined by the two graphs. Thus, the theorem generalizes a variation of the H. Whitney condition for a graph to be planar.

A graph, for purposes here, is a finite set of edges, i.e., homeomorphs of a closed line segment, and an identification of the edge endpoints into a number of equivalence classes called vertices. We call edge endpoints simply *ends* of the edge. We say that an edge end *meets* the vertex which is its image.

The edges and vertices of a polygonal disk, bounded by one or more edges, form a graph—a particular kind, called a circuit. When we identify in pairs, according to homeomorphisms, the edges of a finite set of polygonal disks, we form (by composing these identifications with those in the disks) one or more closed surfaces. The pair $S_U=(S, G_U)$, consisting of one such surface, S, and the graph, G_U, whose edges and vertices are the distinct images in S of disk edges and vertices, is called a *polyhedral surface*, or a *map*, or a *polyhedral surface embedding of* G, where G is a homeomorph of G_U.

The vertices and edges of the graph G_U are also called the vertices and edges of S_U. The disks are called the faces of S_U. The graph G_U of a map is connected in the usual sense. Each of its edges is the image of two face-edges. A vertex of G_U is the image of one or more face vertices and also the image of the same number of edge ends of edges of G_U.

Associated with any map $S_U=(S, G_U)$ is a (topologically unique) dual map $S_W=(S, G_W)$ with the properties that (a) a map is the dual map of its dual map, (b) each vertex of a map is interior to a face of its dual map, (c) each face of a map contains exactly one vertex of its dual map, and (d) each edge of a map crosses once exactly one edge of the dual map.

By an *edge-correspondence* between two graphs, we mean a 1–1 correspondence between the edges of the two graphs. An edge-correspondence is called a *surface duality correspondence* or briefly a *duality* if the two graphs respectively have homeomorphisms with the graphs of dual maps so that the images of corresponding edges cross each other.

Our main objective, Theorem 1, will give a necessary and sufficient condition for an edge-correspondence between two graphs G_U and G_W to be a surface duality correspondence.

Using Euler's formula, the characteristic x of the surface for a surface duality correspondence is determined by the two graphs. Where V_U, E_U, and F_U are the numbers of vertices, edges, and faces in map S_U and V_W, E_W, and F_W are the numbers of vertices, edges, and faces in dual map S_W, we have $x(S)=V_U-E_U+F_U=V_W-E_W+F_W$ and $V_U=F_W$, $E_U=E_W$, and $F_U=V_W$.

Hence,

$$x(S)=V_U-E_U+V_W. \tag{1}$$

COROLLARY 1 (to Theorem 1): A necessary and sufficient condition for a connected graph G_U to have a polyhedral surface embedding of given characteristic x is that it have an edge-correspondence with some graph G_W such that (a) the condition of Theorem 1 is satisfied and (b) formula 1 is satisfied.

Corollary 1 is analogous to Hassler Whitney's well-known theorem on the planar duality of graphs[1]. He defines two graphs to be dual if there is an edge-correspondence between them that satisfies certain combinatorial conditions. His theorem states that a graph is planar if and only if it has a dual.

That a connected graph be planar is equivalent to its having a polyhedral surface embedding which is spherical. A polyhedral surface is a sphere if and only if it has characteristic equal two. Hence, for $x=2$ conditions (a) and (b) of Corollary 1 are equivalent (for connected graphs) to Whitney's notion of duality.

THEOREM 1: A 1-1 correspondence between the edges of two connected graphs is a duality with respect to some polyhedral surface embedding if and only if for each vertex v of each graph, the edges which meet v correspond in the other graph to the edges of

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[1] H. Whitney, Non-separable and planar graphs, Trans. Am. Math. Soc. **34**, 339–362 (1932).

a subgraph G_v which is connected and which has an even number of its edge-ends to each of its vertices (where if an edge meets v at both ends its image in G_v is counted twice).

Proof of necessity. A graph B_v, formed by the edges and vertices of a polygonal disk D_v before identifications in a map, is connected and has exactly two edge-ends meeting each of its vertices. In other words, B_v is a *circuit* or a *simple closed path.* Under the identification which occurs when D_v is regarded as a face of a map $S_U = (S, G_U)$, the image of B_v, call it G_v, is a connected subgraph of G_U.

