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On the Intensity Distribution $\frac{2R}{\sqrt{\alpha\beta}}\exp\left[-\frac{R^2}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right] I_0\left(\frac{R^2}{2}\left[\frac{1}{\beta}-\frac{1}{\alpha}\right]\right)$ and Its Application to Signal Statistics

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A few derivation methods of the distribution in the title are discussed from some different viewpoints. It is found that this type of fading may be caused by the interferences of two correlated waves. A transitional aspect of the distribution due to the change of a parameter is illustrated in detail. Some applications of the distribution to signal statistics are discussed. The functional similarity is found between the characteristic function and the error probability of signals. Using this basic relation some error probabilities are estimated for various cases.

1. Introduction

Since about 25 years ago, the author has engaged in an extensive investigation of the characteristics of radio fading phenomena. At an earlier stage of our series of observations, a kind of fading much severer than the Rayleigh type was sometimes experienced in particular circuits, especially in the Taipei–Tokyo circuit operating at the frequency of about 10 Mc/s.

The appearance of such a type of fading was quite unexpected. Therefore our considerable interest was directed to finding the causes of this type of fading. Unfortunately, however, any convincing explanation about the mechanisms of the fading seemed to be beyond our knowledge at that time.

In due course, we fortunately had an opportunity to make careful observations on several communication circuits in order to find possible mechanisms of fading, using much improved equipments and advanced techniques of observation. Through this well-prepared experiment, the existence of a type of fading stated above was clearly confirmed.

Meanwhile, our theoretical study of fading was proceeding a little ahead of our experimental research. In the course of our statistical approach, three basic distributions were found as particular solutions of the so-called problem of random interference. One is the distribution referred to in the title, tentatively named the "q-distribution" [Nakagami and Sasaki, 1943]. Soon after that, it was found that the formula accounted for the type of observed fading which had shown much larger fluctuation than the Rayleigh fading.

It seems to the author, however, that most of the researchers in this field still have a misconception that the Rayleigh distribution exhibits the largest fluctuations among all types of fading.

In the following, some discussions will be made on the q-distribution. In particular, we shall derive the distribution on the assumption of some propagational modes which may be supposed to exist in actual fading.

2. Derivation of the q-Distribution

We shall derive the q-distribution and its generalized form from gaussian distribution.

Now, let us start from the two-dimensional gaussian distribution

$$p(x,y)=\frac{1}{2\pi\sqrt{\sigma_1\sigma_2(1-\rho^2)}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1}-\frac{2\rho xy}{\sqrt{\sigma_1\sigma_2}}+\frac{y^2}{\sigma_2}\right)\right\}. \tag{1}$$

By changing the variables (x, y) to $(R\cos\theta, R\sin\theta)$, and performing the integration with respect to θ from 0 to 2π, we have

$$p(R)=\int_0^{2\pi} p(R\cos\theta, R\sin\theta)\left|\frac{\partial(x,y)}{\partial(R,\theta)}\right|d\theta$$

$$=\frac{R}{\sqrt{\sigma_1\sigma_2(1-\rho^2)}}\, e^{-\frac{R^2}{4(1-\rho^2)}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)}\, I_0\left\{\frac{R^2}{2(1-\rho^2)}\sqrt{\frac{1}{4}\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)^2+\left(\frac{\rho}{\sqrt{\sigma_1\sigma_2}}\right)^2}\right\}. \tag{2}$$

Here, if we put

$$\left.\begin{aligned}\alpha&=\sigma_1+\sigma_2+\sqrt{(\sigma_2-\sigma_1)^2+4\rho^2\sigma_1\sigma_2},\\ \beta&=\sigma_1+\sigma_2-\sqrt{(\sigma_2-\sigma_1)^2+4\rho^2\sigma_1\sigma_2},\end{aligned}\right\} \tag{3}$$

then (2) takes a standard form of the q-distribution

$$p(R)=\frac{2R}{\sqrt{\alpha\beta}}\, e^{-\frac{R^2}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}\, I_0\left\{\frac{R^2}{2}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\right\}. \tag{4}$$

