RADIO SCIENCE Journal of Research NBS/USNC–URSI
Vol. 68D, No. 9, September 1964

Random Volume Scattering

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(Received December 7, 1963; revised March 25, 1964)

The disturbances produced by a slightly inhomogeneous random medium on a passing wave can be classified into contributions depending on an increasing number of successive scatterings. The individual contributions appear in an expansion of the solution of an integral equation. The first term, the Born approximation, only accounts for a single scattering. Convenient expressions for this approximation result from a saddle point treatment for short distances, and from a Fraunhofer approximation for larger distances. The evaluation of the higher-order contributions, describing plural-scattering effects, leads to mathematical difficulties which are evaded by considering the scattering mechanism as a Markovian process. The corresponding theory can be developed with the aid of an integro-differential diffusion equation; the latter refers to the joint probability density of the lateral and angular deviations suffered by the trajectory of the passing wave. The equation in question can be solved with the aid of four-dimensional operational calculus; it reduces to the simple differential equation of Fokker-Planck under special conditions. The application of the general equation to tropospheric point-to-point radio communication is worked out. It is shown that the far-distance field, associated in this case with multiple scattering, does decrease proportionally to the second or third power of the inverse distance.

1. Introduction

By volume scattering we understand the property that each volume-element of a slightly inhomogeneous medium does deviate a fraction of the energy of any incident radiation from its original propagation direction. In its simplest form such scattering is described by a scalar wave equation with a term consisting of the product of the wave function and the fluctuating component of the refractive index. By solving this equation while treating this term as if it were known we obtain an integral equation. The successive terms of the Neumann expansion of its solution represent the primary field (corresponding to a medium without a random component of the refractive index), the contribution due to a single scattering, that produced by two successive scatterings, three successive scatterings, and so on.

Most attention has been paid to the single-scattering contribution which is known as the Born approximation. Convenient expressions can be deduced from its rigorous representation, either by applying a saddlepoint method, or by taking account of the dimensions of the relevant coherently scattering parts of space ("blobs") which are small compared to their distances to the transmitter and the receiver. The saddlepoint method involves geometric-optical approximations for the distortions of the wave fronts which would leave the transmitter undisturbed in a nonfluctuating medium. The other method (Fraunhofer approximation) leads to the distribution of the scattered energy over the various directions around the original propagation direction. In either method the correlation between the field strengths observed at two different (or identical) places at two different (or identical) moments can be derived from a corresponding correlation function for the random fluctuations of the refractive index.

Mathematical difficulties till now prevented a similar treatment of the higher-order scattering contributions. Therefore, the successive scatterings have simply been assumed as completely independent of each other, which is certainly justified if they do not take place too close to each other. This Markovian scattering process can be described with the aid of a joint probability density h_z which combines the chances for special, both lateral and angular, deviations away from the rectilinear propagation path covered in the case of a homogeneous medium. This probability density satisfies a transport or diffusion equation which can be solved rigorously if the equation is first simplified in view of the assumption of small final angular deviations.

The equation in question reduces to a second-order Fokker-Planck equation if terms depending on the average fourth, sixth, etc., power of the angular deviations (connected with a single scattering) may be neglected.

The probability density h_z being known, it is possible to compute the field strength for tropospheric radio propagation if the radiation diagrams of both the transmitter and the receiver are given. The explicit evaluation is difficult but approximations for large distances, which include the effects of scatterings of any order, can be worked out for special models, for instance for Norton's modified Bessel model. Such calculations show the insufficiency of the corresponding expressions derived from the above-mentioned Fokker Planck equation.

The various items indicated here are discussed in more detail in the subsequent sections.

2. Integral Equation of a Slightly Inhomogeneous Medium

The elimination of the magnetic field from Maxwell's equations for a medium with a dielectric constant

$$\epsilon=\epsilon_0+\delta\epsilon(x,y,z)$$

leads for time harmonic solutions proportional to $e^{-i\omega t}$ to the equation:

$$\nabla^2\vec{E}+\nabla(\nabla \ln \epsilon\cdot\vec{E})+\omega^2\mu_0\epsilon\vec{E}=0.$$

The gradient term can be neglected provided that the changes of $\delta\epsilon/\epsilon_0$ are small over one wavelength. The equation can then be put in the following form, for each component of $\vec{E}$:

$$\nabla^2 E+\omega^2\mu_0\epsilon_0 E=-\omega^2\mu_0\delta\epsilon E.$$

By treating this wave equation as if the right-hand side were a known function, we get the following "solution"

$$E(P)=E_{pr}(P)+\frac{\omega^2\mu_0}{4\pi}\int d\tau_Q\delta\epsilon(Q)E(Q)\,\frac{e^{i\omega\sqrt{\mu_0\epsilon_0}QP}}{QP}, \tag{1}$$

which, as a matter of fact, constitutes a scalar integral equation for each component of the unknown wave function $\vec{E}(P)$. The integration extends over all volume-elements $d\tau_Q$ of the space containing $\delta\epsilon$ fluctuations. The term $E_{pr}(P)$ represents the primary field, that is the solution in the absence of the random component $\delta\epsilon$; QP is the distance from the integration point Q to the point of observation P (receiving antenna).

The Neumann Liouville expansion of the solution of (1) starts with

$$\begin{aligned}E(P)=E_{pr}(P)&+\frac{\omega^2\mu_0}{4\pi}\int d\tau_Q\delta\epsilon(Q)E_{pr}(Q)\,\frac{e^{ik_0QP}}{QP}\\&+\left(\frac{\omega^2\mu_0}{4\pi}\right)^2\int d\tau_Q\delta\epsilon(Q)\int d\tau_{Q'}\delta\epsilon(Q')E_{pr}(Q')\,\frac{e^{ik_0(Q'Q+QP)}}{Q'Q\cdot QP}+\cdots\\&=E_{pr}(P)+E^{(1)}(P)+E^{(2)}(P)+\ldots,\text{ say.}\qquad(k_0=\omega\sqrt{\mu_0\epsilon_0}).\end{aligned} \tag{2}$$

The second term represents the contribution due to single scatterings and constitutes the Born approximation for the total scattered field. The third term $E^{(2)}(P)$ is recognized as the effect of two successive scatterings at two volume-elements $d\tau_Q$ and $d\tau_{Q'}$; the fourth term likewise represents the contribution of third-order scatterings, and so on. Most literature only concerns the Born approximation.

