RADIO SCIENCE Journal of Research NBS/USNC–URSI
Vol. 68D, No. 9, September 1964

Some Nonlinear Problems Arising in the Study of Random Processes

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(Received November 20, 1963; revised February 5, 1964)

Two problems of a nonlinear character concerned with random processes are discussed. In both cases the processes are assumed to be stationary.

The first problem is concerned with the representation of a discrete time parameter stationary random process as a one-sided function (nonlinear generally) of independent random variables and its shifts. This is a representation one might expect if the process is purely nondeterministic. Comments are made on the continuous parameter version of this problem, indicating that it is likely to be much more difficult and perhaps less important from a practical point of view. The second problem is concerned with the harmonic resolution of the moments (of degree two or higher) of stationary random processes. The harmonic resolution of third order moments (the "bispectrum") is considered in some detail and remarks are made about statistical estimates of the bispectrum.

1. Introduction

1.1. Representation of Discrete Time Parameter Processes

Let $\{X_k; k=0, \pm 1, \ldots\}$ be a discrete time parameter stationary random process. It will be convenient to introduce the following notation. Let $\mathcal{B}_n(X)$ be the Borel field of events generated by $X_n, X_{n-1}, \ldots$

$$\mathcal{B}_n(X)=\mathcal{B}\{X_k;\, k\leq n\}; \tag{1}$$

$\mathcal{B}_n(X)$ is the smallest collection of events containing the events

$$\{X_k\leq x_k\}, \qquad k=n, n-1, \ldots,$$

with the x_k's any real numbers, that is closed under countable union and complementation of events. If time n is considered the present, $\mathcal{B}_n(X)$ can be thought of as the carrier of the information (nonlinear) from the infinite past to the present. If the intersection $\prod_n \mathcal{B}_n(X)$ of the Borel fields $\mathcal{B}_n(X)$ is the trivial Borel field consisting only of the empty set (up to a set of probability zero) and its complement, we call the process $\{X_k\}$ *purely nondeterministic.* This means that the infinite past contains no information about the present. Notice that the concept of a purely nondeterministic process is intimately associated with a sense of time direction. A process may be purely nondeterministic and yet if we look at it with the time direction reversed, it may become purely deterministic (the infinite future contains all the information about the present). This contrasts strongly with the linear (or weak) version of the notion of a purely nondeterministic real-valued process. The following well-known process is easily seen to be purely nondeterministic looking forward in time and purely deterministic looking backward in time.[1] $\{X_k\}$ is a stationary Markov process on the unit interval $0\leq x\leq 1$ with instantaneous distribution uniform and the following simple transition mechanism

$$x \begin{cases} \nearrow \tfrac{1}{2}x & \text{with probability } \tfrac{1}{2} \\ \searrow \tfrac{1}{2}+\tfrac{1}{2}x & \text{with probability } \tfrac{1}{2} \end{cases} \quad 0\leq x\leq 1. \tag{2}$$

Let us now discuss the representation problem of interest. We wish to find out when one can find a one-sided Borel function f of independent random variables $\{\xi_k; k=0, \pm 1, \ldots\}$ uniformly distributed on $[0,1]$ such that the process $\{X_k'\}$

$$X_k'=f(\xi_k, \xi_{k-1}, \xi_{k-2}, \ldots), \qquad k=0, \pm 1, \ldots, \tag{3}$$

has the same probability structure as $\{X_k; k=0, \pm 1, \ldots\}$. One can easily show that such a representation cannot hold unless $\{X_k\}$ is purely nondeterministic [Rosenblatt, 1959]. In fact, if $\{X_k\}$ is a countable state Markov chain the condition that the process be purely nondeterministic is necessary and sufficient for such a representation [Rosenblatt, 1960; Hanson, 1963]. It is a reasonable conjecture that this condition is necessary and sufficient for such a representation in the case of a general stationary process. As far as I know this has not yet been shown to be true. The following result for Markov processes (see [Hanson, 1963]) is an interesting step in this direction. Assume that $\{X_k\}$ is a purely nondeterministic Markov process with in-

[1] This was pointed out to the author by B. Jamison.

stantaneous distribution $\overline{P}(\cdot)$ and transition function $P(\cdot, \cdot)$. Further let there be a measure ϕ and events A, B with $\overline{P}(A)>0$, $\phi(B)>0$ such that

$$P(x, B') \geq \phi(B') \tag{4}$$

for all $x \epsilon A$ and all $B' \subset B$. Then a representation of the desired type exists for the process $\{X_k\}$.

