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Propagation of Plasma Waves in a "Spoke-Wheel" Magnetic Field

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A study of the cold plasma cylindrical waves that may propagate in a specific type of two-dimensional magnetic field is initiated in this paper. The plasma is assumed to be of uniform density and collisionless, and a "spoke-wheel" magnetic field is considered which is both anisotropic and inhomogeneous (varying as the inverse radius), as defined in the text. Perturbation series solutions are obtained for the first Fourier component of the electric field for the four extreme cases: large and small magnetic field; large and small plasma densities.

1. Introduction

Interest in the analysis of wave propagation through magneto-active plasmas has, in the recent past, received renewed stimulation due to the emergence of the problem of communication "black out" to or from reentry vehicles.

With regard to this problem, in the present paper, we shall initiate a study of the cylndrical cold plasma [Allis, 1959; Auer et al., 1958; Mason and Gold, 1962; Gold, 1963] waves that may propagate in a specific type of two-dimensional inhomogeneous magnetic field. The plasma is assumed to be of uniform density, and collisions are neglected. A so-called "spoke-wheel" magnetic field is considered which consists of two orthogonal (r, θ) components. The circular $\hat{\boldsymbol{\theta}}$ component [2] is the field generated by a straight wire, while the remaining $\hat{\mathbf{r}}$ component may may be approximated by the field generated at the end of a flattened solenoid of infinite radius.

Both the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ fields decay as the inverse radius. The problems we shall investigate pertain to a specific (i.e., only r dependent, for the first Fourier component) class of waves which are sustained in a plasma that is situated in the region of two-dimensional symmetry of these fields. The related field lines are depicted in figure 1 (constant z).

The r^{-1} dependence of the B_θ field generated by a straight wire is classical. The similar dependence of the B_r field follows from simple geometrical consideration. If there are N B_r lines issuing from the source of radius r_0, so that the B field there is $\alpha N/2\pi r_0$ (where α is a constant), then at the radius r, the value of the field is $\alpha N/2\pi r$, since the same number of lines cross all circles.

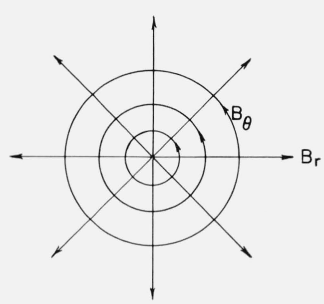

FIGURE 1. *Spoke-wheel magnetic field.*

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[2] The roof notation denotes a unit vector.

Similar studies of guided cylindrical waves in a cold plasma may be found in the literature [Åström, 1950; Buchsbaum et al., 1960]. However, for the most part, the steady B field in these investigations is in the z-direction and is, of course, constant in space. The striking dissimilarity between such a B_z field and the $B_{r,\theta}$ fields herein stated is that in the former case the components of the related electric fields are decoupled, while in the latter case, even for constant-in-space $B_{r,\theta}$ fields, the $E_{r,\theta,z}$ fields are severely coupled. A class of r-dependent solutions is examined for the magnetic field $(B_r,0,0)$. In this case, the E_r field is completely decoupled from $E_{\theta,z}$ and is nontrivial only if the signal frequency is equal to the plasma frequency. The $(0,B_\theta,0)$ problem is not of current interest although it is amenable to a similar analysis.

In this introductory discourse, we shall generate perturbation series for the electric fields. The two physical variables at our disposal are the plasma density and the steady magnetic field intensity. Of the four extremes that are naturally suggested, only the one of large plasma density prohibits perturbation analysis, since in that case the zero-order solution (infinite plasma density) yields zero electric field. In the remaining cases, quadratures are obtained for the first-order terms in the perturbation parameter.

2. Analysis

2.1. Equations

The defining characteristic of a cold plasma is that the constitutive particles suffer no spread in velocity space, viz, the distribution function $f(\boldsymbol{\xi})$ is a delta function $\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0)$. We shall consider also a two-component plasma whose ion mass far exceeds the electron mass so that in any subsequent perturbation from equilibrium, only the electronic motion contributes to the current $\mathbf{J}$. The related wave equation for the electric field $\mathbf{E}$ (MKS units) is,

$$(\omega^2-\omega_0^2A^{-1}-c^2\nabla\times\nabla\times)\mathbf{E}=0 \tag{1}$$

where ω is the mode frequency, ω_0 is the plasma frequency, and c is the speed of light.