Each vertex u in G_v is the image of a set $V_{v,u}$ of one or more vertices of B_v. Each edge in G_v is the image of either one or two edges in B_v; in the later case call it a *double edge* in G_v. Conversely, the image of every edge-end and vertex which meet in B_v is an edge-end and vertex which meet in G_v. Hence, if we count twice the edge-ends of each double edge in G_v, then u meets an even number of edge-ends of edges in G_v, two for each member of $V_{v,u}$.

Each edge of G_v crosses exactly one edge of G_W where $S_W = (S, G_W)$ is the dual of S_U. These edges of G_W are exactly those which meet the one vertex v of G_W which is interior to disk D_v. Furthermore, one of these edges meets v at both ends if and only if the edge of G_v it intersects is double. Thus, the "only if" part of the theorem is proved.

A *closed path* in a graph G is a mapping C of a circuit B into G which takes edge-interiors of B homeomorphically to edge-interiors of G and which takes edges to edges and vertices to vertices so that edge-vertex incidence relations are preserved. Equivalently, a *closed path* is a cyclic order $(\ldots v_iE_iv_{i+1}E_{i+1} \ldots)$, i taken modulo some n, of vertices and edges in G such that for all i edge E_i meets vertex v_i at one end and vertex v_{i+1} at the other end. There is no distinguished position or orientation of the cyclic order. E_i and E_{i+1} are said to be *adjacent at* vertex v_{i+1} in the cyclic order.

An *Euler path* in a graph G is defined as a closed path which contains each edge of G exactly once. (Here, a "double-edge" is to be regarded as two edges.)

Euler's "Konigsberg Bridge" theorem states that a graph G has an Euler path if and only if G is connected and has an even number of edge-ends to each vertex. Theorem 1 is really a strengthened form of Euler's theorem. It states conditions for the existence of a whole family of Euler paths which fit together in a certain way.

To construct in G, connected and even at each vertex, an Euler path, first construct a family of closed paths which together contain each edge of G exactly once. Try doing this and you can't miss. Because G is connected, if the family of paths has more than one member then some two members C_a and C_b have at least one vertex v_0 in common. If C_a and C_b can always be replaced in the family by one closed path, then by iteration an Euler path can be obtained.

Suppose $(\ldots E_1^a v_0 E_2^a \ldots)$ in C_a and $(\ldots E_1^b v_0 E_2^b \ldots)$ in C_b. There are *two ways* to change edge-adjacencies at v_0, either to $E_1^a v_0 E_1^b$ and $E_2^a v_0 E_2^b$ or to $E_1^a v_0 E_2^b$ and $E_2^a v_0 E_1^b$, so that with the other adjacencies unchanged C_a and C_b join to a single closed path. That is, of the three possible adjacencies for E_1^a at v_0, with E_2^a or E_1^b or E_2^b, one yields two closed paths and two yield one closed path. This extra freedom in constructing an Euler path is important to Theorem 1.

Proof of sufficiency. Let there be an edge-correspondence between connected graphs G_U and G_W which satisfies the condition in Theorem 1. For each vertex v in G_W, find an Euler path C_v in G_v, where G_v is the subgraph of G_U which corresponds to the edges meeting v. Let each of these Euler paths bound a disk. More precisely, for each v let the circuit, which is the pre-image of mapping C_v, bound a disk D_v; and then let the boundary of D_v be identified with G_v according to the mapping C_v. Because edges of the disks D_v identify together in pairs according to continuous mappings that are 1–1 in their interiors, the resulting connected complex K is locally planar except possibly at the vertices of G_U.

But what happens at a vertex u of G_U? Each vertex of each disk D_v meets two edge-ends in the disk and these edge-ends (for all of the disks D_v) identify in pairs to edge-ends of G_U in K. This causes disk-vertices (or disk-vertex "corners") and the edge-ends of G_U to arrange themselves in a number of cyclic orders. Let H_u be the set of these cyclic orders whose disk-vertices are pre-images of u—that is, whose edge-ends meet u.

By construction of the cyclic orders, a pair of edges in G_U, with edge-ends adjacent at a disk vertex d in cyclic order $h \epsilon H_u$, correspond to a pair of edges of G_W which meet the vertex, say v, corresponding to the disk D_v which contains d. Thus the edges of G_U with edge-ends in $h \epsilon H_u$ correspond to a set of edges in G_W which form a closed path C_h in G_W with cyclic order determined by h.