It is of great importance to note that either in case of σ_1 equal to σ_2, or ρ^2 equal to zero, (2) retains the same standard form as (4). However, if the above two conditions are satisfied simultaneously, the distribution turns out to become the Rayleigh form

$$p(R)=\frac{2R}{\sigma_1+\sigma_2}\, e^{-\frac{R^2}{\sigma_1+\sigma_2}}. \tag{5}$$

Further, if ρ tends to unity, then β approaches to zero, and (4) takes the following form

$$p(R)=\frac{1}{\sqrt{2\pi(\sigma_1+\sigma_2)}}\, e^{-\frac{R^2}{\sigma_1+\sigma_2}}. \tag{6}$$

It is of much interest for us to see that the limiting form (6) is identical with the limiting case of the observed m-distribution [Nakagami, 1943],

$$p(R)=\frac{2R^{2m-1}m^m}{\Gamma(m)\,\Omega^m}\, e^{-\frac{mR^2}{\Omega}}\equiv \mathcal{M}(R, m, \Omega), \qquad m\geq\frac{1}{2}, \tag{7}$$

as the parameter m approaches ½. In this regard, we shall discuss the fading mechanisms further in the following section.

2.1. Derivations of a Generalized and an Additive Form of the q-Distribution

A generalized and an additive form of the distribution may be deduced in several ways. The simplest cases, however, are shown as follows:

First, if we let

$$p(x)=\frac{|x|^{2m-1}}{\alpha^m\Gamma(m)}\, e^{-\frac{x^2}{\alpha}},\ p(y)=\frac{|y|^{2m-1}}{\beta^m\Gamma(m)}\, e^{-\frac{y^2}{\beta}}, \tag{8}$$

then we may have the standard form

$$p(R)=\frac{2\sqrt{\pi}R^{2m}e^{-\frac{R^2}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}}{\Gamma(m)\,(\alpha\beta)^m\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)^{m-1/2}}\, I_{m-1/2}\left\{\frac{R^2}{2}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\right\}. \tag{9}$$

The detailed properties of this type of distribution are fully discussed in Nakagami and Nishio [1954a].

Next, we shall derive a simplest additive form of distribution. Now we start with the distribution

$$p(x, y)=\frac{(xy)^2}{2\pi\sigma^3(1+2\rho^2)\sqrt{\rho(1-\rho^2)}}\exp\left\{-\frac{x-2\rho xy+y^2}{2(1-\rho^2)\sigma}\right\}. \tag{10}$$

By the transformation of variables from (x, y) to $(R\cos\theta, R\sin\theta)$, we have the distribution

$$p(R)=\frac{4R^5}{[(\alpha+\beta)^2+2(\alpha-\beta)^2]\sqrt{\alpha\beta}}e^{-\frac{R^2}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}\left[I_0\left\{\frac{R^2}{2}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\right\}+I_2\left\{\frac{R^2}{2}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\right\}\right], \tag{11}$$

where

$$\alpha=2\sigma(1+\rho),\ \beta=2\sigma(1-\rho). \tag{12}$$

3. Derivation of the q-Distribution From an Actual Mode of Propagation

We are now in a position to derive the q-distribution from the two correlated Rayleigh or the m-type of fading. At first we shall begin with the Rayleigh type of fading in the following.

3.1. Case of Correlated Rayleigh Fading

As is well known, the joint distribution of the Rayleigh type assumes the form

$$p(r_1, r_2)=\frac{4r_1r_2}{\Omega_1\Omega_2(1-k^2)}e^{-\frac{\Omega_2r_1^2+\Omega_1r_2^2}{\Omega_1\Omega_2(1-k^2)}}I_0\left(\frac{2kr_1r_2}{\sqrt{\Omega_1\Omega_2}(1-k^2)}\right), \tag{13}$$

where $\Omega_1=\overline{r_1^2}$ and $\Omega_2=\overline{r_2^2}$, respectively, and k^2 denotes the correlation coefficient between r_1^2 and r_2^2.

