RADIO SCIENCE Journal of Research NBS/USNC–URSI
Vol. 68D, No. 9, September 1964

Modified Gaussian Distributions for Slightly Nonlinear Variables

M. S. Longuet-Higgins

National Institute of Oceanography, Wormley, Surrey, England

(Received December 27, 1963)

Many random variables are almost linear, in the sense that they can be represented approximately as the sum of independent components in random phase. Such variables (for example, the surface elevation in a random sea) may have a gaussian distribution in the first approximation. However in higher approximations the phases of the different components become correlated, due to nonlinear interactions. The purpose of this paper is to show theoretically what is the effect of such nonlinearities on the basic gaussian distribution.

The modified distribution is derived both for a single variable and for two or more related variables (such as the components of slope of a random surface). The results are applied in the first place to sea waves, and are compared with observations. However the analysis is applicable quite generally to any such nonlinear variables.

Two further problems are solved for weakly nonlinear variables: the mean number of zeros per unit time of a stationary random function $\zeta(t)$ and the distribution of the maximum values of $\zeta(t)$. These solutions are essentially generalizations of the well-known results of Rice for gaussian variables.

1. Introduction

Many random variables occurring in physical problems may be considered as the sum of a large number of independent components; thus we write

$$\zeta=\sum_{i=1}^{N} \alpha_i\xi_i \tag{1}$$

where the α_i are constants and the ξ_i are independent random variables symmetically distributed about 0 with variance V_i, say. Under certain conditions as $N\rightarrow\infty$ and each $V_i\rightarrow 0$, the distribution of ζ tends to a gaussian distribution with variance $\sum V_i$.

In particular it is possible to consider some stationary stochastic processes as the limit of a sum such as (1), the ξ_i being components corresponding to particular frequencies σ_i or wave numbers k_i; there may exist a continuous function $E(\sigma)$, the spectral density of ζ, such that in any small interval $(\sigma,\sigma+d\sigma)$ the sum of the variances V_i is given by

$$\lim_{N\rightarrow\infty}\left(\sum_{\sigma<\sigma_i<\sigma+d\sigma} V_i\right)=E(\sigma)d\sigma+0(d\sigma)^2.$$

Such a representation (equivalent to a stochastic integral) has been widely used in the theory of noise in electrical circuits [Rice, 1944 and 1945], in the theory of random sea surfaces, microseisms, turbulence, and other physical phenomena [Longuet-Higgins, 1960]. The representation gives most satisfactory results when the variable ζ satisfies a linear differential equation and can be shown physically to be the resultant of large number of independent contributions.

In some instances, however, the assumption of linearity is only approximately justified, and the variable in question may actually satisfy a nonlinear differential equation whose nonlinear terms are small but not completely negligible. Such is the case with variables ζ associated with sea waves. The distribution $p(\zeta)$ is then nearly gaussian, but not exactly so. The question we wish to discuss is: what is the effect of the nonlinearities on $p(\zeta)$, and how can they be calculated in terms of the differential equation satisfied by ζ?

In a recent paper [Longuet-Higgins, 1963], the representation (1) was generalized in the following way. Consider the variable

$$\zeta=\alpha_i\xi_i+\alpha_{ij}\xi_i\xi_j+\alpha_{ijk}\xi_i\xi_j\xi_k+\ldots \tag{2}$$

where the α_i, α_{ij}, α_{ijk}, etc., are constants and the ξ_i as before are independent random variables with variance V_i. (The repetition of any subscript will be taken to imply summation.) If the α_i are given, then by substitution in the differential equation for ζ it may be possible to determine by successive approximation the values of $\alpha_{ij}, \alpha_{ijk}$, etc. Assuming that these are bounded, then as $N\rightarrow\infty$ the contributions to ζ from successive terms in the series can be expected to be of decreasing order of magnitude.

To calculate $p(\zeta)$ from (2) it is assumed that $p(\zeta)$ is uniquely determined by the sequence of its cumulants, which can be found from (2) to any order required.

This generalization[1] of the representation (1) leads to a distribution $p(\zeta)$ which is a generalization of the gaussian distribution; in fact it is the gaussian distribution multiplied by a sequence of Hermite polynomials, of increasing degree but decreasing order of magnitude. The series may be only valid asymptotically, and nonuniformly with ζ. Nevertheless when applied to sea waves the second approximation has been shown in [Longuet-Higgins, 1963] to give reasonably good agreement with observation.

In the present paper what we propose to do is to state without detailed proof the general results of Longuet-Higgins [1963], and then to apply the results to the solution of two related problems: to determine (a) the mean number of zero-crossings of ζ per unit time and (b) the distribution of the values of ζ at a maximum.

2. Distribution of a Single Variable

We may start by writing down the moments of $p(\zeta)$. Thus taking mean values in (2) one has

$$\overline{\zeta}=\alpha_i\overline{\xi_i}+\alpha_{ij}\overline{\xi_i\xi_j}+\alpha_{ijk}\overline{\xi_i\xi_j\xi_k}+\ldots.$$

The mean values of all odd-order terms vanish, while in the terms of even order only those remain in which each ζ_i is paired with a similar ζ_i. Thus

$$\overline{\zeta}=\alpha_{ii}V_i+3\alpha_{iijj}V_iV_j+\ldots$$

it being assumed that the α's are symmetric in their suffices. Terms involving $\overline{\zeta}\,^4_i$, $\overline{\zeta}\,^6_i$, etc. are assumed to become negligible in the limit as $N\rightarrow\infty$. Similarly since

$$\zeta^2=(\alpha_i\xi_i+\alpha_{ij}\xi_i\xi_j+\ldots)(\alpha_k\xi_k+\alpha_{kl}\xi_k\xi_l+\ldots)$$

we have on taking mean values

$$\overline{\zeta^2}=\alpha_i\alpha_iV_i+(2\alpha_{ij}\alpha_{ji}+\alpha_{ii}\alpha_{jj})V_iV_j+6\alpha_i\alpha_{ijj}V_iV_j+\ldots$$

and so on for higher moments. Now some of the terms in the last expression can be factorized, e.g.,

$$\alpha_{ii}\alpha_{jj}V_iV_j=(\alpha_{ii}V_i)(\alpha_{jj}V_j)$$

while others, e.g., $\alpha_{ij}\alpha_{ji}V_iV_j$, cannot. We call terms which cannot be factorized "irreducible". For simplicity the following notation is introduced. $\alpha_{ij\ldots l}$ is denoted simply by A_ζ, where ζ is the number of suffices $i, j, \ldots l$; and the sum of all *irreducible* terms in the product