3. Statistical Properties of the Born Approximation

The random character of the fluctuations in slightly inhomogeneous media suggests to introduce statistical quantities right at the beginning, thus applying methods labelled as "dishonest" in Keller's paper [Keller, 1962] on the subject. The statistical properties concern, e.g., the average field strength and the correlation between the fields observed at two neighboring points P_1 and P_2, possibly at different moments t_1 and t_2. These properties can be derived from the quantity

$$\langle E^{(1)}(P_1, t_1) E^{(1)*}(P_2, t_2)\rangle$$

for which we obtain, in view of the above expression for $E^{(1)}$,

$$\left(\frac{\omega^2\mu_0}{4\pi}\right)^2 \int d\tau_{Q_1} E_{pr}(Q_1) \frac{e^{ik_0Q_1P_1}}{Q_1P_1} \int d\tau_{Q_2} E^*_{pr}(Q_2) \frac{e^{-ik_0Q_2P_2}}{Q_2P_2} \langle \delta\epsilon(Q_1, t_1)\delta\epsilon(Q_2, t_2)\rangle. \tag{3}$$

A further evaluation needs an explicit statistical assumption concerning the average of the product of the dielectric-constant fluctuations at two different places. It is customary to introduce here the hypothesis of "homogeneous turbulence" according to which this average only depends on the relative positions of Q_1 and Q_2, and the time difference t_1-t_2. Neglecting further the time variations this amounts analytically to a dependence on the differences of the coordinates of Q_1 and Q_2, or on the vector $\overrightarrow{Q_1Q_2}$. The homogeneous turbulence can then be fixed completely with the aid of the following normalized autocorrelation function:

$$C_{\delta\epsilon}(\overrightarrow{Q_1Q_2}) = \frac{\langle\delta\epsilon(Q_1)\delta\epsilon(Q_2)\rangle}{\langle\delta\epsilon^2(Q)\rangle}. \tag{4}$$

We next assume a primary field due to a point source at T, and normalized according to the formula

$$E_{pr}(Q) = \frac{e^{ik_0TQ}}{TQ}. \tag{5}$$

The substitution of (4) and (5) into (3) yields:

$$\langle E^{(1)}(P_1) E^{(1)*}(P_2)\rangle = \left(\frac{\omega^2\mu_0}{4\pi}\right)^2 \langle\delta\epsilon^2\rangle \int d\tau_{Q_1} \int d\tau_{Q_2} \frac{e^{ik_0(TQ_1+Q_1P_1-TQ_2-Q_2P_2)}}{TQ_1 \cdot Q_1P_1 \cdot TQ_2 \cdot Q_2P_2} C_{\delta\epsilon}(\overrightarrow{Q_1Q_2}). \tag{6}$$

This expression with a double integration over the entire randomly fluctuating medium is basic for all investigations starting from the Born approximation.

4. A Saddlepoint Method Applied to the Born Approximation

Saddlepoint approximations are always applicable for frequencies which are sufficiently high. Its application to the Born approximation $E^{(1)}$ (P) as defined by (2) amounts to replacing the integration point Q in the denominator by its projection Q' on the line TP connecting the point source with the point of observation, and to expanding the exponential up to second-order terms with respect to the coordinates y_q and z_q; the y and z-axes are here assumed as perpendicular to the line TP constituting the x-axis. We then obtain, also using (5),

$$E^{(1)}(P) = \frac{\omega^2\mu_0}{4\pi} e^{ik_0TP} \int d\tau_Q \delta\epsilon(Q) \frac{e^{\frac{ik_0}{2}\left(\frac{1}{Q'T}+\frac{1}{Q'P}\right)\left(y_Q^2+z_Q^2\right)}}{Q'T \cdot Q'P}.$$

We pass from the dielectric constant $\epsilon_0+\delta\epsilon$ to the corresponding expression

$$n^2 = \epsilon/\epsilon_0 = 1+2\delta n$$

for the refractive index n, so as to have $\delta\epsilon = 2\epsilon_0\delta n$. A substitution of the Taylor expansion (up to second-order terms) for δn_Q as a function of Y_Q and Z_Q then leads to a result derived [Bremmer, 1958], which can be interpreted by a phase shift

$$\delta\phi = k_0 \int_{x_T}^{x_P} dx_{Q'} \delta n(Q'), \tag{7}$$

and a relative change of the amplitude given by:

$$\frac{\delta A}{A} = -\frac{1}{2}\int_{x_T}^{x_P} dx_{Q'}\,\frac{Q'T\cdot Q'P}{TP}\left(\frac{\partial^2\delta n}{\partial y^2}+\frac{\partial^2\delta n}{\partial z^2}\right)_{Q'}. \tag{8}$$

The phase correction is quite obvious in view of the fluctuations of the refractive index n. The amplitude change points to a lens effect. In fact, the regions with a negative or positive value of $(\partial^2/\partial y^2+\partial^2/\partial z^2)\delta n$ cause a small increase or decrease of the field which is equivalent to a focusing or a defocusing.

It is also possible to derive statistical averages for these geometric-optical quantities when assuming homogeneous turbulence. For instance, the variance $\langle\delta\phi^2\rangle$ of the phase fluctuations at a special point, or the correlation between these fluctuations at two different points are obtained from a double integral consisting of the product of two expressions (7), substituting the autocorrelation function corresponding to (4) for the refractive-index fluctuations. This autocorrelation function may be represented by

$$C_{\delta n}\,(\overrightarrow{Q_1Q_2}) = C_{\delta n}\left(\frac{x_{Q_1}-x_{Q_2}}{l}\right)$$

in which we introduce the scale of turbulence l in order to get a function of a dimensionless parameter. The variance in question then becomes as follows at a point at a distance $TP=d$ from the point source, provided that $d \rangle\rangle\, l$:

$$\langle\delta\phi^2\rangle = 2k_0^2\langle\delta n^2\rangle l d\int_0^\infty dy C_{\delta n}(y) = 8d/d_0, \qquad \text{say.} \tag{9}$$

A similar computation, starting from (8) and to published elsewhere, leads to the following corresponding expression for the variance of the amplitude fluctuations:

$$\left\langle\left(\frac{\delta A}{A}\right)^2\right\rangle = \frac{2}{15}\int_0^\infty\left\{\frac{C''_{\delta n}(s)}{s^2}-\frac{C'_{\delta n}(s)}{s^3}\right\} ds\cdot\langle\delta n^2\rangle\,\frac{d^3}{l^3}. \tag{10}$$

Variances of other geometric-optical quantities, such as angular deviations from ray trajectories which are rectilinear in the absence of the random fluctuations, can be computed in a similar way. We refer in particular to the investigation [Muchmore and Wheelon, 1955] and to a discussion of the wave-front disturbances caused by turbulent random fluctuations [Bremmer, 1963].

5. Fraunhofer Approximations of the Born Approximation

The limited range of noticeable coherence of the $\delta\epsilon$-fluctuations involves a splitting up of the integral of (6) into terms comprising a double integration over individual incoherently scattering volume elements. These elements or "blobs" may have dimensions of the order of the parameter l introduced above in the autocorrelation function $C_{\delta n}$.

In the application to far-distance tropospheric propagation the effective blobs are situated in a rather small part of space situated simultaneously above the tangenting plane to the earth (horizon plane) through the transmitter, and above that through the receiver. The resulting conventional theory is summarized below, emphasizing the influence of the finiteness of the blob dimensions.