Then, the resultant intensity distribution of the sum of two vectors $\dot{r}_1$ and $\dot{r}_2$ and a scattered component can be easily expressed by the Hankel form of integral:

$$p(R)=R\int_0^\infty \lambda J_0(\lambda R)F(\lambda)d\lambda, \tag{14}$$

where $F(\lambda)$ is tentatively called the amplitude characteristic function. If we assume that the phase difference of $\dot{r}_1$ and $\dot{r}_2$ is random, then $F(\lambda)$ becomes

$$F(\lambda)=\overline{J_0(\lambda r_1)\,J_0(\lambda r_2)\,e^{-\frac{\delta}{4}\lambda^2}}, \tag{15}$$

where δ indicates the scattered power, and

$$\overline{J_0(\lambda r_1)J_0(\lambda r_2)}=\iint_0^\infty J_0(\lambda r_1)J_0(\lambda r_2)p(r_1, r_2)dr_1dr_2. \tag{16}$$

This form of integral may be readily evaluated as

$$\overline{J_0(\lambda r_1)J_0(\lambda r_2)}=e^{-\frac{\lambda^2}{4}(\Omega_1+\Omega_2)}I_0\left(\frac{\sqrt{\Omega_1\Omega_2}}{2}k\lambda^2\right). \tag{17}$$

Hence, we may arrive at final result

$$p(R)=R\int_0^\infty \lambda J_0(\lambda R)I_0\left(\frac{\sqrt{\Omega_1\Omega_2}}{2}k\lambda^2\right)e^{-\frac{\lambda^2}{4}(\Omega_1+\Omega_2+\delta)}d\lambda$$

$$=\frac{2R}{\sqrt{\alpha\beta}}e^{-\frac{R^2}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}I_0\left\{\frac{R^2}{2}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\right\}, \tag{18}$$

where

$$\left.\begin{aligned}\alpha&=(\Omega_1+\Omega_2+\delta)+2\sqrt{\Omega_1\Omega_2}\,k,\\ \beta&=(\Omega_1+\Omega_2+\delta)-2\sqrt{\Omega_1\Omega_2}\,k.\end{aligned}\right\}. \tag{19}$$

In a limiting case of k equal to zero, $p(R)$ assumes the Rayleigh form

$$p(R)=\frac{2R}{\Omega_1+\Omega_2+\delta}e^{-\frac{R^2}{\Omega_1+\Omega_2+\delta}}. \tag{20}$$

Under the conditions $\Omega_1=\Omega_2=\Omega$, $k\to 1$, and $\delta\to 0$, (18) turns to the half Gaussian form

$$p(R)=\frac{1}{\sqrt{\pi\Omega}}e^{-\frac{R^2}{4\Omega}}. \tag{21}$$

It is of great interest to see (20) and (21), especially the latter distribution; because these two distributions are both situated at the extreme boundaries of the q-distribution, as was already proved from a different viewpoint.

In figure 1, a family of curves of the distribution are shown with $\epsilon=\delta/\Omega$ as a parameter. They are very suggestive of the transitional aspect of the shape of distribution.

The comparison of the q-distribution with the m-distribution is shown in figure 2 under the extreme conditions respectively. From this figure we may find a good agreement of the two formulas even at the extremities. This functional similarity was already discussed in detail.

Some examples of the deepest type of fading observed in a microwave communication circuit in Japan are shown in figure 3. The foregoing discussions seem to maintain that the observed distribution shown in figure 3 might be caused by correlated interferences.

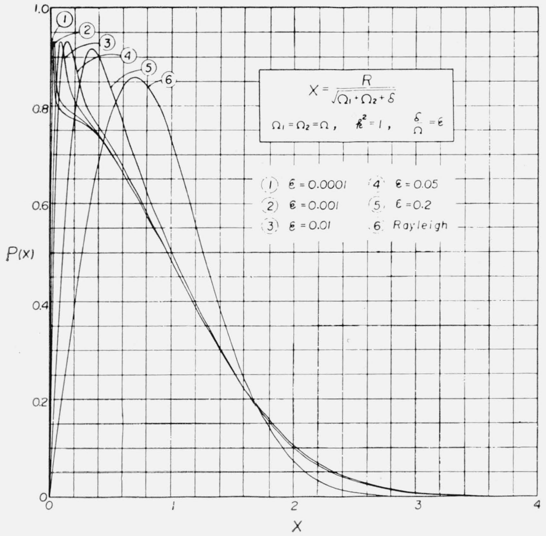

FIGURE 1. *Transitional aspect of the shape of the probability density function p(x).*