[1] A less direct approach is suggested by Wiener [1958]. It should be noted that in the present paper the individual ξ_i are not assumed necessarily to be gaussian.

$$\overline{(A_p\xi_{i_1}\ldots\xi_{i_p})(A_q\xi_{j_1}\ldots\xi_{j_q})\ldots(A_s\xi_{l_1}\ldots\xi_{l_s})}$$

is denoted by $(A_pA_q \ldots A_s)$. Then we have

$$\bar{\zeta}=\sum_p (A_p)$$

$$\overline{\zeta^2}=\sum_{p,q} [(A_pA_q)+(A_p)(A_q)]$$

etc. However, much simpler than the moments of $p(\zeta)$ are the cumulants κ_u, defined by

$$\int_{-\infty}^{\infty} p(\zeta)e^{it\zeta}d\zeta=\exp\left[\frac{K_1}{1!}(it)+\frac{K_2}{2!}(it)^2+\ldots\right]. \tag{3}$$

We have

$$K_1=\bar{\zeta}=\sum_p (A_p)$$

$$K_2=\overline{\zeta^2}-(\bar{\zeta})^2=\sum_{p,q}(A_pA_q)$$

and in general it can be shown that [Longuet-Higgins, 1963]

$$\kappa_u=\sum_{p,q,\ldots s}(A_pA_q\ldots A_s) \tag{4}$$

where $p, q, \ldots s$ run through all values from 1 to ∞ independently. Thus if we retain terms up to the third order in the V_i we have

$$\left.\begin{aligned} K_1&=(A_2)+(A_4)+(A_6)\\ K_2&=(A_1^2)+(A_2^2)+(A_3^2)+2(A_1A_3)+2(A_1A_5)+2(A_2A_4)\\ K_3&=(A_2^3)+3(A_1^2A_2)+3(A_1^2A_4)+6(A_1A_2A_3)\\ K_4&=(A_1^4)+4(A_1^3A_3)+6(A_1^2A_2^2)\end{aligned}\right\} \tag{5}$$

We see that K_1 and K_2 are both of order V in general, but when $n\geq 2$ the lowest nonvanishing term in κ_u is $O(V^{n-1})$. If we define the coefficients

$$\lambda_3=K_3/K_2^{3/2}$$

$$\lambda_4=K_4/K_2^2 \tag{6}$$

and generally

$$\lambda_n=K_n/K_2^{n/2} \tag{7}$$

we see that λ_n is of order $V^{n/2-1}$ in general.

The density $p(\zeta)$ is now found by inverting the expression (3):

$$p(\zeta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[\frac{K_1}{1!}(it)+\frac{K_2}{2!}(it)^2+\ldots\right]e^{-it\zeta}dt:$$

Writing

$$\frac{\zeta-K_1}{K_2^{1/2}}=f \tag{8}$$

we find [Longuet-Higgins, 1963] that

$$p(\zeta)=\frac{e^{-\frac{1}{2}f^2}}{(2\pi K_2)^{\frac{1}{2}}}\left[1+\frac{1}{6}\lambda_3 H_3+\left(\frac{1}{24}\lambda_4 H_4+\frac{1}{72}\lambda_3^2 H_6\right)+\ldots\right] \tag{9}$$

where $H_n(f)$ denotes the Hermite polynomial of degree n:

$$H_n=f^n-\frac{n(n-1)}{1!}\frac{f^{n-2}}{2}+\frac{n(n-1)(n-2)(n-3)}{2!}\frac{f_1^{n-4}}{2^2}-\ldots \tag{10}$$

This is the required distribution. In the first approximation λ_3, λ_4, . . . are neglected and the distribution is gaussian with mean K_1 and variance K_2. In the next approximation a term $\lambda_3 H_3$ is included, where H_3 is the cubic polynomial (f^3-3f). In the third approximation the quartic polynomial H_4 and the sextic polynomial H_6 are both involved. Higher approximations can be written down at will.

Equation (9) will be recognized as essentially similar to Edgeworth's form of the Gram-Charlier distribution [Edgeworth, 1906].

3. Application to a Stochastic Variable

Suppose now that the variable ζ is a stationary random function of the space coordinate $\mathbf{x}$ and the time t, satisfying a nonlinear partial differential equation (or boundary condition) in $\mathbf{x}$ and t. How is the distribution $p(\zeta)$ related to the spectral density of ζ?

Let the equation satisfied by ζ be represented symbolically by

$$L(\zeta)+Q(\zeta)+C(\zeta)+\ \ldots\ =0 \tag{11}$$

where L, Q, C, etc., represent operators that are linear, quadratic, cubic, etc., in ζ. To solve (11) for small perturbations we naturally substitute

$$\zeta=\epsilon\zeta^{(1)}+\epsilon^2\zeta^{(2)}+\epsilon^3\zeta^{(3)}+\ \ldots, \tag{12}$$

where ϵ is a small parameter. Writing (11) in the form

$$L(\zeta)=-Q(\zeta)-C(\zeta)-\ \ldots \tag{13}$$

and equating coefficients of ϵ, ϵ^2, ϵ^3, . . . on the two sides of the equation we have successively

$$L(\zeta^{(1)})=0 \tag{14}$$

$$L(\zeta^{(2)})=-Q(\zeta^{(1)}) \tag{15}$$

$$L(\zeta^{(3)})=-Q(\zeta^{(1)},\zeta^{(2)})-C(\zeta^{(1)}) \tag{16}$$

etc., where $Q(\zeta^{(1)},\zeta^{(2)})$ denotes an expression that is bilinear in $\zeta^{(1)}$ and $\zeta^{(2)}$. If there are wave-like solutions to (14) then we can write

$$\zeta^{(1)}=\sum_{n=1}^{N'} a_n \cos(\mathbf{k}_n\cdot\mathbf{x}-\sigma_n t+\theta_n) \tag{17}$$

where a_n and θ_n are amplitude and phase constants and $\mathbf{k}_n$ and σ_n are wave numbers and frequencies respectively. Generally (14) implies a relation between $\mathbf{k}_n$ and σ_n. Thus in the case of gravity waves on water we have

$$\sigma_n^2=g|k_n|.$$

Now (17) can be written

$$\zeta^{(1)}=\sum_{n=1}^{N'}\{a_n\cos\theta_n\cos(\mathbf{k}_n\cdot\mathbf{x}-\sigma_n t)+a_n\sin\theta_n\sin(\mathbf{k}_n\mathbf{x}-\sigma_n t)\}.$$

If now we suppose that $a_n \cos \theta_n$, $a_n \sin \theta_n$ are normally and independently distributed then we have

$$\zeta^{(1)} = \sum_{n=1}^{2N'} \alpha_n \xi_n$$

where

$$\left.\begin{aligned} \xi_n &= a_n \cos \theta_n, & \alpha_n &= \cos (\mathbf{k}_n \cdot \mathbf{x} - \sigma_n t) \\ \xi_{N+n} &= a_n \sin \theta_n, & \alpha_{N+n} &= \sin (\mathbf{k}_n \cdot \mathbf{x} - \sigma_n t) \end{aligned}\right\} n=1, N'.$$

Thus $\zeta^{(1)}$ is expressed in the same form as in (1).