Let us consider the operator A in more detail. Two cases follow:

$$A_r\mathbf{J}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -p_r\\ 0 & p_r & 1\end{pmatrix}\begin{pmatrix}J_r\\ J_\theta\\ J_z\end{pmatrix},$$

where we have set

$$p_r\equiv\frac{1}{K_r r},\qquad p_\theta\equiv\frac{1}{K_\theta r}\equiv 0.$$

If $\boldsymbol{\Omega}$ is the magentic Larmor frequency $[e\mathbf{B}/m]$, and $\mathbf{b}$ is a unit vector $\mathbf{B}/B$, then K is the magentic wave number, defined through

$$\frac{i\boldsymbol{\Omega}}{\omega}=\frac{\mathbf{b}}{Kr}.$$

In similar manner, there follows for A_θ

$$A_\theta\mathbf{J}=\begin{pmatrix}1 & 0 & p_\theta\\ 0 & 1 & 0\\ -p_\theta & 0 & 1\end{pmatrix}\begin{pmatrix}J_r\\ J_\theta\\ J_z\end{pmatrix}.$$

For the inverses we obtain

$$A_r^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{1+p_r^2} & \dfrac{p_r}{1+p_r^2} \\ 0 & \dfrac{-p_r}{1+p_r^2} & \dfrac{1}{1+p_r^2} \end{bmatrix},$$

$$A_\theta^{-1}=\begin{bmatrix} \dfrac{1}{1+p_\theta^2} & 0 & \dfrac{-p_\theta}{1+p_\theta^2} \\ 0 & 1 & 0 \\ \dfrac{p_\theta}{1+p_\theta^2} & 0 & \dfrac{1}{1+p_\theta^2} \end{bmatrix}.$$

Returning to the general analysis, we may write (1) in component form. The θ dependence of the solution is absorbed into the circular harmonic form,

$$\mathbf{E}=\sum_{l=0}^{\infty} \mathbf{E}^l(r)e^{\pm il\theta}.$$

For case 1 ($p=1/K_r r$), (1) appears as

$$\left(\omega^2-\omega_0^2-\frac{l^2c^2}{r^2}\right)E_r-(\pm il)\frac{c^2}{r^2}\frac{\partial}{\partial r}(rE_\theta)=0,$$

$$\left[\left(\omega^2-\frac{\omega_0^2}{1+p^2}\right)+c^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}r\right)\right]E_\theta-\frac{\omega_0^2 p}{1+p^2}E_z-c^2(\pm il)\frac{\partial}{\partial r}\left(\frac{E_r}{r}\right)=0,$$

$$\left[-\frac{l^2c^2}{r^2}+\omega^2-\frac{\omega_0^2}{1+p^2}+\frac{c^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]E_z+\frac{\omega_0^2 p}{1+p^2}E_\theta=0,$$

where $E_{r,\theta}$ has been written for $E^l_{r,\theta}$. The equations constituting case 2 ($p\equiv 1/K_\theta r$) are similar in form but will not be considered in detail at the present time.

2.2. Solution

In this analysis, we will restrict our considerations to the $l=0$ mode. In this case, the E_r field is completely decoupled from $E_{\theta,z}$ and is nontrivial only for $\omega=\omega_0$. The remaining two equations for $E_{\theta,z}$ appear as:

$$\left[1+\frac{1}{x^2}L_1(x)\right]E_\theta=\alpha^2\left(\frac{y^2}{1+y^2}\right)\left(E_\theta+\frac{1}{y}E_z\right)$$

$$\left[1+\frac{1}{x^2}L_0(x)\right]E_z=\alpha^2\left(\frac{y^2}{1+y^2}\right)\left(E_z-\frac{1}{y}E_\theta\right) \qquad (2)$$

where we have set

$$\alpha^2=\frac{\omega_0^2}{\omega^2};\ y=Kr=p^{-1};\ x=k_0r;\ k_0=\frac{\omega}{c}.$$

The L operators are

$$L_n\equiv x^2\frac{d^2}{dx^2}+x\frac{d}{dx}-n^2,$$

so that

$$(L_n+x^2)[AJ_n(x)+BY_n(x)]=0,$$

where J_n and Y_n are nth order Bessel functions of the first and second kinds, respectively.