The size of the blobs being small compared to the distances Q_1T, Q_2T, Q_1P_1, Q_2P_2 from their inner points to the transmitter and the points of observation, it is justified to replace these inner points by a fixed central point Q_0 of the blob in question when considering the denominator in (6). Moreover, these distances may be approximated in the exponential by expressions which are linear with respect to the coordinates of Q_1 and Q_2 (the origin being at Q_0), that is the components of two vectors $\vec{x}_1$ and $\vec{x}_2$. We further introduce new mixed coordinates (components of two vectors $\vec{\xi}$ and $\vec{\eta}$) according to:

$$\vec{\xi}=\vec{x}_2+\vec{x}_1,\ \vec{\eta}=\vec{x}_2-\vec{x}_1.$$

In the case of spherical blobs of radius $L/2$ around their central point Q_0 the splitting of (6) into contributions over the various blobs then can finally be represented by:

$$\langle E^{(1)}(P_1)E^{(1)*}(P_2)\rangle=\frac{\langle(\omega^2\mu_0)^2\delta\epsilon^2\rangle}{128\pi^2}\sum_{Q_0}\frac{e^{ik_0(P_1Q_0-P_2Q_0)}}{TQ_0^2\cdot Q_0P_1\cdot Q_0P_2}\int\limits_{|\vec{\eta}\mp\vec{\xi}|<L}d\vec{\xi}\int d\vec{\eta}e^{ik_0(\vec{V}\vec{\xi}+\vec{W}\vec{\eta})}C_{\delta\epsilon}\left(\frac{\vec{\eta}}{l}\right). \tag{11}$$

Here we have introduced two other new vectors $\vec{V}$ and $\vec{W}$ defined by:

$$\vec{V}=\frac{1}{2}(\vec{u}_{Q_0P_2}-\vec{u}_{Q_0P_1}),$$

$$\vec{W}=\cos\frac{\psi}{2}\vec{u}_b-\vec{u}_{TQ_0},$$

where $\vec{U}_{Q_0P_2}$ marks a unit vector in the direction of Q_0P_2, the other unit vectors being defined likewise; b represents the bissectrix of the angle $\psi=P_1Q_0P_2$.

The special case $P_1\equiv P_2$ leads at once to the differential scattering cross section σ which represents the energy scattered by a unit scattering volume into a unit infinitesimal solid angle along the direction Q_0P, for a unit energy-current density of the incident radiation. This definition proves to be equivalent with that used later on (see sec. 7). The former definition implies the following value for the total scattering cross section connected with the N incoherently scattering blobs contained in a unit volume-element:

$$\sigma=N\ TQ_0^2\ Q_0P^2\langle E^{(1)}(P)\ E^{(1)*}(P)\rangle_1;$$

the index (1) here refers to the contribution connected with a single blob. Hence, in view of (11),

$$\sigma=\frac{N(\omega^2\mu_0)^2\langle\delta\epsilon^2\rangle}{128\pi^2}\int\limits_{|\vec{\eta}\mp\vec{\xi}|<L}d\vec{\xi}\int d\vec{\eta}e^{ik_0(\vec{u}_{sc}-\vec{u}_i)\vec{\eta}}C_{\delta\epsilon}\left(\frac{\vec{\eta}}{l}\right),$$

in which the new unit vectors are directed along the scattered and the incident radiation (along Q_0P and TQ_0), respectively.

The integral over $\vec{\xi}$ amounts to the volume of the common space of two spheres with radius L and centers separated by a distance $2|\vec{\eta}|$. This common space only exists if $|\vec{\eta}|<L$. The blobs (the spherical form of which has only been assumed in order to show the influence of the blob sizes in general) may be considered as adjacent so as to have $N=\{(4/3)\pi L^3\}^{-1}$. This factor disappears when substituting the mentioned volume for the $\vec{\xi}$ integral. We thus arrive at:

$$\sigma=\frac{(\omega^2\mu_0)^2\langle\delta\epsilon^2\rangle}{128\pi^2}\int\limits_{|\vec{\eta}|<L}d\vec{\eta}e^{ik_0(\vec{u}_{sc}-\vec{u}_i)\vec{\eta}}C_{\delta\epsilon}\left(\frac{\vec{\eta}}{l}\right)\left(1-\frac{3}{2}\frac{|\vec{\eta}|}{L}+\frac{1}{2}\frac{|\vec{\eta}|^3}{L^3}\right).$$

The last factor of the integrand points to an influence of the blob dimensions which is not accounted for by the autocorrelation function. It turns out that the effect of these dimensions L can only be neglected if L (being of the order of the scale of turbulence l) exceeds the maximal relevant value of $|\vec{\eta}|$. In view of the oscillating character of the other exponential factor this maximum is of the order of

$$2\pi\left\{k_0|\vec{u}_{sc}-\vec{u}_i|\right\}^{-1}=2\pi\left(2k_0\sin\frac{\vartheta}{2}\right)^{-1},$$

ϑ being the "scattering angle" between the propagation directions TQ_0 of the incident radiation and Q_0P of the scattered energy. This angle being small in practice, we arrive at the condition $l>2\pi(k_0\vartheta)^{-1}$, or $\vartheta>\lambda/l$, in order that the blob-size effect may be neglected. This inequality characterizes the region beyond that of the predominant forward scattering, that is the region of most interest for propagation up to distances well beyond the horizon of the transmitter. In this latter region it is also justified to extend the $\vec{\eta}$ integration up to infinity. The conventional formula for the scattering coefficient, viz,

$$\sigma=\frac{(\omega^2\mu_0)^2\langle\delta\epsilon\rangle^2}{128\pi^2}\int\limits_{0<|\vec{\eta}|<\infty}d\vec{\eta}e^{ik_0(\vec{u}_{sc}-\vec{u}_i)\vec{\eta}}C_{\delta\epsilon}\left(\frac{\vec{\eta}}{l}\right), \tag{12}$$

then results; it refers to scattering angles included by the unit vectors $\vec{u}_{sc}$ and $\vec{u}_i$.

The relation between this expression and the power spectrum $P(\vec{\omega})$ of the random fluctuations, to be defined with the aid of the three-dimensional Fourier transform of the spatial distribution of these fluctuations is well known. If $P(\vec{\omega})$ is normalized such as to obtain the unit constant when integrated (over all real values of ω_1, ω_2, and ω_3), we find

$$\sigma=\frac{\pi}{16}(\omega^2\mu_0)^2\langle\delta\epsilon^2\rangle P\{k_0(\vec{u}_{sc}-\vec{u}_i)\}.$$

We next consider the correlation, according to the Fraunhofer approximation, between the fields at two neighboring points P_1 and P_2. In view of the short distance of these points in practice, the $\vec{\eta}$ integral in (11) is hardly changed from its value for the previous case in which the angle ψ vanished. However, the $\vec{\xi}$ integration becomes more complicated, the vector $\vec{V}$ now being different from zero. Its evaluation indicates noticeable correlation between the fields at P_1 and P_2 up to distances for which the angle $\psi=P_1Q_0P_2$ becomes of the order λ/l. In particular, the correlation is well established between points situated on one and the same line through the scattering source Q_0, and least in directions perpendicular to such a line.

6. Scattering Considered as a Markovian Process

Statistical properties could in principle be derived for the higher-order scattering contributions in a similar way as indicated for the Born approximation referring to first-order scatterings. The results would depend on the averages $<\delta\epsilon(Q_1)\ \delta\epsilon(Q_2)\ \ldots\ \delta\epsilon(Q_n)>$ of multifold products. The corresponding final expressions holding for the combination of all terms of the Neumann expansion (2) would become extremely complicated though progress in this direction has been obtained [Hoffman, 1959, and Furutsu, 1963]. The difficulties here arising can be avoided when the successive scattering contributions, represented by the various terms of (2), may be considered as independent of each other. The Markovian character of the scattering mechanism, then to be assumed, may be made plausible as follows.