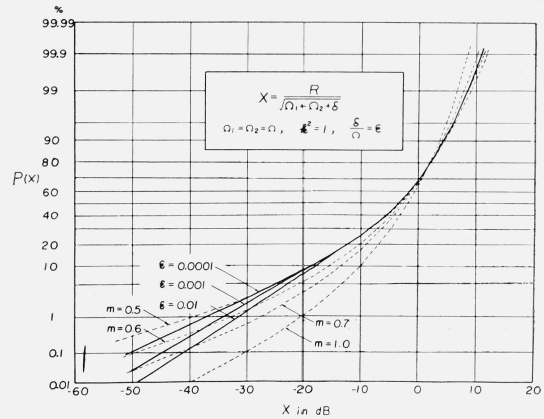

FIGURE 2. *Comparison of the q-distribution with the m-distribution having the parameter m close to ½.*

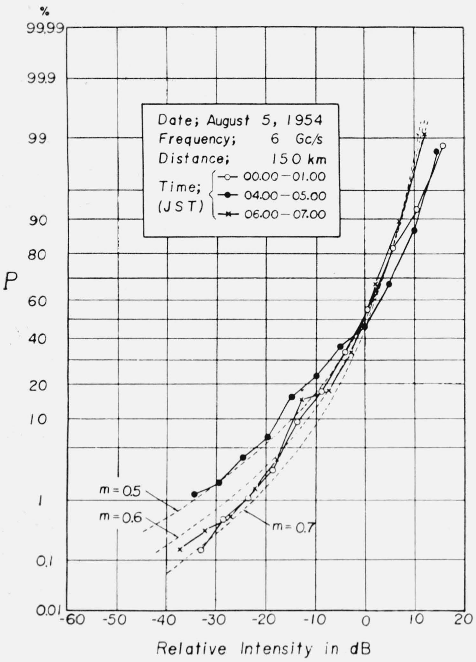

FIGURE 3. *Comparison of the observed distribution in a microwave circuit with the* m*-distribution having the parameter* m *smaller than 1.*

3.2. Derivation From the Correlated m-Distribution

We now proceed to a more general case of two correlated m-variables. As is well known, the joint probability density function assumes the form [Nakagami and Nishio, 1954b, 1955]

$$p_m(r_1,r_2)=\frac{4(r_1r_2)^m e^{-\frac{\sigma_2 r_1^2+\sigma_1 r_2^2}{\sigma_1\sigma_2(1-k^2)}}}{\Gamma(m)\sigma_1\sigma_2(1-k^2)\,(\sqrt{\sigma_1\sigma_2}k)^{m-1}} I_{m-1}\left\{\frac{2kr_1r_2}{\sqrt{\sigma_1\sigma_1}(1-k^2)}\right\}, \tag{22}$$

where

$\sigma_1 m=\overline{r_1^2}$, $\sigma_2 m=\overline{r_2^2}$. Therefore

$$p(R)=R\int_0^\infty \lambda J_0(\lambda R)\overline{J_0(\lambda r_1)J_0(\lambda r_2)}\,e^{-\frac{\delta}{4}\lambda^2}d\lambda, \tag{23}$$

where

$$\overline{J_0(\lambda r_1)J_0(\lambda r_2)}=\iint_0^\infty J_0(\lambda r_1)J_0(\lambda r_2)\,p_m(r_1,r_2)dr_1dr_2. \tag{24}$$

For the sake of simplicity, we treat the simpler case of $m=2$ and $\delta=0$. The calculation of (23) is somewhat complicated even under the simplified conditions, $\sigma_1=\sigma_2$ and $m=2$, respectively. The final result is

$$p(R)=\frac{R}{\Omega\sqrt{1-k^2}}e^{-\frac{R^2}{\Omega(1-k^2)}}I_0\left(\frac{k}{1-k^2}\cdot\frac{R^2}{\Omega}\right)+\frac{R^3}{\Omega^2\sqrt{1-k^2}k}e^{-\frac{R}{\Omega(1-k^2)}}I_1\left(\frac{k}{1-k^2}\cdot\frac{R^2}{\Omega}\right). \tag{25}$$