To find $\zeta^{(2)}$ we now substitute for $\zeta^{(1)}$ in (15). Since the terms on the right are quadratic they can in general be expressed as a series of sum or difference wave numbers, i.e., harmonic terms in $\{(\mathbf{k}_n \pm \mathbf{k}_m)x - (\sigma_n \pm \sigma_m)t\}$. Hence we can in general find expressions for $\zeta^{(2)}$. On writing the solutions again as products of the original harmonics constituents we find

$$\zeta^{(2)} = \sum_{n,\, m} \alpha_{n,\, m} \xi_n \xi_m.$$

Thus $\zeta^{(1)} + \zeta^{(2)}$ has the form of (2), as far as the second-order terms. The evaluation of $\zeta^{(3)}$, etc., proceeds similarly.

The application to gravity waves on water is given in Longuet-Higgins [1963]. It is convenient to use the assumption of homogeneity so that we can consider the waves at the special point $x=0$ and time $t=0$. That is, we may take

$$\alpha_i = \left\{\begin{matrix} 1, & i=1, 2, \ldots N' \\ 0, & i=N'+1, \ldots 2N' \end{matrix}\right\}.$$

Then for gravity waves it is found that

$$\alpha_{ij} = \begin{cases} \dfrac{1}{\sqrt{k_i k_j}} (B_{ij}^- + B_{ij}^+ - \mathbf{k}_i \cdot \mathbf{k}_j + (k_i + k_j)\sqrt{k_i k_j}), & i, j = 1, \ldots N' \\ \dfrac{1}{\sqrt{k_i k_j}} (B_{ij}^- - B_{ij}^+ - k_i k_j), & i, j = N'+1, \ldots N \\ 0 \end{cases}$$

where

$$B_{ij}^- - \frac{(\sqrt{k_i} - \sqrt{k_j})^2 (\mathbf{k}_i \cdot \mathbf{k}_j + k_i k_j)}{(\sqrt{k_i} - \sqrt{k_j})^2 - |\mathbf{k}_i - \mathbf{k}_j|},$$

$$B_{ij}^+ = \frac{(\sqrt{k_i} + \sqrt{k_j})^2 (\mathbf{k}_i \cdot \mathbf{k}_j - k_i k_j)}{(\sqrt{k_i} + \sqrt{k_j})^2 - |\mathbf{k}_i + \mathbf{k}_j|}.$$

The diagonal terms α_{ii} are simply given by

$$\alpha_{ii} = \begin{cases} k_i, & i=1, \ldots N', \\ -k_i, & i=N'H, \ldots N. \end{cases}$$

So we find, for example, that, to the second order of approximation

$$K_1 = \alpha_{ii} V_i = \sum_{i=1}^{N'} k_i V_i + \sum_{i=N'+1}^{N} (-k_i) V_i = 0 \tag{18}$$

as was to be expected; while

$$K_2 = \alpha_i \alpha_i V_i = \sum_{i=1}^{N'} V_i = \iint E(\mathbf{k}) dk \tag{19}$$

and

$$K_3 = 6\,\alpha_i\alpha_j\alpha_{ij}V_iV_j$$

$$= 6\sum_{i,j=1}^{N'}\alpha_{ij}V_iV_j$$

$$= 6\iiiint K(\mathbf{k},\,\mathbf{k}')E(\mathbf{k})E(\mathbf{k}')d\mathbf{k}d\mathbf{k}' \tag{20}$$

where K is a function of $\mathbf{k}$ and $\mathbf{k}'$.

In the special case when the waves travel all in the same direction the above expressions become very simple, and it turns out that

$$K(k,\,k') = \min\,(k,\,k').$$

Hence introducing the frequency spectrum $F(\sigma)$ of ζ we have

$$K_2 = \int F(\sigma)d\sigma \tag{21}$$

$$K_3 = 6\iint \min\,(k,\,k')F(\sigma)F(\sigma')d\sigma d\sigma'$$

$$= 12\int_0^\infty \left\{\int_0^{\sigma'}\frac{\sigma^2}{g}F(\sigma)d\sigma\right\}F(\sigma')d\sigma' \tag{22}$$

since $k = \sigma^2/g$.

Returning to the general case, when the waves are not unidirectional, it can be shown that the expression on the right of (22), which we denote by I, is related to K_3 by the inequalities

$$0.44\;I \leq K_3 \leq 1.01\;\mathrm{I}.$$

Combined with (21) this gives us bounds for the skewness coefficient $\lambda_3 = K_3/K_2^{3/2}$, in terms of the frequency spectrum $F(\sigma)$ and irrespective of the directional properties of the spectrum.

These results have been compared with the observations of Kinsman [1960; Longuet-Higgins, 1963]. It is found that the inequality on the skewness is satisfied in most cases of observed wave spectra.

Some observed distributions of wave height have been compared by Kinsman with a Gram-Charlier distribution based on the measured coefficients of skewness and kurtosis. As shown in figure 1, these are a better fit to the observations than the simple uncorrected gaussian distribution. Kinsman's suggested distribution differs from the form of the Gram-Charlier distribution found in section 2 of this paper, since it does not include a term in H_6. Nevertheless the difference resulting from the terms in H_4 and H_6 is so small in his measurements that the correction is essentially given by the second term λ_3H_3. Hence his figure can be used effectively as an assessment of the present theory.

If one attempts to carry the formal calculation of the moments to third and higher approximations in the application to water waves, the calculation breaks down. This is because of the occurrence of resonant interactions at the third approximation [Phillips, 1960; Hasselmann, 1962; and Benney, 1962], which render some of the third-order terms α_{ijk} in the series (2) slowly dependent on the time t. Thus the present method of calculation is consistent only as far as the second approximation in water waves.

In general, however, there seems no reason why the process should not be carried further and the higher-order terms in (2) be evaluated.