Graph G_W can be embedded in K so that each vertex v in G_W is contained in the interior of corresponding disk D_v and so that edges of G_W cross corresponding edges of G_U. Then K is a map (S, G_U) with dual (S, G_W), if and only if for each u of G_U the family H_u contains one member. Where H_u contains k_u cyclic orders, a neighborhood of u in K consists of k_u open disks which intersect only at u. When $k_u = 1$ for all u, the face of (S, G_W) containing u is a polygonal disk bounded by the pre-image of the mapping C_h, where h is the only member of H_u.

The edges of G_U which meet u correspond to a subgraph, say G_u, of G_W. Since by hypothesis G_u is connected, when $k_u > 1$ there exist two closed paths C_g and C_h, g and $h \epsilon H_u$, which in G_u have a common vertex, say v. Hence, closed path C_v passes through u at least twice, once at a vertex d_g of D_v where d_g is a member of cyclic order $g \epsilon H_u$ and once at a vertex d_h of D_v where d_h is a member of cyclic order $h \epsilon H_u$.

Suppose $(\ldots E_1^a u E_1^b \ldots E_2^b u E_2^a \ldots)$ describes the closed path C_v where ends of edges E_1^a and E_1^b of G_U are adjacent at d_g in g and where ends of edges E_2^b and E_2^a of G_U are adjacent at d_h in h. From C_v, by changing the adjacencies $E_1^a u E_1^b$ and $E_2^b u E_2^a$ to $E_1^a u E_2^b$ and $E_1^b u E_2^a$ and leaving unchanged the other adjacencies in C_v, we obtain another Euler path C_v' of G_v. Replacing disk D_v by new disk D_v', we get a new complex K'. At the same time, cyclic orders g and h combine to a single cyclic order with the same edge-end adjacencies except that, here also, the E_1^a, E_1^b and E_2^b, E_2^a adjacencies are replaced by E_1^a, E_2^b and E_1^b, E_2^a adjacencies.

The new family H_u' of cyclic orders associated with u contains one less cyclic order than H_u. The families of cyclic orders associated with the other vertices of G_U remain the same. Repeating the process just described enough times we eventually get only one cyclic order in each family. Therefore, Theorem 1 is proved.

A corollary of Theorem 1 is that every connected graph G can be embedded in some closed surface S so that $S-G$ is an open disk. (For example, a circuit has such an embedding in the projective plane.)

Another corollary is that every connected graph G which has an even number of edge-ends to each vertex is the graph of a map formed by identifying the edges of one polygonal disk with the edges of another. Notice how this corollary strengthens Euler's theorem—from asserting the existence of one Euler path in G to asserting the existence of two Euler paths in G (distinct unless G is a circuit) which fit together in a certain way.

The characteristic x of a closed surface determines uniquely its topology only if x is two or an odd number (less than two). For $x=2$ the surface is a sphere and for odd values the surface is nonorientable. For each of the other possible values of x, nonpositive and even, there are two closed surfaces, one orientable and one nonorientable. Though the two graphs of a surface duality correspondence determine the surface characteristic, Theorem 1 provides no control over which of the two surfaces the maps might be.

In trying to find an analog of Theorem 1 for orientable maps, it is natural to use the notion of *oriented edge-correspondence* between two graphs, G_U and G_W. That is a 1–1 correspondence between the edges of G_U with orientations and the edges of G_W with orientations such that if two edges with orientations correspond then the same edges with opposite orientations correspond.

For any G_U and G_W which are the graphs of dual oriented maps there is determined an oriented edge-correspondence with the following property. P_0: For each vertex v of each graph, the edges directed away from v correspond in the other graph to a directed subgraph which is connected and which has the same number of its edges directed away from as directed toward each of its vertices. Property P_0 follows from the fact that the faces of the oriented maps determine a *directed Euler path* in each of these directed subgraphs.

However, an oriented edge-correspondence with property P_0 between two connected graphs is not necessarily an oriented surface duality correspondence. That is we cannot always find directed Euler paths which fit together into single cyclic orders at each vertex. In particular, there are lots of connected graphs G_U, even where V_U-E_U is odd, which are not graphs of oriented maps with one face, even though for any connected graph one can set up an oriented edge-correspondence satisfying P_0 with a graph which consists of one vertex and the proper number of edges. The directed-graph analog of Euler's theorem is true, but not true with enough leeway for the corresponding oriented surface-duality theorem to be true.

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