This form of distribution resembles the distribution (11). In this case, if k^2 tends to zero, (25) takes the form

$$p(R)=\tfrac{1}{2}\{\mathcal{M}(R,1,\Omega)+\mathcal{M}(R,3,\Omega)\}. \tag{26}$$

4. Some Applications of the Distributions to Signal Statistics

4.1. Criterion of Signal Improvement

In the field of signal statistics, it is important to evaluate the performance improvement attained in various communication systems. For this purpose, two criteria may be used. One is expressed by the cumulative distribution, $P(R)$, and the other by the so-called error probability, P_e.

The former is expressed as

$$P(R)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{zR^2}\phi(z)\frac{dz}{z}, \tag{27}$$

$$\phi(z)=\int_0^\infty e^{-zR^2}p(R)dR, \tag{28}$$

where $P(R)$ and $p(R)$ are the probability and the probability density function of signal intensity R. This criterion apparently seems to be irrelevant to noise information. The latter criterion, P_e, however, contains noise information explicitly. Though it is a quantity defined with respect to digital communication systems, it can be used in other systems as well. As we are here concerned with fading signals, we may better choose the average error probability $\overline{P}_e$ as an adequate measure of signal quality.

In general, P_e takes various forms depending on the methods of signal detections. In the case of unknown phase information [Turin, 1958], it is simply expressed by

$$\overline{P}_e=\frac{1}{2}e^{-\frac{E}{2N}}, \tag{29}$$

where E is the signal energy and N the white noise power density, respectively. The average error probability, $\overline{P}_e$ in this case, is simply expressed by

$$\overline{P}_e=\frac{1}{2}\int_0^\infty e^{-\frac{R^2}{4N}}p(R)dR \tag{30}$$

$$=\frac{1}{2}[\phi(z)]_{z=\frac{T}{4N}}, \tag{31}$$

where $E=\frac{1}{2}R^2T$, T=element length in seconds.

It is worthy of notice that $\overline{P}_e$ assumes the identical form with the characteristic function, $\phi(z)$.

Through the author's experience, it may be said that $\phi(z)$ usually takes a simpler form than $p(R)$. This fact will be of great help when the treatment of $p(R)$ is complicated.

On the other hand, it would be more convenient, in many cases, to use the Hankel type of integral rather than the Laplace transformation. However, as is mentioned above, $\phi(z)$ is usually more suitable to estimate $\overline{P}_e$. Therefore the following well-known transformation from $F(\lambda)$ to $\phi(z)$

$$\phi(z)=\frac{1}{4z}\int_0^\infty e^{-\frac{\lambda^2}{4z}}F(\lambda)d\lambda \tag{32}$$

will be useful.

4.2. Error Probability

For the estimation of average error probability, as given by (31), it is sufficient to find the characteristic function, $\phi(z)$. On this basis, we shall proceed to calculate the error probability of combined signals in diversity reception.

a. Case of Squared Addition

Let us begin with the case of the sum of squared signals:

$$R^2=R_1^2+R_2^2+\ldots+R_n^2. \tag{33}$$

The characteristic function, $\phi(z)$, is expressed by

$$\phi(z)=\int_0^\infty e^{-z(R_1^2+R_2^2+\cdots+R_n^2)}\,p\,(R_1, R_2, \ldots, R_n)\,dR_1dR_2\ldots dR_n, \tag{34}$$

where $p(R_1, R_2, \ldots, R_n)$ is the joint probability density function of n-diversity signals.

For simplicity, we assume that the signals are independent of each other, then (34) takes the form

$$\phi(z)=\prod_{i=1}^{n}\phi_i(z). \tag{35}$$

Accordingly, $\overline{P}_e$ is simply expressed as

$$\overline{P}_e=2^{n-1}\prod_{i=1}^{n}{}_i\overline{P}_e, \tag{36}$$

where ${}_i\overline{P}_e$ shows the error probability of the ith branch of signal.