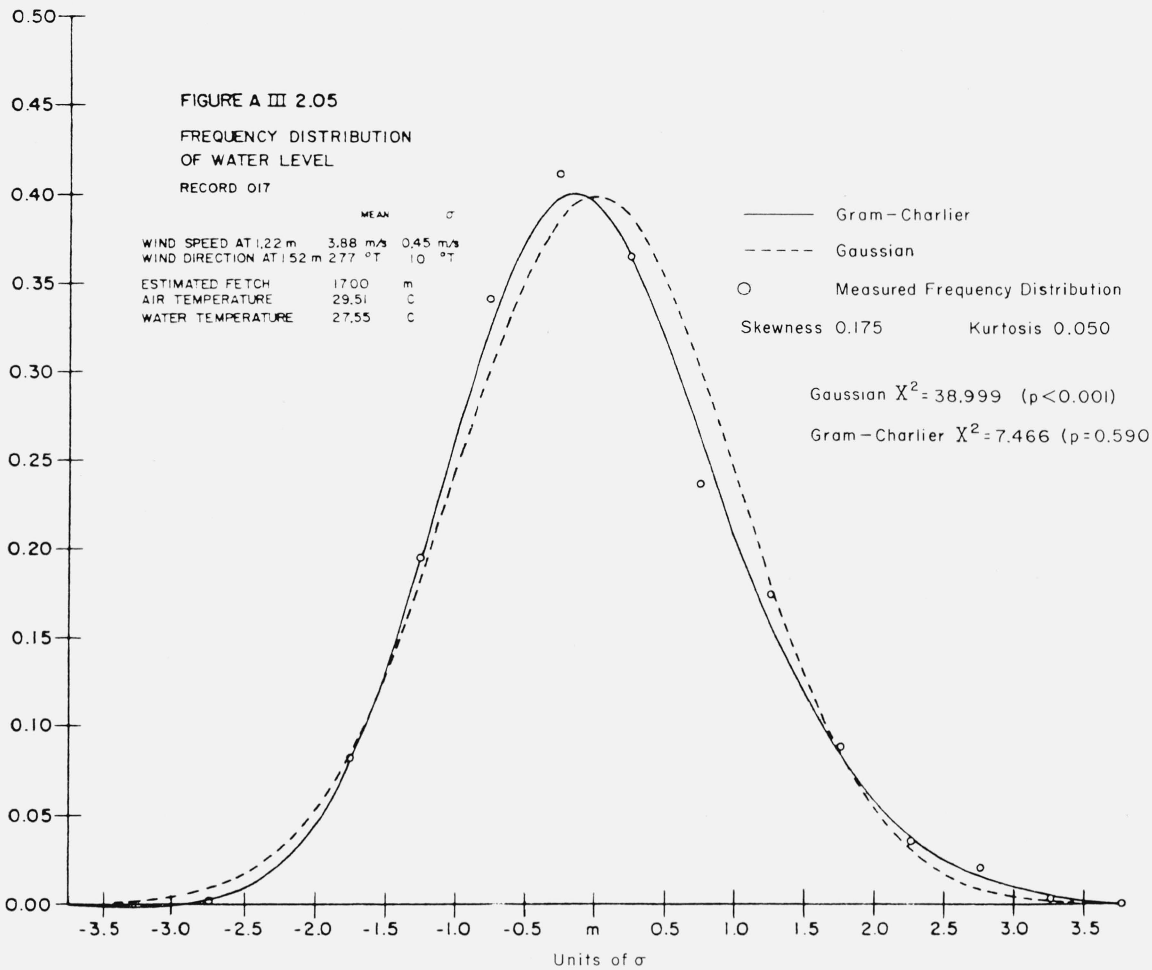

FIGURE 1. *From Kinsman 1960. Comparison of an observed distribution of wave height in sea waves with the normal distribution (broken line) and with a Gram-Charlier distribution (solid line).*

4. The Joint Distribution of Two Variables

The method of section 2 can be extended to the determination of the joint distribution of any number of variables of the type

$$\zeta=\alpha_i\xi_i+\alpha_{ij}\xi_i\xi_j+\alpha_{ijk}\xi_i\xi_j\xi_k+\ldots\quad \eta=\beta_i\xi_i+\beta_{ij}\xi_i\xi_j+\beta_{ijk}\xi_i\xi_j\xi_k+\ldots. \tag{23}$$

where the α's and β's are known constants and the ξ_i are random variables defined as before. For simplicity we state the results for two variables ζ, η.

Let us denote by A_p and B_q the terms $\alpha_{ij\ldots l}$ and $B_{ij\ldots m}$ where p and q are the numbers of suffices $i, j \ldots l$ and $i, j \ldots m$; and let $(A_pB_q \ldots)$ denote the irreducible part of the mean product

$$(A_p\xi_{i_1}\xi_{j_1}\ldots\xi_{l_1})(B_q\xi_{i_2}\xi_{j_2}\ldots\xi_{m_2})\ldots.$$

Then it can be shown [Longuet-Higgins, 1963] that the cumulants K_{mn} in the joint distribution of ζ and η are given by

$$K_{mn}=\sum_{p_1\ldots p_m q_1\ldots q_n}(A_{p_1}\ldots A_{p_m}B_{q_1}\ldots B_{q_n}) \tag{24}$$

the summation extending over all positive integral values of the p_i and q_j. In particular we have $K_{m0}=K_m$, the cumulant of $p(\zeta)$, as found in section 2, and as far as the terms in V^3 we have

$$K_{11}=(A_1B_1)+[(A_1B_3)+(A_2B_2)+(A_3B_1)]+[(A_1B_5)+(A_2B_4)+\ldots+(A_5B_1)],$$

$$K_{21}=[(A_1^2B_2)+2(A_1A_2B_1)]+[(A_1^2B_4)+2(A_1A_2B_3)+2(A_1A_3B_2)+(A_2^2B_2)+2(A_1A_4B_1)+2(A_2A_3B_1^2)],$$

$$K_{22}=[2(A_1^2B_1B_3)+(A_1^2B_2^2)+4(A_1A_2B_1B_2)+(A_2^2B_1^2)+2(A_1A_3B_1^2)],$$

$$K_{31}=[(A_1^3B_3+3(A_1^2A_2B_2)+3(A_1A_2^2B_1)].$$

The joint distribution of ζ, η is now given by

$$p(\zeta,\eta)=\frac{\exp\left[-\frac{1}{2}(f^2-2\rho ff'+f'^2)/(1-\rho^2\right]}{2\pi(K_{20}K_{02}-K_{11}^2)^{\frac{1}{2}}}\left[1+\frac{1}{6}(\lambda_{30}H_{30}+3\lambda_{21}H_{21}+3\lambda_{12}H_{12}+\lambda_{03}H_{03})+\ldots\right] \tag{25}$$

where

$$\left.\begin{aligned} f&=\frac{\zeta-K_{10}}{K_{20}^{1/2}}, & f'&=\frac{\eta-K_{01}}{K_{02}^{1/2}} \\ \lambda_{mn}&=\frac{K_{mn}}{(K_{20}^mK_{02}^n)^{\frac{1}{2}}}, & \rho&=\lambda_{11} \end{aligned}\right\} \tag{26}$$

and $H_{mn}(f,f';\rho)$ is a two-dimensional analog of the Hermite polynomial, viz.