We will now consider four classes of perturbation solutions. The parameter of smallness for the four cases is defined in the following table:

(a) $y=\frac{K}{k_0}x=\epsilon x$:	Large B_0 field ($k_0>>K$)
(b) $y=\frac{K}{k_0}x=\epsilon^{-1}x$:	Small B_0 field ($k_0<<K$)
(c) $\alpha^2=\epsilon$:	Rare plasma ($\omega_0<<\omega$)
(d) $\alpha^{-2}=\epsilon$:	Dense plasma ($\omega_0>>\omega$).

Case a, large B_0 field. In all of the following cases, the fields are expanded in powers of ϵ:

$$\mathbf{E}=\sum_{n=0}^{\infty}\mathbf{E}^{(n)}\epsilon^n.$$

In addition, all functions of ϵ in the set (2) are expanded about $\epsilon=0$. For the case under consideration, there results ($y=\epsilon x$),

$$(x^2+L_1)\sum E_\theta^{(n)}\epsilon^n=\alpha^2\epsilon^2x^4\left(\sum(-)^n\epsilon^{2n}x^{2n}\right)\left(\sum E_\theta^{(n)}\epsilon^n+\frac{1}{\epsilon x}\sum E_z^{(n)}\epsilon^n\right),$$

$$(x^2+L_0)\sum E_z^{(n)}\epsilon^n=\alpha^2\epsilon^2x^4\left(\sum(-)^n\epsilon^{2n}x^{2n}\right)\left(\sum E_z^{(n)}\epsilon^n-\frac{1}{\epsilon x}\sum E_\theta^{(n)}\epsilon^n\right).$$

The recursive equations for $[E_z^{(n)};\ E_\theta^{(n)}]$ are obtained by equating the coefficients of equal powers of ϵ. The first two of these equations appear as (assuming that α is of order one)

$$(x^2+L_1)E_\theta^{(0)}=0;\ (x^2+L_0)E_z^{(0)}=0; \tag{3a, b}$$

$$(x^2+L_1)E_\theta^{(1)}=\alpha^2x^3E_z^{(0)};\ (x^2+L_0)E_z^{(1)}=-\alpha^2x^3E_\theta^{(0)}. \tag{4a, b}$$

These two equations determine the solution to $0(\epsilon)$, which, in turn, appears as

$$\mathbf{E}=\mathbf{E}^{(0)}+\epsilon\mathbf{E}^{(1)};\ \epsilon=\frac{K}{k_0}=\frac{c}{i\Omega r};\ \Omega=e\frac{B_0}{m};\ B_0\sim r^{-1}.$$

The zeroth order solutions are the free space solutions, or, more accurately, the infinite magnetic field solutions. The simulation of the free space case is due to the rigid ordering of the particles by the B field, so that it becomes impossible for the electric fields to do work on them. For $\mathbf{E}^{(0)}$, we find,

$$\left.\begin{aligned}E_z^{(0)}&=AJ_0(x)+BY_0(x)\equiv j_0\\E_\theta^{(0)}&=CJ_1(x)+DY_1(x)\equiv j_1\end{aligned}\right\} \tag{5}$$

where we recall $x=k_0r$, and J_n and Y_n are Bessel functions.

The next order terms are composed of two parts: a particular integral and a complementary function. The complementary functions for (4a) through (4b) are all the same as the solution to the zeroth order (3). A particular solution of (4b) is obtained by setting

$$E_z^{(1)}=\mathcal{E}_z j_0.$$

There results the following first-order equation for $\mathcal{E}_z'$:

$$\mathcal{E}_z''+\mathcal{E}_z'\frac{d}{dx}[\ln(xj_0^2)]=-\frac{\alpha^2j_1x}{j_0},$$

where a prime denotes a derivative with respect to x. Multiplying through by exp $[\ln(xj_0^2)]=xj_0^2$ gives

$$\frac{d}{dx}(\mathcal{E}_z' x j_0^2)=-\alpha^2 x^2 j_0 j_1=-\frac{\alpha^2}{2}\frac{d}{dx}(xj_1)^2, \tag{6}$$

the last equality following if the constants of j_0 are set equal to those of j_1 since $(d/dx)(xj_1)\equiv xj_0$. Integrating (6) twice gives the desired result in quadrature form:

$$\mathcal{E}_z(x)=-\frac{\alpha^2}{2}\int_{x_0}^{x}\frac{tj_1^2(t)}{j_0^2(t)}\,dt=-\frac{\alpha^2}{2}\left[\frac{tj_1(t)}{j_0(t)}-\frac{t^2}{2}\right]_{x_0}^{x}. \tag{7a}$$