A narrow pencil of rays leaving some point source only produces a noticeable scattered field after it has covered some minimal distance. The order of magnitude of the latter can be estimated by investigating the scattered field produced by an antenna which radiates isotropically, as given by (5), into an infinite random medium without boundaries. In the case of homogeneous turbulence the variance of this field follows rigorously from (6) by taking $P_1=P_2$, the integrations extending over the entire space. Its computation has been worked out in [Fannin, 1956] assuming an isotropic auto-correlation function $<\delta\epsilon(Q_1)\ \delta\epsilon(Q_2)>$ of gaussian form with respect to the distance Q_1Q_2; this function could also account for an anisotropy caused by wind effects. Fannin's main result (for zero wind velocity) can be interpreted thus that the variance of $E^{(1)}$, divided by the squared primary field, does increase in proportion to $d^3/(d_0d_{cr}^2)$ up to distances of the order of d_{cr}, while tending to a limiting value of the order of d/d_0 [see (8)] for $d\gg d_{cr}$; the critical distance $d_{cr}=l^2/\lambda$ is introduced here as a function of the scale of turbulence l and the wavelength λ. The increase of the variance for small distances shows that the scattered field only becomes appreciable when the radiation has traveled over a distance of the order of $d_0^{1/3}\ d_{cr}^{2/3}$ if this quantity proves to be smaller than d_{cr}, that is if $d_0<d_{cr}$. This latter very general result proves to be independent of the special correlation-function model chosen by Fannin, as can be shown with the aid of (10).

The importance of the dimensionless parameter $d/d_{cr}=d\lambda/l^2$, here becoming obvious, can be understood as follows. Forward scattering is mainly restricted to a cone of angular deviations (around the undisturbed propagation direction) which are not larger than λ/l. An originally sharp ray having traveled over a distance d in a random medium is therefore broadened to a beam with a cross section having a dimension of the order of $d\lambda/l$. This cross dimension just equals the size of a turbulent blob if $d\lambda/l=l$, that is if $d=d_{cr}$. Therefore, if $d>d_{cr}$, the radiation due to the scattering will have traversed, on the average, at least one blob. Also, d/d_{cr} constitutes a measure of the number of blobs passed after traveling over a distance d in the forward direction. The geometric optical approximations of section 4 refer to small values of d/d_{cr}.

In view of the above remarks individual scatterings take place, on the average, once along each section of a length of the order of $d_0^{1/3}\, d_{cr}^{2/3}$ if $d_0 < d_{cr}$, but of d_0 if $d_0 > d_{cr}$. Hence the scattering angles associated with the successive individual scatterings may be considered as independent of each other if the correlation between the refractive-index fluctuations at the end of such a section proves to be negligible. This leads to the condition $d_0^{1/3}\, d_{cr}^{2/3} > l$ if $d < d_{cr}$, and $d_0 > l$ if $d > d_{cr}$. It is therefore always sufficient to have $d_0 > l$ if $\lambda < l$ (a condition satisfied for forward scattering in order that the scattering angles connected with the nth and $(n-1)$st scattering should be statistically independent. The angular spread caused by the nth scattering then only depends on the propagation direction obtained after the $(n-1)$st scattering, and not on those obtained after the preceding scatterings. This makes plausible the Markovian character of tropospheric multiple scattering (when occurring at all).

7. Equations Characterizing Markovian Small-Angle Scattering

Physical phenomena associated with scattering generally depend on both angular and lateral deviations suffered by the energy bent aside by the irregularities of the medium. This suggests to introduce a joint probability density h_Z fixing the distributions of both types of deviations. Let us consider an energy unit (or a particle in the case of scattering of a stream of incident electrons or atomic particles by other particles having random positions in the medium traversed) which leaves an origin 0 in the Z-direction along a path which would be rectilinear in the absence of scattering irregularities. We then define the function h_Z such that $h_Z\,(X, Y, x, y)\, dXdYdxdy$ constitutes the probability for the unit in question to pass the special plane $Z=$constant through a prescribed surface element $dXdY$ while traveling there in a direction comprised in an infinitestimal cone, likewise prescribed. This cone subtends a solid-angle element $dx\, dy$ if x and y are direction cosines with respect to the X and Y axes (both perpendicular to the Z-axis). The transport equation concerns the change of the function h_Z along a line element $ds=(dX^2+dY^2+dZ^2)^{1/2}$.

Our analysis will be confined to *forward* scattering which implies that all relevant angular deviations (with respect to the Z-axis) may be considered as small. The limiting case of isotropic Rayleigh scattering is then excluded. The theory dealing with scattering angles of any magnitude usually resort at once to expansions in terms of Legendre functions, as discussed, e.g., [Lewis, 1950].

In the small-angle case the angle between two directions fixed by the cosines (x_1, y_1) and (x_2, y_2) can be approximated by

$$\vartheta \sim \sin\vartheta \sim \{(x_1-x_2)^2+(y_1-y_2)^2\}^{1/2}.$$

Moreover, all direction cosines with respect to the Z-axis may be replaced by unity.

The small-angle approximations involve a considerable simplification. As an example of its significance we first consider the equation which expresses the property that scattering over a distance Z can be split into a pair of completely independent scatterings along any two subsections $Z-Z_1$ and Z_1, such in view of the Markovian hypothesis. The number of units passing the plane at a distance $Z-Z_1$ from 0, with lateral deviations ξ, η, and angular deviations u, v, is determined by the quantity $h_{Z-Z_1}(\xi, \eta, u, v)$. A fraction of these units will arrive, after traveling over a further forward distance Z_1, within a prescribed range $dXdYdxdy$ of lateral and angular deviations at the Z plane. During its propagation over Z_1 this fraction suffers a total lateral deviation

$$\{(X-Z_1\, u-\xi)^2+(Y-Z_1\, v-\eta)^2\}^{1/2}$$

relative to the rectilinear path covered if it would have continued in the direction (u, v) up to the plane at the distance Z; in fact, along this latter path it would have arrived there at a point with coordinates $Z_1u+\xi$, $Z_1v+\eta$, Z. The corresponding angular deviation amounts to $\{(x-u)^2+(y-v)^2\}^{1/2}$. In view of the axially symmetric character of the scattering, assumed throughout, the probability for these latter deviations would be the same as that in the case of lateral displacements $X-Z_1u-\xi$ and $Y-Z_1v-\eta$ in the X and Y directions respectively, combined with angular displacements corresponding to a diviation away from the Z-axis into a direction with cosines $x-u$ and $y-v$. Hence the probability for arriving within the range $dXdYdxdy$ at the Z plane, after having passed the $Z-Z_1$ plane in the range $d\xi\, d\eta\, dudv$, can be represented by

$$h_{Z_1}(X-Z_1u-\xi,\ Y-Z_1v-\eta,\ x-u,\ y-v)\ dX\, dY\, dx\, dy.$$

An integration over all possible situations in the $(Z-Z_1)$ plane, while taking account of the probability density $h_{Z-Z_1}(\xi, \eta, u, v)$ for each of these situations, leads to the desired equation, viz,

$$h_Z(X, Y, x, y) = \iiiint_{-\infty}^{\infty} d\xi d\eta du dv\, h_{Z-Z_1}(\xi, \eta, u, v) h_{Z_1}(X-\xi-Z_1u, Y-\eta-Z_1v, x-u, y-v). \tag{13}$$

The integration over the direction cosines has been extended here up to infinity. This is allowed since the integrand will already be negligibly small near the limiting values ± 1 of the direction cosines with respect to the X and Y axis, which would correspond to very large angular deviations. Therefore, the integrations can just as well be continued over the complex directions fixed by cosines beyond these limiting values. Such infinite integration limits will be used throughout in our further analysis when the integration should comprise all real propagation directions.