$$H_{mn}\exp\left[-\tfrac{1}{2}(f^2-2\rho ff'+f'^2)/(1-\rho^2)\right]=(-1)^{m+n}\frac{\partial^{m+n}}{\partial f^m\partial f'^n}\exp\left[-\tfrac{1}{2}(f^2-2\rho ff'+f'^2)/(1-\rho^2)\right] \tag{27}$$

In particular

$$\left.\begin{aligned} H_{\infty}&=1\\ H_{10}&=(f-\rho f')/(1-\rho^2)\\ H_{01}&=(f'-\rho f)/(1-\rho^2)\\ H_{20}&=(f-\rho f')^2/(1-\rho^2)-1\\ H_{11}&=(f-\rho f')(f'-\rho f)/(1-\rho^2)+\rho\\ H_{02}&=(f'-\rho f)^2/(1-\rho^2)-1. \end{aligned}\right\} \tag{28}$$

The first approximation, in which λ_{mn} $(m+n>2)$ are neglected, is the familiar bivariate gaussian distribution, as we would expect. In the second approximation cubic polynomials in f, f' must be included, which introduce various types of skewness depending on the coefficients λ_{30}, λ_{21}, λ_{12}, and λ_{03}. In higher approximations we encounter further terms in a bivariate Gram-Charlier series.

In the paper [Longuet-Higgins, 1963] this distribution was applied to the joint distribution of the two components of slope of a random surface. In the present paper we shall apply it to two different problems: the mean number of zeros per unit time of a nonlinear random function $\zeta(t)$; and the distribution of the maximum values of $\zeta(t)$.

5. Mean Number Of Zeros Per Unit Time

We now consider the problem of determining the average number of zero-crossings (or of maxima and minima, which are zeros of the derivative) per unit time in a wave record. We assume that the process is stationary to all orders.

A general formula for the number of crossings of a level ζ per unit time t is due to Kac [1943].

$$N(\zeta)=\int_{-\infty}^{\infty} p(\zeta,\, \zeta_t)|\zeta_t|d\zeta_t, \tag{29}$$

where $p(\zeta,\, \zeta_t)$ denotes the joint distribution of ζ and its time-derivative ζ_t. Now for this distribution we have

$$K_{10}=\overline{\zeta}=0$$

$$K_{01}=\overline{\zeta_t}=0$$

and

$$K_{11}=\overline{\zeta\zeta_t}-\overline{\zeta}\,\overline{\zeta_t}=\overline{\frac{\partial}{\partial t}\frac{1}{2}(\zeta^2)}=0$$

for any stationary process ζ, so that $\rho=\lambda_{11}=0$ and from (25)

$$p(\zeta,\, \zeta_t)=\frac{e^{-\frac{1}{2}(f^2+f'^2)}}{2\pi(K_{20}K_{02})^{\frac{1}{2}}}\left[1+\frac{1}{6}\{\lambda_{30}H_3(f)H_0(f')+\ldots\}+\ldots\right] \tag{30}$$

where

$$f=\frac{\zeta}{(K_{20})^{\frac{1}{2}}}, \qquad f'=\frac{\zeta_t}{(K_{02})^{\frac{1}{2}}}.$$

Now since $e^{-\frac{1}{2}f'^2}|f'|$ is an even function of f', the integral of this function multiplied by $H_n(f')$ will vanish whenever n is odd. Moreover, when n is even and greater than 0

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}f'^2}|f'|H_n(f')df'=2\int_{-\infty}^{\infty} f'(-1)^n\frac{d^n}{df'^n}(e^{-\frac{1}{2}f'^2})df'$$

$$=2H_{n-2}(0)$$

on integration by parts. So on substitution in (29) we find

$$N(\zeta)=\frac{e^{-\frac{1}{2}f^2}}{\pi(K_{20}K_{02})^{\frac{1}{2}}}\left[1+\frac{1}{6}\{\lambda_{30}(f^3-3f)+3\lambda_{12}f\}+\ldots\right]. \tag{31}$$

Thus the number of zeros per unit time, considered as a function of ζ, has a skewness λ_{30} similar to that of $p(\zeta)$, but a mean value

$$\frac{1}{2}\lambda_{12}K_{20}^{1/2}=\frac{1}{2}K_{12}/K_{02}$$

different from the mean of ζ.

If we take $f=0$ in (31) then the third-order terms vanish and we have, to fourth order

$$N(0)=\frac{1}{\pi(K_{20}K_{02})^{\frac{1}{2}}}\left[1+\frac{1}{24}\{\lambda_{40}H_4(0)+6_{22}H_2(0)-\lambda_{04}H_0(0)\}\right]$$

$$=\frac{1}{\pi(K_{20}K_{02})^{\frac{1}{2}}}\left[1+\frac{1}{8}\lambda_{40}-\frac{1}{4}\lambda_{22}-\frac{1}{24}\lambda_{04}\right]. \tag{32}$$

6. Distribution of Heights of Maxima

The distribution $p(\zeta_{\max})$ of the heights of the maxima of ζ is given quite generally by

$$p(\zeta_{\max})=\int_{-\infty}^{0} p(\zeta,\, 0,\, \zeta_{tt})|\zeta_{tt}|d\zeta_{tt} \tag{33}$$

where $p(\zeta,\, \zeta_t,\, \zeta_{tt})$ denotes the joint distribution of ζ and its first two derivatives with respect to t [Rice, 1944–1945; Cartwright and Longuet-Higgins, 1956]. Now if K_{ijk} denotes the (i, j, k)th joint cumulant of ζ, ζ_t, ζ_{tt}, it is clear that K_{100}, K_{010}, K_{001} all vanish, as do K_{110} and K_{011}. Hence we have (as in sec. 3)

$$p(\zeta, \zeta_t, \zeta_{tt}) = \frac{1}{(2\pi)^3} \iiint \exp\Big[-i(\zeta t + \zeta_t t' + \zeta_{tt} t'') - \frac{1}{2}(K_{200}t^2 + K_{020}t'^2$$

$$+ K_{002}t''^2 + 2K_{101}tt'') + \frac{i^3}{6}(K_{300}t^3 + 3K_{210}t^2t' + \ldots) + \ldots\Big] dt\,dt'\,dt''.$$