In a similar manner, we obtain for $\mathcal{E}_\theta$ (where $E_\theta^{(1)}=\mathcal{E}_\theta j_1$),

$$\mathcal{E}_\theta=\frac{\alpha^2 x^2}{4}. \tag{7b}$$

To order ϵ the fields now appear as:

$$E_z=E_z^{(0)}(1+\epsilon\mathcal{E}_z),$$

$$E_\theta=E_\theta^{(0)}+\epsilon\tilde{E}_\theta^{(0)}\mathcal{E}_\theta$$

with the zero order solutions given by (5) and the functions $\mathcal{E}(x)$ given by (7). It should be noted first that the solutions to the first-order homogeneous equations have been absorbed in the unity factor, and second that the constants of $E_z^{(0)}$ are not the same as those of $E_\theta^{(0)}$; however, $A=C$, $B=D$ for the $0(\epsilon)$ solutions. Reference to (5) indicates that four arbitrary constants remain, two residing in E_z and two in E_θ.

Case b, small B_0 field. This case is characterized by setting

$$\frac{k_0}{K}=\epsilon.$$

Equations (2) then appear as

$$(x^2+L_1)E_\theta=x^2\alpha^2\left[\frac{1}{1+(\epsilon^2/x^2)}\right]\left(E_\theta+\frac{\epsilon}{x}E_z\right),$$

$$(x^2+L_0)E_z=x^2\alpha^2\left[\frac{1}{1+(\epsilon^2/x^2)}\right]\left(E_z-\frac{\epsilon}{x}E_\theta\right).$$

Expansion of the relevant functions about $\epsilon=0$ and ordering, results in a sequence of coupled equations for the field components $\mathbf{E}^{(n)}$. The first two equations appear as [for $\alpha=0(1)$]

$$[L_1+(1-\alpha^2)x^2]E_\theta^{(0)}=0;\ [L_0+(1-\alpha^2)x^2]E_z^{(0)}=0;$$

$$[L_1+(1-\alpha^2)x^2]E_\theta^{(1)}=\alpha^2 xE_z^{(0)};\ [L_0+(1-\alpha^2)x^2]E_z^{(1)}=-\alpha^2 xE_\theta^{(0)}.$$

The solutions to the zeroth order equations are given by

$$E_\theta^{(0)}=AJ_1(\alpha' x)+BY_1(\alpha' x)\equiv j_1;\ E_z^{(0)}=CJ_0(\alpha' x)+DY_0(\alpha' x)\equiv j_0,$$

where

$$\alpha'^2=1-\alpha^2=1-\frac{\omega_0^2}{\omega^2}.$$

The next order solutions are obtained in a manner exactly the same as for case a; viz, we set

$$E_z^{(1)}=j_0\mathcal{E}_z;\ E_\theta^{(1)}=j_1\mathcal{E}_\theta.$$

There results:

$$\mathcal{E}_z=\frac{\alpha^2}{2\alpha'}\ln x;\ \mathcal{E}_\theta=-\frac{\alpha^2}{2\alpha'}\int_{x_0}^{x}\frac{j_0^2(\alpha' t)}{tj_1^2(\alpha' t)}\,dt=\frac{\alpha^2}{2\alpha'}\left[\ln(\alpha t)+\frac{j_0(\alpha' t)}{\alpha' t j_1(\alpha' t)}\right]_{x_0}^{x}$$

using the identity $(d/dx)[j_0(\alpha' x)] \equiv -\alpha' j_1(\alpha' x)$ and requiring that $A=C$, $B=D$. To within the stated order, the solution now appears as

$$E_z = E_z^{(0)}(1+\epsilon\mathcal{E}_z),$$

$$E_\theta = E_\theta^{(0)} + \epsilon \tilde{E}_\theta^{(0)} \mathcal{E}_\theta.$$