Any equation characterizing a special scattering medium should depend on its differential scattering cross section $\sigma(\sqrt{x^2+y^2})$, $\sigma\, dZ\, dx\, dy$ constituting the probability for first-order scattering, during propagation over a forward distance dZ, into an infinitesimal range of directions (fixed by the cosines x and y) around a central axis in the Z direction. In view of the axial symmetry σ only depends on the small angle $(x^2+y^2)^{1/2}$ of this direction with the Z axis, but it proves to be convenient to define also a function $\sigma(x, y)$ of two variables connected with that of one variable according to

$$\sigma(x, y) \equiv \sigma(\sqrt{x^2+y^2}).$$

The final equation completely fixing all scattering phenomena can then be represented by

$$\frac{dh_Z}{ds} = \left(x\frac{\partial}{\partial X} + y\frac{\partial}{\partial Y} + \frac{\partial}{\partial Z}\right) h_Z(X, Y, x, y) = \iint_{-\infty}^{\infty} d\xi d\eta\, \sigma(\xi, \eta) h_Z(X, Y, x-\xi, y-\eta) - \frac{h_Z(X, Y, x, y)}{d_0}. \tag{14}$$

The first two members of this special case of Boltzmann's transport equation determine the change of h_Z along the line element ds, the coefficient in front of ∂/∂_Z being unity in view of the small-angle assumption. The integral term represents the energy units that arrive at the plane $Z=$constant at prescribed points (X, Y), traveling in directions prescribed by cosines $x-\xi$ and $y-\eta$, and which are scattered near this plane into the direction fixed by the cosines x and y under consideration. The last term of (14) represents the loss due to the chance $\sigma(x, y)dxdyds$ for the units traveling along ds to be scattered aside into a new direction making an angle $(x^2+y^2)^{\frac{1}{2}}$ with the original direction. Extending the integrations up to infinity, the total amount of these losses becomes

$$ds \iint_{-\infty}^{\infty} dxdy\, \sigma(x, y) = \frac{d_s}{d_0}, \text{ say.} \tag{15}$$

Working out the integrations with the aid of (12), the quantity d_0 defined here proves to be the same as that entering in the relation (8) for the variance of the phase fluctuations. According to (15) d_0 represents the distance along which any energy unit becomes a chance 1:1 for being scattered aside completely. Therefore, this distance has been termed "mean free path for scattering." All terms of the transport equation (14) have thus been made clear.

8. Solution of the Transport Equation in Terms of Laplace Transforms

The integral term in the transport equation (14) constitutes a convolution product. This special form, which can only be obtained in the small-angle approximation, enables the solving of the equation with the aid of Laplace transforms; this has been worked out [Snyder and Scott, 1949] for the projection of the scattering paths on a plane through the Z-axis. In a completely four-dimensional treatment the transform in question may be defined by

$$\text{Lapl}\, h_Z(X,Y,x,y) = \int\int_{-\infty}^{\infty}\int\int dXdYdxdy\, h_Z(X,Y,x,y)\, e^{-p_1X-q_1Y-p_2x-q_2y}.$$

Above we introduced the function h_z by considering a unit element leaving the origin along the Z-axis. This implies the boundary condition $h_z=\delta(X)\delta(Y)\delta(x)\delta(y)$ at $Z=0$. The Laplace transform of the solution satisfying this condition proves to be given (in so far as existing for special values of p_1, q_1, p_2, q_2) by:

$$\text{Lapl}\, h_Z(X,Y,x,y) = e^{-Z/d_0} e^{d_0 \int_0^{Z/d_0} djH(d_0p_1j+p_2,\, d_0q_1j+q_2)}, \tag{16}$$

as shown [Bremmer, 1963]. The function $H(p,q)$ is defined here as the two-dimensional Laplace transform of $\sigma(x,y)$. The function (16) also satisfies the Laplace transform of (13), as it should.

The general solution obtained here contains as special cases the distribution functions connected with the lateral deviations only, or with the angular deviations only. The former, $h_Z(X,Y)$ say, results after integrating $h_Z(X,Y,x,y)$ over all possible values $-\infty<x,y<\infty$ of the direction cosines x and y. This integration corresponds to taking $p_2=q_2=0$ in the four-dimensional general Laplace transform. We thus arrive at the following Laplace transform for this function of two variables only:

$$\text{Lapl}\, h_Z(X,Y) = e^{-Z/d_0} e^{d_0 \int_0^{Z/d} d_0 jH(d_0pj,\, d_0qj)}. \tag{17}$$

The corresponding two-dimensional Laplace transform for the distribution function $h_Z(x,y)$ of the angular deviations is obtained by substituting $p_1=q_1=0$ in (16). The final result reads:

$$\text{Lapl}\, h_Z(x,y) = e^{Z\{H(p,q)-1/d_0\}}. \tag{18}$$

An integral expression equivalent to this relation has been derived in [Molière, 1948].

9. Poisson Distribution Connected With Multiple Scattering

The function h_Z includes the scattering contributions of any order. Each individual scattering being associated with the scattering coefficient σ, the contribution of all scatterings of nth order can be recognized by the occurrence of n factors σ. The role of this function is taken over by that of H in the Laplace transforms. Hence the expansion with respect to H of any result expressed in these transforms will show the distribution over the contributions connected with specific numbers of scatterings.

As an example we consider all energy units that have passed through a plane Z=constant. The units applied in the definition of h_Z represent probabilities, hence their total number should equal unity. This total number is obtained, e.g., by integrating $h_Z(x,y)$ over all values $-\infty<x,\, y<\infty$. The result equals the two-dimensional Laplace transform of $\sigma(x,y)$ at $p=q=0$. Hence we should have

$$1=\int_{-\infty}^{\infty}\int dxdy h_Z(x,y) = \{\text{Lapl}\, h_Z(x,y)\}_{p=q=0},$$

or, in view of (18),

$$1=e^{Z\{H(0,0)-1/d_0\}} = e^{-Z/d_0}\sum_{n=0}^{\infty}\frac{Z^n\{H(0,0)\}^n}{n!}.$$

The nth term then constitutes the fraction of units that has suffered n successive scatterings. Moreover,

$$H(0,0)=1/d_0 \tag{19}$$

holds in view of the definition of d_0. The probability $P(n)$ for n scatterings during forward propagation over a distance Z can therefore be represented by

$$P(n)=e^{-Z/d_0}\frac{(Z/d_0)^n}{n!} \tag{20}$$

This Poisson distribution here directly results as a consequence of the assumption of a Markovian scattering mechanism. Equivalent derivations have been given by Dexter and Beeman [1949] and Fejer [1953].