Writing now

$$\left.\begin{array}{ll} t = s/K_{200}^{1/2}, & \zeta = K_{200}^{1/2} f, \\ t' = s'/K_{020}^{1/2}, & \zeta_t = K_{020}^{1/2} f', \\ t'' = s''/K_{002}^{1/2}, & \zeta_{tt} = K_{002}^{1/2} f'', \end{array}\right\}$$

we have

$$p(\zeta, \zeta_t, \zeta_{tt}) = \frac{1}{(2\pi)^3 (K_{200}K_{020}K_{002})^{1/2}} \iiint e^{-i(fs + f's' + f''s'')} \times \exp\Big[-\frac{1}{2}(s^2 + s'^2 + s''^2 + 2\rho ss'')$$

$$+ \frac{1}{6}\{\lambda_{300}(is)^3 + 3\lambda_{210}(is)^2(is') + \ldots\} + \ldots\Big] ds\,ds'\,ds''$$

where $\rho = K_{101}/(K_{200}K_{002})^{1/2}$. Now in

$$\frac{1}{(2\pi)^{3/2}} \iiint e^{-i(fs+f's'+f''s'') - \frac{1}{2}(s^2+s'^2+s''^2+2\rho ss'')} \times (is)^r (is')^{r'} (is'')^{r''} ds\,ds'\,ds''$$

the terms in s' are separable from the terms in s and s'' and hence as in sections 2 and 5 the above expression equals

$$\frac{1}{(1-\rho^2)^{1/2}} H_{r,r''}(f, f''; \rho) H_{r'}(f') e^{-\frac{1}{2}[f'^2 + (f^2 - 2\rho ff'' + f''^2)/(1-\rho^2)]}.$$

Hence we find

$$p(\zeta, \zeta_t, \zeta_{tt}) = \frac{e^{-\frac{1}{2}[f'^2 + (f^2 - 2\rho ff'' + f''^2)/(1-\rho^2)]}}{(2\pi)^{3/2}(K_{200}K_{020}K_{002})^{1/2}(1-\rho^2)^{1/2}}$$

$$\times\Big[1 + \frac{1}{6}\{\lambda_{300}H_{30}(f, f''; \rho)H_0(f') + 3\lambda_{210}H_{20}(f, f''; \rho)H_1(f) + \ldots\} + \ldots\Big]. \quad (34)$$

To find $p(\zeta_{\max})$ from (33) we set $\zeta_t = 0 = f'$, multiply by $|\zeta_{tt}|$ and integrate with respect to ζ_{tt} from $-\infty$ to 0. Writing

$$\left.\begin{array}{l} f = (1-\rho^2)^{1/2} x \\ f'' = -(1-\rho^2)^{1/2} y \end{array}\right\}$$

in the resulting integral we have

$$p(\zeta_{\max}) = \frac{K_{020}^{1/2}(1-\rho^2)^{1/2}}{(2\pi)^{3/2}(K_{200}K_{002})^{1/2}} \int_0^\infty y e^{-\frac{1}{2}(x^2 + 2\rho xy + y^2)}$$

$$\times\Big[1 + \frac{1}{6}\{(\lambda_{300}H_{30} + 3\lambda_{210}H_{21} + 3\lambda_{102}H_{12} + \lambda_{003}H_{03}) - 3\lambda_{120}H_{10} - 3\lambda_{021}H_{01}\} + \ldots\Big] dy \quad (35)$$

where

$$\left.\begin{array}{l} H_{10} = x + \rho y \\ H_{01} = \rho x + y \\ H_{20} = (x + \rho y)^2 - 1 \\ H_{11} = (x + \rho y)(\rho x + y) - \rho \\ H_{02} = (\rho x + y)^2 - 1 \end{array}\right\}$$

and generally

$$H_{m,n}e^{-\frac{1}{2}(x^2+2\rho xy+y^2)}=(-1)^{m+n}\frac{\partial^{m+n}}{\partial x^m\partial y^n}e^{-\frac{1}{2}(x^2+2\rho xy+y^2)} \tag{36}$$

To evaluate the integrals we use the following results. Straightforward integration by parts gives

$$\int_0^\infty yH_{m,n}e^{-\frac{1}{2}(x^2+2\rho xy+y^2)}dy=(H_{m,n-2})_{y=0}e^{-\frac{1}{2}x^2} \tag{37}$$

whenever $n\geq 2$. To deal with the cases $n=1, 0$, we note that

$$-\left(\frac{\partial}{\partial x}-\rho\frac{\partial}{\partial y}\right)e^{-\frac{1}{2}(x^2+2\rho xy+y^2)}=x(1-\rho^2)e^{-\frac{1}{2}(x^2+2\rho xy+y^2)}.$$

So by repeated application of the operator $-\left(\frac{\partial}{\partial x}-\rho\frac{\partial}{\partial y}\right)$:

$$H_{10}-\rho H_{01}=x(1-\rho^2)=G_1$$

$$H_{20}-2\rho H_{11}+\rho^2 H_{02}=x^2(1-\rho^2)^2-(1-\rho^2)=G_2$$

$$H_{30}-3\rho H_{21}+3\rho^2 H_{12}-\rho^3 H_{03}=x^3(1-\rho^2)^3-3x(1-\rho^2)^2=G_3$$

and in general

$$H_{n,0}-\binom{n}{1}\rho H_{n-1,1}+\ldots=(1-\rho^2)^{\frac{1}{2}n}H_n\{x(1-\rho^2)^{\frac{1}{2}}\}=G_n,$$

say. Thus in general,

$$\begin{aligned}\int_0^\infty yH_{n,1}e^{-\frac{1}{2}(x^2+2\rho xy+y^2)}dy &=\int_0^\infty H_{n,0}e^{-\frac{1}{2}(x^2+2\rho xy+y^2)}dy\\ &=\int_0^\infty\left[\binom{n}{1}\rho H_{n-1,1}-\binom{n}{2}\rho^2 H_{n-2,2}+\ldots+(-1)^{n-1}\rho^n H_{0,n}+G_n\right]\\ &\quad\times e^{-\frac{1}{2}(x^2+2\rho xy+y^2)}dy\\ &=\left[\binom{n}{1}\rho H_{n-1,0}-\binom{n}{2}\rho^2 H_{n-2,1}+\ldots+(-1)^{n-1}\rho^n H_{0,n-1}\right]_{y=0}\\ &\quad+G_nF(x;\rho)\end{aligned} \tag{38}$$

where

$$F(x;\rho)=\int_0^\infty e^{-\frac{1}{2}(x^2+2\rho xy+y^2)}dy=e^{-\frac{1}{2}x^2(1-\rho^2)}\int_{\rho x}^\infty e^{-\frac{1}{2}z^2}dz. \tag{39}$$