The problem we have discussed above can be analyzed in greater detail with some benefit. Let us allow $\{\xi_n\}$ to be any sequence of independent identically distributed random variables. For example, the common distribution function might be discrete. If

$$X'_k = f(\xi_k, \xi_{k-1}, \ldots) \qquad k=0, \pm 1, \ldots \tag{5}$$

with some given f, and $\{X'_k\}$ has the same probability structure as $\{X_k\}$ we shall say as before that we have a one-sided representation of the process $\{X_k\}$. Now let $L_2(X;n)$ be the family of square integrable functions measurable with respect to $\mathcal{B}_n(X)$. Further let $L_2(X)\theta L_2(X;n)$ be the square integrable functions measurable with respect to $\mathcal{B}(X) = \mathcal{B}_\infty(X)$ that are orthogonal to $L_2(X;n)$. We shall say that the representation (5) is a one-sided *canonical representation of* X if $L_2(\xi;n)$ is orthogonal to $L_2(X')\theta L_2(X';n)$. Essentially this says that there is no information about the present and past of the ξ process in the future of the X' process. If $\mathcal{B}_n(\xi) = \mathcal{B}_n(X')$, we will call (5) a one-sided *properly canonical representation of* X. In Rosenblatt [1959], it is shown that a one-sided *properly canonical representation* is extremely rare for finite state Markov chains. Let $P=(p_{ij})$ be the transition probability matrix of the chain. Then a necessary (but not sufficient) condition for such a representation is that all the row probability distributions $\{p_{ij}, j=1, 2, \ldots\}$ be the same except for permutation. However, a one-sided canonical representation is always possible for a purely nondeterministic chain (see [Rosenblatt, 1959] for details). The impossibility of a properly canonical representation, even though a canonical representation is possible, appears to be due to the fact that the probability space of the process is not rich enough.

Wiener has discussed one-sided properly canonical representations in his interesting book on nonlinear problems ([1958] lectures 12 and 13 on coding and decoding). Let $\{X_n\}$ be a stationary purely nondeterministic process with

$$F(x|x_{n-1}, x_{n-2}, \ldots) = P(X_n \leq x | X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots) \tag{6}$$

the conditional distribution function of X_n given the past of the process relative to n. Wiener states that a properly canonical one-sided representation of $\{X_n\}$ is possible as long as $F(x|x_{n-1}, x_{n-2}, \ldots)$ is *properly increasing* for almost every past $x_{n-1}, x_{n-2}, \ldots$. It is not quite clear what is meant by the term properly increasing. However, from the context it would appear to be the case that a distribution function is properly increasing if it is absolutely continuous with a unimodal density function. Wiener's conjectured result may very well be valid but the way in which he proposes to construct the independent random variables ξ_n in terms of which the properly canonical representation is to be set up will not generally work. He proposes to set

$$\xi_n = F(X_n | X_{n-1}, X_{n-2}, \ldots) \tag{7}$$

and the random variables ξ_n will be independent uniformly distributed random variables with

$$\mathcal{B}_n(\xi) \subset \mathcal{B}_n(X) \tag{8}$$

if $F(x|x_{n-1}, x_{n-2}, \ldots)$ is a continuous function in x for almost every past $x_{n-1}, x_{n-2}, \ldots$. The following example is one in which it is clear that $\mathcal{B}_n(\xi) \neq \mathcal{B}_n(X)$ so that X_n cannot be fully reconstructed from $\xi_n, \xi_{n-1}, \ldots$ with ξ_n given by (7).