10. Fokker-Planck Equation as an Approximation of the General Transport Equation

The influence of the medium under consideration on the transport equation (14) results from the occurrence of the function $\sigma(\xi, \eta)$. However, the equation only proves to depend on the discrete set of parameters:

$$2\pi d_0\int_{-\infty}^{\infty} d\rho\, \sigma(\rho)\,\rho^{2n+1}=\frac{\int_{-\infty}^{\infty}\int dxdy\; \sigma(x,y)\,(x^2+y^2)^n}{\int_{-\infty}^{\infty}\int dxdy\; \sigma(x,y)}=\langle\vartheta_1^{2n}\rangle,\text{ say,} \tag{21}$$

which constitute the averages of the even powers of the scattering angles $\vartheta_1=(x^2+y^2)^{1/2}$ connected with a single scattering. In fact, a Taylor expansion of the integral term in (14), as worked out by Bremmer [1964], leads to the following alternative representation of the transport equation:

$$\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+\frac{\partial}{\partial Z}\right)h_Z(X,Y,x,y)=\frac{1}{d_0}\sum_{n=1}^{\infty}\frac{\langle\vartheta_1^{2n}\rangle}{4^n(n!)^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^n h_Z(X,Y,x,y). \tag{22}$$

This equation suggests to consider the approximation obtained by restricting the right-hand side to its first term the significance of which has been discussed by Middleton [1960]. The approximation in question leads to a diffusion equation of the Fokker-Planck type, viz,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h_Z-\frac{4d_0}{\langle\vartheta_1^2\rangle}\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+\frac{\partial}{\partial Z}\right)h_Z=0.$$

Such an equation has been applied to particle scattering as early as 1929 [Bothe, 1929], at least insofar as the angular deviations are concerned.

The exact solution satisfying this F. P. equation and also the previous boundary condition, that is $h_z=\delta(X)\delta(Y)\delta(x)\delta(y)$ for $Z=0$, reads as follows:

$$\{h_Z(X,Y,x,y)\}_{\text{F.P.}}=\frac{12d_0^2}{\pi^2\langle\vartheta_1^2\rangle^2Z^4}\,e^{-\frac{12d_0}{\langle\vartheta_1^2\rangle^2Z^3}\left\{X^2+Y^2-Z(Xx+Yy)+\frac{Z^2}{3}(x^2+y^2)\right\}}. \tag{23}$$

This expression is equivalent to a similar one derived [Rossi and Greisen, 1941] for scattering by particles. Obviously, its applicability requires sufficiently small values of the higher-order moments $\langle\vartheta_1^4\rangle$, $\langle\vartheta_1^6\rangle$, and so on. For the sake of completeness we also mention the corresponding probability densities for the lateral and angular deviations only:

$$h_Z(X,Y)=\frac{3d_0}{\pi\langle\vartheta_1^2\rangle Z^3}\,e^{-\frac{3d_0}{\langle\vartheta_1^2\rangle Z^3}(X^2+Y^2)},$$

$$h_Z(x,y)=\frac{d_0}{\pi\langle\vartheta_1^2\rangle Z}\,e^{-\frac{d_0}{\langle\vartheta_1^2\rangle Z}(x^2+y^2)}. \tag{24}$$

We infer an increase of the average lateral deviations which is roughly in proportion with $Z^{3/2}$, and of the angular deviations with $Z^{1/2}$.

11. Evaluation of Scattered Fields for Special Antenna Systems

Tropospheric radio propagation depends, among other things, on the radiation diagrams of the transmitter and the receiver. That of the former may be given by the gain $g_{tr}(\zeta, \psi)$ as a function of two independent angles fixing the directions of the rays leaving the transmitter. We shall define ζ as the angle between such a ray and the local horizontal plane through the transmitter, and ψ as the angle between (a) the vertical plane through the ray in question, and (b) the other plane through the transmitter, the receiver and the center of the earth. The radiation diagram of the receiver can be fixed by the gain $g_r(\zeta', \psi')$ of the latter for radiations reaching it along directions fixed by two other angles ζ' and ψ'; these angles may conveniently be defined in an identical way with respect to the receiver. The analysis sketched below leads to the following ratio of the received and transmitted energies:

$$\frac{P_{rec}}{P_{tr}}=\frac{\lambda^2}{16\pi^2}\int d\zeta d\psi g_{tr}(\zeta,\psi)\int d\zeta' d\psi' g_{rec}(\zeta',\psi')\cdot h_{a\theta}\left\{a\vartheta\left(\frac{\vartheta}{2}+\gamma\right), a\vartheta_{\psi}, \vartheta+\zeta+\zeta', \psi+\psi'\right\}. \tag{25}$$

The lateral and angular deviations, away from the original direction fixed by ζ and ψ, are completely fixed by the four angles ζ, ψ, ζ', and ψ' when observing in a plane through the receiver that is perpendicular to the original direction. We can determine the components X, Y, x, y of these deviations with respect to a coordinate system the X and Y axes of which are perpendicular to the (ζ, ψ) ray leaving the transmitter, the X axis being situated in the plane mentioned under (b). An elementary, though tedious geometrical analysis, shows that these components are given by

$$a\vartheta\left(\frac{\vartheta}{2}+\zeta\right), a\vartheta\psi, \vartheta+\zeta+\zeta', \psi+\psi', \tag{26}$$

respectively, a being the earth's radius and ϑ the angular distance from the transmitter to the receiver. These latter expressions only result as approximations if ϑ, ζ, ψ, ζ' and ψ' are assumed as small quantities of one and the same order of magnitude. The propagation distance covered in forward direction may then be taken equal to $a\,\vartheta$.

The products $d\zeta d\psi$ and $d\zeta' d\psi'$ constitute solid-angle elements of infinitesimal pencils of rays leaving the transmitter or reaching the receiver respectively. Therefore, in view of (26), the probability for an energy unit to leave the transmitter along a given (ζ, ψ) direction and to reach the receiver within a prescribed $d\zeta' d\psi'$ range is given by

$$h_{a\vartheta}\left\{a\vartheta\left(\frac{\vartheta}{2}+\zeta\right), a\vartheta\psi, \vartheta+\zeta+\zeta', \psi+\psi'\right\}A_{rec}(\zeta', \psi')d\zeta' d\psi', \tag{27}$$

in which

$$A_{rec}(\zeta', \psi')=\frac{\lambda^2}{4\pi}g_{rec}(\zeta', \psi')$$

constitutes the effective area of the receiver for the radiation arriving from the direction (ζ', ψ').