Similarly, when $m=0$,

$$\begin{aligned}\int_0^\infty yH_{n,0}e^{-\frac{1}{2}(x^2+2\rho xy+y^2)}dy &=\int_0^\infty y\left[\binom{n}{1}\rho H_{n-1,1}-\binom{n}{2}\rho^2 H_{n-2,2}+\ldots+G_n\right]e^{-\frac{1}{2}(x^2+2\rho xy+y^2)}dy\\ &=n\rho\left[\left\{\binom{n-1}{1}\rho H_{n-2,0}-\binom{n-1}{2}\rho^2 H_{n-3,1}+\ldots\right\}_{y=0}e^{-\frac{1}{2}x^2}+G_{n-1}F(x;\rho)\right]\\ &\quad+\left[-\binom{n}{2}\rho^2 H_{n-2,0}+\binom{n}{3}\rho^3 H_{n-3,1}-\ldots\right]_{y-0}e^{-\frac{1}{2}x^2}+G_n[e^{-\frac{1}{2}x^2}-\rho xF(x;\rho)]\\ &=\left[1\binom{n}{2}\rho^2 H_{n-2,0}-2\binom{n}{3}\rho^3 H_{n-3,1}+\ldots+G_n\right]e^{-\frac{1}{2}x^2}\\ &\qquad+\rho(nG_{n-1}-xG_n)F(x;\rho).\end{aligned} \tag{40}$$

Using formulas (7.5) to (7.8) we find altogether

$$
\begin{aligned}
p(\zeta_{\max}) = {} & \frac{[K_{020}(K_{200}K_{002}-K_{101}^2)^{1/2}]}{(2\pi)^{3/2}K_{200}K_{002}} \times \Big[e^{-\frac{1}{2}x^2} - \rho x F(x;\,\rho) \\
& + \frac{1}{6}\lambda_{300}\{((1-\rho^2)^3x^3-(3-9\rho^2+5\rho^4)x)\,e^{-\frac{1}{2}x^2}-(\rho(1-\rho^2)^3x^4-6(1-\rho^2)^2x^2+3\rho(1-\rho^2))\,F(x;\,\rho)\} \\
& + \frac{1}{2}\lambda_{201}\{\rho(2-\rho^2)xe^{-\frac{1}{2}x^2}+((1-\rho^2)^2x^2-(1-\rho^2))\,F(x;\,\rho)\} \\
& + \frac{1}{2}\lambda_{102}xe^{-\frac{1}{2}x^2}+\frac{1}{6}\lambda_{003}\rho xe^{-\frac{1}{2}x^2}-\frac{1}{2}\lambda_{120}\{(1-\rho^2)xe^{-\frac{1}{2}x^2}+(\rho-\rho x+\rho^3x)\,F(x;\,\rho)\}-\frac{1}{2}\lambda_{021}F(x;\,\rho)\Big]
\end{aligned}
\tag{41}
$$

where

$$
\left.
\begin{aligned}
x &= \frac{f}{(1-\rho^2)^{1/2}} = \frac{\zeta_{\max}}{[K_{200}(1-\rho^2)]^{1/2}} \\
\rho &= \frac{K_{101}}{(K_{200}K_{002})^{1/2}}
\end{aligned}
\right\}.
$$

The cumulants can be expressed in terms of the spectrum in the following way. If

$$
\left.
\begin{aligned}
\zeta &= \alpha_i\xi_i+\alpha_{ij}\xi_i\xi_j+\alpha_{ijk}\xi_i\xi_j\xi_k+\ldots \\
\zeta_t &= \beta_i\xi_i+\beta_{ij}\xi_i\xi_j+\alpha_{ijk}\xi_i\xi_j\xi_k+\ldots \\
\zeta_{tt} &= \gamma_i\xi_i+\gamma_{ij}\xi_i\xi_j+\gamma_{ijk}\xi_i\xi_j\xi_k+\ldots
\end{aligned}
\right\}
$$

then we may take

$$
\left.
\begin{aligned}
\alpha_i &= (1,1,\ldots 1;0,0,\ldots 0) \\
\beta_i &= (0,0,\ldots 0;-\sigma_1,-\sigma_2,\ldots -\sigma_N) \\
\gamma_i &= (\sigma_1^2,\sigma_2^2,\ldots \sigma_N^2;\,0,0,\ldots 0)
\end{aligned}
\right\}.
$$

Also if

$$
(\alpha_{ij}) = \begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1,N} & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
\alpha_{N,1} & \cdots & \alpha_{N,N} & 0 & \cdots & 0 \\
0 & \cdots & 0 & \alpha_{N+1,N+1} & \cdots & \alpha_{N+1,2N} \\
\vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & \alpha_{2N} & \cdots & \alpha_{2N,2N}
\end{pmatrix}
$$

then

$$
(\beta_{ij})=\begin{pmatrix}
0\cdot & \cdots & \cdot 0 & (-\sigma, \alpha_{N+1,N+1}+\sigma_1\alpha_{11}) & \cdots & (-\sigma_1\alpha_{N+1,2N}+\sigma_N\alpha_{1,N}) \\
\vdots & & \vdots & \vdots & & \\
0\cdot & \cdots & \cdot 0 & (-\sigma_N\alpha_{2N,N+1}+\sigma_1\alpha_{N,1}) & \cdots & (-\sigma_N\alpha_{2N,2N}+\sigma_N\alpha_{NN}) \\
(\sigma_1\alpha_{11}-\sigma_1\alpha_{N+1,N+1}) & \cdots & (\sigma_1\alpha_{1N}-\sigma_N\alpha_{N+1,2N}) & 0\cdot & \cdots & \cdot 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
(\sigma_N\alpha_{N,1}-\sigma_1\alpha_{2N,N+1}) & \cdots & (\sigma_N\alpha_{NN}-\sigma_N\alpha_{2N,2N}) & 0\cdot & \cdots & \cdot 0
\end{pmatrix}
$$

and

$$
(\gamma_{ij})=\begin{pmatrix}\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}\end{pmatrix}
$$