Let $F(x)$ be an absolutely continuous distribution function with unimodal positive continuous density function. The process $\{X_n\}$ is taken to be a stationary Markov process with conditional distribution function

$$F(x|x_{n-1}) = \begin{cases} F(x) \text{ if } 2j-1 \leq x_{n-1} < 2j \\ F(x-1) \text{ if } 2j \leq x_{n-1} < 2j+1 \end{cases} \qquad j=0, \pm 1, \pm 2, \ldots \tag{9}$$

One could, for example, take $F(x)=\Phi(x)$ with $\Phi(x)$ the standard gaussian distribution function. In such a case the process $\{X_n\}$ will certainly be purely nondeterministic. Then, using the construction (7),

$$\xi_n = \begin{cases} F(X_n) \text{ if } 2j-1 \leq X_{n-1} < 2j \\ F(X_{n-1}) \text{ if } 2j \leq X_{n-1} < 2j+1 \end{cases} \qquad j=0, \pm 1, \ldots \tag{10}$$

However, using knowledge of ξ_k, $k \leq n$, we can not determine whether the greatest integer less than or equal to X_n is even or odd.

The problem of obtaining one-sided representations of processes amounts to an investigation of the class of processes one can obtain from independent random variables by one-sided nonlinear filtering. In the case of a one-sided properly canonical representation, the process is obtained by an invertible nonlinear filter whose inverse is one sided.

1.2. Representation of Continuous Time Parameter Processes

N. Weiner was also concerned with the one-sided representation of processes in terms of the Brownian motion (or Wiener) process in his book on nonlinear methods in random theory [1958]. We shall describe some recent work of M. Nisio [1961] on aspects of this problem. Let $\{X(t), -\infty < \mathrm{t} < \infty\}$ be a stationary purely nondeterministic process and $dB=\{dB(t),\ -\infty<t<\infty\}$ be the Wiener random measure. Consider a sequence of functions $f=\{f_n,\ n=0,1,2,\ldots\}$, where each f_n is a symmetric L_2 function of n variables on the domain $(-\infty, 0]^n$. Assume that $\sum_{n=0}^{\infty} n!\,||f_n||^2<\infty$ where $||f_n||$ is the L_2 norm of f_n. Given any such sequence f, we define the corresponding process $X'=\{X'(t),\ -\infty<t<\infty\}$ as

$$X'(t)=\sum_{n=0}^{\infty}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t} f_n(t_1-t,\ldots,t_n-t)\,dB(t_1)\ldots dB(t_n) \tag{11}$$

where the nth term is an n-dimensional Wiener integral. If $\{X'(t)\}$ has the same probability structure as $\{X(t)\}$ we say that (11) is a *one-sided representation of* X *in terms of the Brownian motion with* f *the kernel of the representation.* As before we say that the representation is *canonical* if $L_2(dB,\ t)\perp L_2(X')\theta L_2(X',\ t)$. The representation is *properly canonical* if $L_2(dB,\ t)=L_2(X',\ t)$. The terminology on representations used in this paper was introduced by M. Nisio in the continuous time context.

Miss Nisio has shown that a class of one dimensional diffusion processes has properly canonical one-sided representations in terms of Brownian motion. What is much more surprising is that she has shown that if $P(t)$, $-\infty<t<\infty$, is a Poisson process with parameter λ then $P(t)-P(t-1)$ $(-\infty<t<\infty)$ has no canonical one-sided representation in terms of Brownian motion. It is a very interesting question as to whether there is any one-sided representation of $P(t)-P(t-1)$ in terms of Brownian motion. One is inclined to doubt whether this is so on the basis of her result. In the case of a discrete time parameter process it seemed reasonable to conjecture that a necessary and sufficient condition for a process to have a representation is that it be purely nondeterministic. If there is a representation, it is always possible to set it up in terms of independent uniformly distributed (on [0, 1]) random variables However, if the conjecture on the impossibility of a one-sided representation of $P(t)-P(t-1)$ in terms of Brownian motion is valid, it is likely that in order to set up one-sided representations of purely nondeterministic processes one must consider not only representations in terms of Brownian motion but also in terms of any differential process. Further the appropriate differential process in which to set up the representation is most likely determined by the local properties of the given process $X(t)$.