Case c, rare plasma. Here, we set

$$\left(\frac{\omega_0}{\omega}\right)^2 \equiv \alpha^2 = \epsilon.$$

In expanded form, (2) now appear as

$$(x^2+L_1)\sum E_\theta^{(n)}\epsilon^n = \epsilon\left(\frac{y^2x^2}{1+y^2}\right)\left(\sum E_\theta^{(n)}\epsilon^n + y^{-1}\sum E_\theta^{(n)}\epsilon^n\right),$$

$$(x^2+L_0)\sum E_z^{(n)}\epsilon^n = \epsilon\left(\frac{y^2x^2}{1+y^2}\right)\left(\sum E_z^{(n)}\epsilon^n - y^{-1}\sum E_\theta^{(n)}\epsilon^n\right).$$

For this case, it is particularly simple to write the general nth order recursive equations for $\boldsymbol{E}$, which appear as [provided $K/k_0 = 0(1)$]:

$$(x^2+L_1)E_\theta^{(n+1)} = \left(\frac{y^2x^2}{1+y^2}\right)\left(E_\theta^{(n)} + y^{-1}E_z^{(n)}\right),$$

$$(x^2+L_0)E_z^{(n+1)} = \left(\frac{y^2x^2}{1+y^2}\right)\left(E_z^{(n)} - y^{-1}E_\theta^{(n)}\right).$$

In exactly the same manner as with the previous two cases, we obtain for $\boldsymbol{E}$, to $0(\epsilon)$

$$E_\theta = E_\theta^{(0)}(1+\epsilon\mathcal{E}_\theta),$$

$$E_z = E_z^{(0)} + \epsilon \tilde{E}_z^{(0)} \mathcal{E}_z$$

where

$$E_\theta^{(0)} = j_1(x);\ E_z^{(0)} = j_0(x),$$

and the first-order contributions are the quadratures,

$$\mathcal{E}_\theta = \int_{x_0}^{x} \frac{dt}{t j_1^2(t)} \int_{t_0}^{t} \frac{\lambda^2 z^3 j_1^2(z)}{1+\lambda^2 z^2}\left[1+\frac{j_0(z)}{\lambda z j_1(z)}\right] dz,$$

$$\mathcal{E}_z = \int_{x_0}^{x} \frac{dt}{t j_0^2(t)} \int_{t_0}^{t} \frac{\lambda^2 z^3 j_0^2(z)}{1+\lambda^2 z^2}\left[1-\frac{j_1(z)}{\lambda z j_0(z)}\right] dz,$$

where we have set $\lambda \equiv K/k_0$.

Case d, dense plasma. This final case is characterized by the statment

$$\epsilon = \frac{1}{\alpha^2}.$$

It is easily seen that this equality renders the perturbation series

$$\mathbf{E} = \mathbf{E}^{(0)} + \epsilon\mathbf{E}^{(1)} + \ldots$$

degenerate if λ is of order one, inasmuch as the starting term vanishes, generating, in turn, a null series. The physical significance is, of course, that electromagnetic waves cannot pass through an infinitely dense plasma ($\epsilon = 0$).

3. Conclusions

We have presented an introductory discussion on the class of electromagnetic waves which will pass through a specific type of cylindrical magnetic field embedded in a cold collisionless plasma of uniform density. The magnetic field is both anisotropic and inhomogeneous (varying as the inverse radius). The spoke-wheel magnetic field is defined in the text.

The most significant aspect of the analysis is the presentation of a formalism (i.e., well-known Fourier expansion in the θ variation together with subsequent perturbation expansion of the Fourier components) that permits the analytic investigation of these propagating fields. Four distinct cases were examined for the lowest order Fourier components in the four extremes: large and small magnetic field; large and small plasma densities. These in turn were examined to terms of first order in the parameter of smallness. The results are given in quadrature form. Well-known facts, such as infinite magnetic field being equivalent to vanishing plasma density, are recaptured in the lowest order expressions. The analysis differs from previous similar studies of guided electromagnetic waves through cylindrical plasmas with embedded B_z fields insofar as the electric field components in the present work are severely coupled while in the B_z case they are not.

It remains to fit these solutions to specific geometries. The most physically relevant of these is the class of annular geometries with the steady B_0 field zero in the region about the origin, and of spoke-wheel type exterior to this circular region. The plasma density is either zero or nonzero (finite) in the various annular regions.

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