The energy leaving the transmitter in a prescribed range $d\zeta d\psi$ further amounts to

$$P_{tr}\frac{d\zeta d\psi}{4\pi}g_{tr}(\zeta, \psi).$$

The fraction of this energy that reaches the receiver within the other range $d\zeta' d\psi'$ is obtained by multiplying the latter quantity by (27). The total received energy then results from an integration of this product over all relevant values of ζ, ψ, ζ' and ψ', which leads to the final expression (25).

The ζ, ζ' integrations may be restricted to positive values of ζ and ζ'; in fact, negative values correspond to rays suffering a reflection against the earth's surface while the effect of such rays could be included (at least for far-distance propagation) in the relevant radiation diagram. The conventional computations referring to the Born approximation would correspond to a substitution of the expression

$$h_Z^{(1)}=e^{-Z/d_0}\sigma(x, y)\delta(xY-yX)U\left(\frac{X}{x}\right)$$

for h_Z, $U(\alpha)$ being Heaviside's unit function (unity or zero according as $\alpha>0$ or $\alpha<0$).

Apart from the difficulties connected with the exact determination of h_Z, the influence of the radiation diagrams will involve new mathematical complications. It is therefore recommendable to resort to simple representative situations. For instance we may consider the combination of a very narrow transmitting beam with an isotropic receiver. Under idealized conditions this corresponds to

$$g_{\mathrm{tr}}=4\pi\delta(\zeta)\delta(\psi);\; g_{\mathrm{rec}}=\overline{g_r},$$

$\overline{g}_r$ being independent of ζ' and ψ'. In this special situation the received field only depends on the lateral deviation $a\,\theta^2/2$ away from the single original propagation direction $\zeta=\psi=0$. The resulting field is thus completely conditioned by the function $h_Z(X,Y)$ of (17), and the general formula (25) here reduces to

$$\frac{P_{\mathrm{rec}}}{P_{\mathrm{tr}}}=\frac{\lambda^2}{4\pi}\overline{g_r}h_{a\vartheta}\left(\frac{a\vartheta^2}{2},\,0\right)\cdot \tag{28}$$

An application of this relation is discussed in the next sections.

12. Expansion for the Distribution of the Lateral Deviations Only and That of the Angular Deviations

The last example shows the importance of the function $h_Z(X,Y)$ in tropospheric far-distance propagation. According to (17) this function depends on the Laplace transform $H(p,q)$ of the scattering coefficient $\sigma(x,y)$. Assuming, once again, axially symmetric scattering, σ merely depends on $(x^2+y^2)^{1/2}$ and, consequently, H on $(p^2+q^2)^{1/2}$. For a very general class of scattering models the latter dependence is given by an entire function. The corresponding Maclaurin expansion can be represented by

$$H(p,q)=\frac{f(\sqrt{p^2+q^2})}{d_0}=\frac{1}{d_0}\sum_{n=0}^{\infty}c_n(\sqrt{p^2+q^2})^n, \tag{29}$$

in which $c_0=1$ in view of (19). The introduction of the function $f(s)$ of one variable enables a reduction of (17) to:

$$\mathrm{Lapl}\,h_Z(X,Y)=e^{-Z/d_0}\,e^{\frac{\int_0^{Z\sqrt{p^2+q^2}}ds\,f(s)}{d_0\sqrt{p^2+q^2}}}.$$

In view of the above mentioned Maclaurin series this relation proves to be equivalent to the following symbolical representation:

$$\mathrm{Lapl}\,h_Z(X,Y)=e^{-Z/d_0}\left\{e^{\frac{Z}{d_0}\frac{\int_0^{\partial/\partial\gamma}ds f(s)}{\partial/\partial\gamma}}\,e^{\gamma Z\sqrt{p^2+q^2}}\right\}_{\nu=0}. \tag{30}$$

Each power of $\partial/\partial\nu$ here merely amounts to a multiplication by the same power of $Z\sqrt{p^2+q^2}$.

The transform variables now only occur in the last exponential which, fortunately, can be recognized as the Laplace transform of an elementary function. In order to show this, we start from the relation:

$$\frac{e^{\lambda\sqrt{p^2+q^2}}}{\sqrt{p^2+q^2}}=\mathrm{Lapl}\,\frac{1}{2\pi i\sqrt{x^2+y^2-\lambda^2}}, \tag{31}$$

which holds when the following conditions are fulfilled simultaneously:

(a) $\sqrt{x^2+y^2-\lambda^2}$ has to be defined with a positive real part,

(b) $\sqrt{p^2+q^2}$ has to be positive imaginary,

(c) $\Im m\lambda>0$.

This relation can be checked by introducing polar coordinates ρ, φ in the double Laplace integral of its right-hand side. The φ integration then reduces to a zero-order Bessel function. The remaining ρ integral constitutes a special example of Sommerfeld's integral for the wave function of a point source; this integral represents the decomposition of this wave function into solutions of the wave equation that are separated in cylindrical coordinates.

A differentiation of (31) with respect to λ yields the relation to be applied to the last exponential in (30). We thus arrive at the following representation for the function h_Z itself:

$$h_Z X, Y=\frac{Z}{2\pi i}\left[e^{\frac{Z}{d_0}\left\{\frac{\int_0^{\partial/\partial\gamma} ds f(s)}{\partial/\partial\gamma}-1\right\}}\frac{\gamma}{(X^2+Y^2-\gamma^2Z^2)^{3/2}}\right]_{\nu=0}.$$

After a substitution of the Maclaurin series for $f(s)$ we may work out an expansion of the exponential still occurring with respect to $\partial/\partial\gamma$. We only need the odd powers, the even powers vanishing when applied to the subsequent function of γ. Each odd power of $\partial/\partial\gamma$ only affects a single term of the binomial expansion of this last function with respect to $\gamma^2Z^2/(X^2+Y^2)$. An elementary, though tedious evaluation finally results in an expansion of the function h_Z itself which starts with:

$$h_Z(X,Y)=\frac{Z}{2\pi \mathrm{i}}\left[\frac{c_1}{2}\frac{Z}{d_0(X^2+Y^2)^{3/2}}+9\left(\frac{c_3}{4}\frac{Z}{d_0}+\frac{c_1c_2}{6}\frac{Z^2}{d_0^2}+\frac{c_1^3}{48}\frac{Z^3}{d_0^3}\right)\frac{Z^2}{(X^2+Y^2)^{5/2}}\right.$$
$$\left.+225\left\{\frac{c_5}{6}\frac{Z}{d_0}+\left(\frac{c_1c_4}{10}+\frac{c_2c_3}{12}\right)\frac{Z^2}{d_0^2}+\left(\frac{c_1c_2^2}{36}+\frac{c_1^2c_3}{32}\right)\frac{Z^3}{d_0^3}+\frac{c_1^3c_2}{144}\frac{Z^4}{d_0^4}+\frac{c_1^5}{3840}\frac{Z^5}{d_0^5}\right\}\frac{Z^4}{(X^2+Y^2)^{7/2}}+\ldots\right]. \quad (32)$$

In special applications, such as given below, the convergence of this series proves to be sufficiently rapid.