One should mention that a great deal of the discussion in Wiener's book on nonlinear methods is concerned with the computation of the expectation of polynomial forms in a process with a representation in terms of Brownian motion. This study has been extended by McShane [1962] and more recently Sinai and Shiryaev [1963] to processes with representations of a much more general type.

2. Higher Order Moment Functions and Their Spectral Resolutions

In the case of a weakly stationary process X_t with mean zero, $EX_t\equiv 0$, the second order moments $EX_tX_\tau=r(t-\tau)$ exist and it is well known that there is a spectral resolution

$$r(t)=\int_{-\infty}^{\infty} e^{it\lambda}dF(\lambda) \tag{12}$$

where F is bounded and nondecreasing. Suppose we assume the process X_t is strictly stationary and that the kth moment $E|X_t|^k<\infty$ for some integer $k>2$. Let the mean EX_t again be identically zero. Then

$$E(X_{t_1}\ldots X_{t_k})=m_k(t_1,\ldots,t_k)=r_k(t_2-t_1,\ldots,t_k-t_1) \tag{13}$$

exists for all $t_1,\ldots,t_k$ and we might hope for a spectral representation of $r_k(\tau_1,\ldots,\tau_{k-1})$

$$r_k(\tau_1,\ldots,\tau_{k-1})=\int_{-\infty}^{\infty}\cdots\int e^{i\Sigma\tau_j\lambda_j}dF_k(\lambda_1,\ldots,\lambda_{k-1}) \tag{14}$$

in terms of a complex-valued function $F_k(\lambda_1,\ldots,\lambda_{k-1})$ of bounded variation. Blanc-Lapierre and Fortet [1953] were among the earliest to discuss such higher order spectral representations. Recently Kolmogorov and Sinai gave examples of stationary processes for which $E|X_t|^k<\infty$ and yet r_k has no spectral representation in terms of an F_k of bounded variation. In fact, in Sinai's example [1963] all moments exist. Of course, it is still possible that there may be such representations with F_k of unbounded variation.

In any case, it is still interesting to consider the large class of processes for which such spectral representations of higher order moment functions are possible. The spectral representation of the third order moment function has drawn especial attention (see Tukey [1959] for an example). Such an approach seems to be of some interest in studying certain nonlinear problems in random process theory. For a recent paper using estimation of the third order spectral function (or bispectrum as it is sometimes called) in the analysis of certain ocean wave patterns see Hasselman, Munk, and MacDonald [1963].

We shall now briefly sketch the statement of a result recently obtained by Van Ness [1963] on the estimation of the bispectrum of a stationary process. Let X_t, $EX_t\equiv 0$, be a real-valued stationary process with $EX_t^6<\infty$. Let its second and third order spectral functions be absolutely continuous with continuous derivatives, the spectral density $f(\lambda)$ and bispectral density function $g(\lambda_1,\lambda_2)$ respectively

$$r(t)=\int_{-\infty}^{\infty} e^{it\lambda}f(\lambda)d\lambda$$

$$r_3(t_1,t_2)=\int_{-\infty}^{\infty}\int e^{it_1\lambda_1+it_2\lambda_2}g(\lambda_1,\lambda_2)d\lambda_1d\lambda_2. \tag{15}$$

The third order moment function $r_3(t_1, t_2)$ has the symmetry properties

$$r_3(t_1, t_2) = r_3(t_2, t_1) = r_3(-t_1, t_2 - t_1) \tag{16}$$

because the process X_t is real-valued. Van Ness considers an estimate of $g(\lambda_1, \lambda_2)$ of the form

$$g_N^*(\lambda_1, \lambda_2) = \frac{1}{(2\pi)^2} \int_{-N}^{N} \int e^{-i\lambda_1\nu_1 - i\lambda_2\nu_2} k(B_N\nu_1, B_N\nu_2) \rho_N(\nu_1, \nu_2) d\nu_1 d\nu_2 \tag{17}$$

with $\rho_N(\nu_1, \nu_2)$ the third order sample moment function

$$\rho_N(\nu_1, \nu_2) = \frac{1}{N} \int X_t X_{t+\nu_1} X_{t+\nu_2} dt \tag{18}$$