Further, we shall consider the case of two correlated m-fading signals; then

$$\phi(z)=\iint_0^\infty e^{-z(R_1^2+R_2^2)}\,p_m(R_1, R_2)\,dR_1dR_2, \tag{37}$$

where $p_m(R_1, R_2)$ stands for (22). After some calculations, $\overline{P}_e$ is obtained as

$$\overline{P_e}=\frac{1}{2}\left[\frac{1}{\{1+z(\sigma_1+\sigma_2)+z^2(1-k^2)\sigma_2\sigma_2\}^m}\right]_{z=\frac{T}{4N}}. \tag{38}$$

b. Case of Linear Addition

We shall treat the sum of two independent signals which follow the m-distribution. In this case, $\phi(z)$ takes the following form

$$\phi(z)=\iint_0^\infty e^{-z(R_1+R_2)^2}\,\mathcal{M}\,(R_1, m, \Omega)\,\mathcal{M}\,(R_2, m, \Omega)\,dR_1dR_2. \tag{39}$$

Applying the well-known transformation

$$R_1^2=ue^{-x},\ R_2^2=ue^{x}, \tag{40}$$

(39) is brought to

$$\phi(z)=\frac{4m^{2m}}{\Gamma^2(m)\,\Omega^{2m}}\int_0^\infty u^{2m-1}K_0\left\{2\left(\frac{m}{\Omega}+z\right)u\right\}e^{-2zu}du, \tag{41}$$

where $K_0(x)$ is the modified Bessel function of second kind.

In table 1 are shown some of the calculated results.

TABLE 1

m	$\phi(z)$
$\frac{1}{2}$	$\frac{2}{\pi}\frac{1}{\sqrt{1+4\Omega z}}\cos^{-1}\frac{2\Omega z}{1+2\Omega z}$
1	$\frac{1}{1+2\Omega z}\left[1-\frac{\Omega z}{\sqrt{1+2\Omega z}}\cos^{-1}\frac{\Omega z}{1+\Omega z}\right]$
2	$\frac{1}{16(1+\Omega z)^3}\left[16+16\Omega z+15\Omega^2 z^2 - \frac{3\Omega z(12+12\Omega z+5\Omega^2 z^2)}{2\sqrt{1+\Omega z}}\cos^{-1}\frac{\Omega z}{2+\Omega z}\right]$

c. Case of Vectorial Addition

Next, we shall proceed to a more general case of two vectorial sum

$$R^2=R_1^2+R_2^2+2R_1R_2\cos\theta, \tag{42}$$

where we assume that R_1 and R_2 follow the correlated m-distribution and θ varies at random. Then

$$\phi(z)=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\int e^{-z(R_1^2+R_2^2+2R_1R_2\cos\theta)}p_m(R_1, R_2)dR_1dR_2d\theta \tag{43}$$

which yields, after some calculations.

$$\phi(z)=\frac{4}{\Gamma(m)\sigma^2(1-k^2)(\sigma k)^m}\int_0^{\infty}u^m K_0\{2(a+z)u\}I_{m-1}(2aku)I_0(2uz)du, \tag{44}$$

where

$$\sigma_1=\sigma_2=\sigma,\ a=\frac{1}{(1-k^2)\sigma}.$$

For larger values of m, the calculations become more complex. The results are shown in table 2, where calculations are limited to smaller values of m.

TABLE 2

m	$\phi(z)$
1	$\frac{1}{\sqrt{\{2\sigma z(1+k)+1\}\{2\sigma z(1-k)+1\}}}$
2	$\frac{2\sigma z\{\sigma(1-k^2)z+1\}+1}{[\{2\sigma z(1-k)+1\}\{2\sigma z(1+k)+1\}]^{3/2}}$
3	$\frac{1+4\sigma z\{1+\sigma z(1-k^2)\}+6\sigma^2 z^2\{1+\sigma z(1-k^2)\}^2}{[1+4\sigma z+4\sigma^2 z^2(1-k^2)]^{5/2}}$

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