The method, applied here, can be worked out in a similar way for the angular deviations, starting from (18) instead of (17). The series corresponding to (30) then proves to read:

$$h_Z(x,y)=\frac{1}{2\pi\mathrm{i}}\left[\frac{c_1\dfrac{Z}{d_0}}{(x^2+y^2)^{3/2}}+\frac{g\left(c_3\dfrac{Z}{d_0}+c_1c_2\dfrac{Z^2}{d_0^2}+\dfrac{c_1^3}{6}\dfrac{Z^3}{d_0^3}\right)}{(x^2+y^2)^{5/2}}\right.$$
$$\left.+\frac{225\left\{c_5\dfrac{Z}{d_0}+(c_1c_4+c_2c_3)\dfrac{Z^2}{d_0^2}+\dfrac{c_1(c_2^2+c_1c_3)}{2}\dfrac{Z^3}{d_0^3}+\dfrac{c_1^3c_2}{6}\dfrac{Z^4}{d_0^4}+\dfrac{c_1^5}{120}\dfrac{Z^5}{d_0^5}\right\}}{(x^2+y^2)^{7/2}}+\ldots\right].$$

13. Far-Distance Field in the Case of Norton's Model for Tropospheric Scattering

The derivation of this field in the case of a narrow-beam transmitter and a wide-angle receiver is briefly indicated below.

According to (28) the received energy is proportional to $h_Z(X,Y)=h_Z(\sqrt{X_2-Y_2})$ taken for the arguments

$$Z=a\vartheta,\ X^2+Y^2=\left(\frac{a\vartheta^2}{2}\right)^2=\frac{Z^4}{4a^2}.$$

Substitution into the expansion (32) yields the following dependence on the linear propagation distance Z:

$$h_{a\vartheta}\left(\frac{a\vartheta^2}{2},0\right)=\frac{Z}{2\pi i}\left[4c_1\frac{a^3}{d_0Z^5}+144\left(\frac{c_3}{2}\frac{Z}{d_0}+\frac{c_1c_2}{3}\frac{Z^2}{d_0^2}+\frac{c_1^3}{24}\frac{Z^3}{d_0^3}\right)\right.$$
$$\left.+225.64\left\{\frac{c_5}{3}\frac{Z}{d_0}+\left(\frac{c_1c_4}{5}+\frac{c_2c_3}{6}\right)\frac{Z^2}{d_0^2}+\left(\frac{c_1c_2^2}{18}+\frac{c_1^2c_3}{16}\right)\frac{Z^3}{d_0^3}+\frac{c_1^3c_2}{72}\frac{Z^4}{d_0^4}+\frac{c_1^5}{1920}\frac{Z^5}{d_0^5}\right\}\frac{a^7}{Z^{10}}+\ldots\right]. \quad (33)$$

For $c_1 \neq 0$ we get a dominating contribution, if $Z > d_0$, which decreases in proportion to Z^{-4}. This solution, corresponding to multiple scattering, involves a field strength decreasing as Z^{-2}. On the contrary, for $c_1 = 0$, the dominating term of h_Z decreases as Z^{-6}, and the field strength as Z^{-3}. A vanishing value of c_1 occurs for the "modified Bessel model," originally proposed by Muchmore [Norton and Norton, 1956], which is characterized by the isotropic autocorrelation function

$$C_{\delta\epsilon}(\rho/l) = (\rho, l) K_1(\rho/l).$$

This model is in accordance with many observational data [Norton, 1959], as well as the Villars and Weisskopf [1955] theory of turbulent mixing (turbulent blobs with dimensions conditioned by the vertical gradient of the refractive index). In the small-angle approximations, the modified Bessel model can be characterized by the isotropic case ($\mu = 1$) of the derivation given in appendix 2 of [Norton, 1960]. The scattering function is therefore given by

$$\sigma(x, y) = \frac{3/(\pi d_0 \langle \vartheta_1^2 \rangle)}{\left\{1 + \frac{2}{\langle \vartheta_1^2 \rangle}(x^2 + y^2)\right\}^{5/2}}. \tag{34}$$

The corresponding function $f(s)$ of (29) reads:

$$f(s) = e^{i\sqrt{\frac{\langle \vartheta_1^2 \rangle}{2}}\, s} \left(1 - i\sqrt{\frac{\langle \vartheta_1^2 \rangle}{2}}\, s\right). \tag{35}$$

This can be proved by putting (34) in the form:

$$\sigma(x, y) = \frac{2B^{3/2}}{\pi d_0} \frac{\partial^2}{\partial B^2} \frac{1}{(x^2 + y^2 + B)^{1/2}}$$

$$\text{for } B = \langle \vartheta_1^2 \rangle / 2,$$

and by applying (31) thereafter in order to obtain the two-dimensional Laplace transform.

The Maclaurin expansion of (35) yields the coefficients c_i. Their substitution into (33) results in

$$h_{a\vartheta}\left(\frac{a\vartheta^2}{2}, 0\right) = \frac{6a^5 \langle \vartheta_1^2 \rangle^{3/2}}{\pi \sqrt{2} d_0 Z^6} \left\{1 + \frac{25}{3} \frac{\langle \vartheta_1^2 \rangle a^2}{d_0 Z} \left(1 - \frac{2}{5} \frac{d_0}{Z}\right) + \ldots\right\}.$$

For the transmitter and receiver here considered the decrease of the energy proves to be proportional to Z^{-6} in the plural-scattering range ($d >> d_0$); this corresponds to a field decrease proportional to Z^{-3}.

14. Final Remarks

The conventional theories for volume scattering in random continuous media only consider single scatterings (Born approximation). However, multiple-scattering effects may become relevant under special circumstances, as shown by Bugnolo [1960] and Ament [1960] for far-distance tropospheric radio propagation. It is therefore suggested to introduce in that case first of all a phenomenological theory taking account of the radiation diagrams of the transmitter and the receiver, without specifying the number of scatterings. Such a theory may start from a joint probability density for the lateral and angular deviations suffered by an energy unit which would travel along a rectilinear path in the absence of the random fluctuations of the medium properties. This probability density h_Z has to satisfy an integral equation, the "transport equation"; it reduces to a much simpler second-order differential equation of the Fokker-Planck type provided that very special approximations are justified. The general integral equation depends on the scattering coefficient introduced in the theories dealing with single scattering. The latter equation can be solved rigorously with the aid of Laplace transforms, the solution including the contributions associated with all orders of scattering. These contributions can be identified individually by a proper expansion of the solution. The Born approximation fits in this general theory by taking a proper function for h_Z.

The application of the general theory to the modified Bessel scattering model, which agrees with the Villars and Weisskopf turbulence theory, leads to a decrease of the far-distance field in the multiple-scattering range which is proportional to the inverse third power of the distance. This result is quite different from the exponential decrease conditioned by the Fokker-Planck approximation, or also from the corresponding decrease associated with exponential models (instead of the algebraic model of Norton) for the autocorrelation function fixing the scattering function σ. The above shows that the complete transport equation has to be applied in tropospheric propagation theories in order to get reliable results for the nonexponential autocorrelation functions confirmed by empirical data.

The author acknowledges his contact with Dr. A. E. Goertz (CRPL Laboratory, Boulder, USA) who directed in particular his attention to the comprehensive literature on the subject.

Eindhoven, October 31st, 1963

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(Paper 68D9–396)