where

$$
\mathbf{C}=\begin{pmatrix}
(-2\sigma_1^2\alpha_{11}+2\sigma_1^2\alpha_{N+1,N+1}) & \cdots\cdots & [-(\sigma_1^2+\sigma_N^2)\alpha_{1,N}+2\sigma_1\sigma_N\alpha_{N+1,\ 2N}] \\
\vdots & & \vdots \\
\vdots & & \vdots
\end{pmatrix}
$$

and in $\mathbf{D}$ the suffices of α_{ij} are each increased or decreased by N. The nonvanishing cumulants can now be expressed as follows:

$$
\left.\begin{aligned}
K_{200}&\doteqdot(A_1^2)=\alpha_i\alpha_iV_i=\int E d\sigma=m_0 \\
K_{020}&\doteqdot(B_1^2)=\beta_i\beta_iV_i=\int\sigma^2 E d\sigma=m_2 \\
K_{002}&\doteqdot(C_1^2)=\gamma_i\gamma_iV_i=\int\sigma^4 E d\sigma=m_4 \\
K_{101}&\doteqdot(A_1C_1)=\alpha_i\gamma_iV_i=-\int\sigma^2 E d\sigma=-m_2
\end{aligned}\right\}.
$$

Also

$$
K_{300}\doteqdot 3(A_1^2A_2)=6\alpha_i\alpha_j\alpha_{ij}V_iV_j=6\iint\alpha(\sigma,\sigma')EE'd\sigma d\sigma'
$$

$$
K_{030}\doteqdot 3(B_1^2B_2)=6\beta_i\beta_j\beta_{ij}V_iV_j=0
$$

$$
K_{003}\doteqdot 3(C_1^2C_2)=6\gamma_i\gamma_j\gamma_{ij}V_iV_j=6\iint\sigma^2\sigma'^2\{-(\sigma^2+\sigma'^2)\alpha(\sigma,\sigma')+2\sigma\sigma'\alpha^*(\sigma,\sigma')\}EE'd\sigma d\sigma'
$$

and

$$
K_{201}\doteqdot(A_1^2C_2)=2\alpha_i\alpha_j\gamma_{ij}V_iV_j=2\iint\{-(\sigma^2+\sigma'^2)\alpha(\sigma,\sigma')-2\sigma\sigma'\alpha^*(\sigma,\sigma')\}EE'd\sigma d\sigma'
$$

$$
K_{102}\doteqdot(C_1^2A_2)=2\gamma_i\gamma_j\alpha_{ij}V_iV_j=2\iint\sigma^2\sigma'^2\alpha(\sigma,\sigma')EE'd\sigma d\sigma'
$$

$$K_{120} \doteq (B_1^2 A_2) = 2\beta_i \beta_j \alpha_{ij} V_i V_j = 2 \iint \sigma^2 \alpha'^2 \alpha^*(\sigma, \sigma') EE' d\sigma d\sigma'$$

$$K_{021} \doteq (B_1^2 C_2) = 2\beta_i \beta_j \gamma_{ij} V_i V_j = 2 \iint \sigma\sigma' \{-(\sigma^2 + \sigma'^2)\alpha^*(\sigma, \sigma') + 2\sigma\sigma' \alpha(\sigma, \sigma')\} EE' d\sigma d\sigma'.$$

One can if necessary retain the next term in K_{200}, K_{020}, K_{002}, and K_{101}; thus

$$K_{200} \doteq (A_1^2) + (A_2^2) = \alpha_i \alpha_i V_i + 2\alpha_{ij}\alpha_{ij} V_i V_j = \int E d\sigma + 2 \iint [\alpha(\sigma, \sigma')]^2 EE' d\sigma d\sigma',$$

with similar expressions for K_{020} and K_{002}. (Note that $K_{101} = -K_{020}$ to all orders).

In the first approximation (7.9) gives

$$p(\zeta_{\max}) = \frac{m_2(m_0 m_4 - m_2^2)^{\frac{1}{2}}}{(2\pi)^{3/2} m_0 m_4} [e^{-\frac{1}{2}x^2} - \rho x F(x;\rho)]$$

where

$$x = f/(1-\rho^2)^{\frac{1}{2}}, \qquad \rho = -m_2/(m_0 m_4)^{\frac{1}{2}}$$

This is the distribution obtained by Rice [1944–1945] and studied by Cartwright and Longuet-Higgins [1956].

The remaining terms in (7.9) represent the corrections to this distribution, which are order $V^{1/2}$.

7. References

Benney, D. J. (1962), Nonlinear gravity wave interactions, J. Fluid Mech. **14,** 577–584.

Cartwright, D. E., and M. S. Longuet-Higgins (1956), The statistical distribution of the maxima of a random function, Proc. Roy. Soc. A, **237,** 212–232.

Edgeworth, F. Y. (1906), The generalized law of error, or law of great numbers, J. Roy. Statist. Soc. **69,** 497–530.

Hasselmann, K. (1960), Grundgleichungen der Seegangsvoraussage, Schiffstechnik **7,** 191–195.

Hasselmann, K. (1961), On the nonlinear energy transfer in a wave spectrum, Proc. Conference on Ocean Wave Spectra, Easton, Md.

Hasselmann, K. (1962), On the nonlinear energy transfer in a gravity-wave spectrum. Part I. General theory, J. Fluid Mech. **12,** 481–500.

Kac, M. (1943), On the average number of real roots of a random algebraic equation, Bull. Am. Math. Soc. **49,** 314–320.

Kinsman, B. (1960), Surface waves at short fetches and low wind speed—a field study. Chesapeake Bay Inst. Tech. Report 19.

Longuet-Higgins, M. S. (1960), The statistical geometry of random surfaces, Proc. Symposium Applied Math., Am. Math. Soc. **13,** 105–143.

Longuet-Higgins, M. S. (1963), The effect of nonlinearities on statistical distributions in the theory of sea waves, J. Fluid Mech. **17,** 459–480.

Phillips, O. M. (1960), On the dynamics of unsteady gravity waves of finite amplitude. Part 1. The elementary interactions. J. Fluid Mech. **9,** 193–217.

Phillips, O. M. (1961), On the dynamics of unsteady gravity waves of finite amplitude. Part 2. Local properties of a random wave field, J. Fluid Mech. **11,** 143–155.

Rice, S. O. (1944), The mathematical analysis of random noise, Bell System Tech. J. **23,** 282–332.

Rice, S. O. (1945), The mathematical analysis of random noise, Bell System Tech. J. **24,** 46–156.

Wiener, N. (1958), Nonlinear problems in random theory, p. 131 (Chapman and Hall, London).

(Paper 68D9–406)