with t, $t+\nu_1$, $t+\nu_2$ restricted to the range $[0, N]$ and $k(\nu_1, \nu_2)$ a normalized ($k(0, 0) = 1$) continuous bounded uniformly integrable weight function with the symmetry properties of $r_3(t_1, t_2)$. Here $B_N \to 0$ as $N \to \infty$ so that the Fourier transform of $k(B_N\nu_1, B_N\nu_2)$ behaves asymptotically like a δ-function. Under a variety of boundedness and integrability conditions on the moment functions up to sixth order, $g_N^*(\lambda_1, \lambda_2)$ is shown to be asymptotically unbiased as $N \to \infty$ and

$$\begin{aligned}\lim_{N\to\infty} NB_N^2 \operatorname{cov}[g_N^*(\lambda_1, \lambda_2), g_N^*(\lambda_3, \lambda_4)] &= \frac{1}{2\pi} [f(\lambda_1) f(\lambda_2) f(\lambda_1+\lambda_2) f(\lambda_3) f(\lambda_4) f(\lambda_3+\lambda_4)]^{1/2} \\ &\quad \{k_1 \delta(\lambda_2)\delta(\lambda_4)[1+2\delta(\lambda_1)][1+2\delta(\lambda_3)] \\ &\quad + k_2 \delta(\lambda_1-\lambda_3)\delta(\lambda_2-\lambda_4) \\ &\quad [1+\delta(\lambda_1-\lambda_4)+4\delta(\lambda_1)\delta(\lambda_2)]\}\end{aligned} \tag{19}$$

if $B_N \to 0$, $B_N^2 \to \infty$ where $0 \leq \mu_1$, $\mu_3 < \infty$, $0 \leq \mu_2 \leq \mu_1$, $0 \leq \mu_4 \leq \mu_3$. In formula (19)

$$k_1 = \left[\int_0^\infty k(0, \nu) d\nu\right]^2,$$

$$k_2 = \int_0^\infty \int k^2(\nu_1, \nu_2) d\nu_1 d\nu_2,$$

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

My thanks are due to J. Tukey for suggesting a neater way of writing the asymptotic covariance estimate that led to formula (19).

3. References

Blanc-Lapierre, A., and R. Fortet (1953), Theorie des Fonctions Aleatoires (Masson et Cie Editeurs, Paris, France).

Hanson, D. L. (1953), On the representation problem for stationary stochastic processes with trivial tail field, J. Math. Mech. **12,** 293–301.

Hasselman, K., W. Munk, and G. MacDonald (1963), Bispectrum of ocean waves, Times Series Analysis, Ed. M. Rosenblatt, 125–139. (John Wiley & Sons, Inc., New York, N.Y.)

McShane, E. J. (1962), Stochastic integrals and nonlinear processes, J. Math. Mech. **11,** 235–283.

Nisio, M. (1961), Remarks on the canonical representation of strictly stationary processes, J. Math. Kyoto Univ. **1,** 129–146.

Rosenblatt, M. (1959), Stationary processes as shifts of functions of independent random variables, J. Math. Mech. **8,** 665–681.

Rosenblatt, M. (1960), Stationary Markov chains and independent random variables, J. Math. Mech. **9,** 945–949.

Sinai, Ya. G. (1963), On spectral characteristics of ergodic dynamical systems, (in Russian), Dokl. Akad. Nauk SSSR, **150,** 1235–1237.

Sinai, Ya. G. and A. H. Shiryaev (1963), Nonlinear problems in the theory of stationary processes, A paper given at the 7th All-Union Conference on the Theory of Probability and Statistics in Tbilisi, USSR 1963.

Tukey, J. W. (1959), An introduction to the measurement of spectra, Probability and Statistics, Ed. U. Grenander, 300–330 (John Wiley and Sons, Inc., New York, N.Y.).

Van Ness, J. (1963), Estimates of the bispectrum of stationary time series (private communication).

Wiener, N. (1958), Nonlinear Problems in Random Theory (Cambridge Technology Press of Massachusetts Institute of Technology).

(Paper 68D9–393)