Admittance of Annular Slot Antennas Radiating Into a Plasma Layer

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The slot admittance is calculated for a slot radiating into a waveguide of radius 5 to 10λ. The radius of the wide waveguide is selected so that the slot admittance is equal to the admittance for a semi-infinite half space in absence of plasma. This slot admittance is an approximation to the admittance of an inhomogeneous half space in presence of a plasma layer. The calculated slot susceptance is shown to be nearly constant for various thicknesses of the plasma layer, while the slot conductance and radiation conductances are decreased by increasing plasma thickness. This approximate calculation appears to be particularly good for thin layers of electrically contrasting plasma ($\epsilon_p/\epsilon_0 \neq 1$ and/or $\tan \delta_p \gg 1$). The calculations indicate an admittance similar to that of earlier calculations for a rectangular slot radiating into a wide rectangular waveguide.

1. Introduction

The admittance of annular slots which are backed by coaxial lines or cavities and which radiate into free space have been discussed by Levine and Papas [1951]; Wait [1958]; Cohn and Flesher [1958], and Galejs and Thompson [1962]. The same formulation applies also to slots radiating into a half space filled with a homogeneous dielectric or plasma. There are difficulties in treating the inhomogeneous half space or dielectric and plasma layers of finite thicknesses, although such an analysis has been carried out by Galejs [1963a] for a rectangular slot covered by finite plasma layers. The calculations of the slot admittances are greatly simplified if the slot is assumed to radiate into a wide waveguide instead of an inhomogeneous half space [Galejs, 1963b]. A waveguide of a finite diameter is characterized by discrete modes which can be readily treated in machine calculations.

Radiation from a coaxial line into a waveguide of infinite diameter has been discussed by Cohn and Flesher [1958]. Their results agree with an analytical formulation not employing such artificial boundaries [Levine and Papas, 1951]. In the present note a waveguide of finite diameter will be used and it will be shown that the admittance of this waveguide approximates the admittance for an inhomogeneous half space. This was also the case for rectangular slots provided the guide diameter exceeded 5 to 10λ [Galejs, 1963b]. A similar behavior can be also expected for annular slots radiating into a circular guide because the fields in the vicinity of the slot should not be affected by the presence of the waveguide walls if the distance between the walls is large enough. Hence, this geometry will yield realistic susceptances and also realistic estimates of losses in

the plasma layer. However, the radiation fields of the slot will be modified by the waveguide walls. The radion patterns of the slot cannot be determined in this model and the effects of guide diameter on the radiation conductance must be also examined in detail.

The analytical formulation of the admittance for a cavity backed slot can be considered as straightforward, and only the principal steps of the development are summarized in section 2. Convergence considerations and the numerical results will be discussed in section 3.

2. Slot Admittance

The waveguide and slot geometry is shown in figure 1. The same basic geometry has been considered by Galejs and Thompson [1962], and the configuration of figure 1 may be treated by a suitable modification of this work. For an $\exp(j\omega t)$ time variation of the ϕ symmetrical fields the magnetic field components $H_\phi(\rho, z)$ are given by

$$H_{\phi i} = \sum_{m=1}^{\infty} A_m R_m(\rho)(e^{-\gamma_m z} + B_m e^{\gamma_m z}) \qquad (1)$$

$$H_{\phi j} = \sum_{n=1}^{\infty} A_{nj} R_n(\rho)(e^{-\gamma_{nj} z} + B_{nj} e^{\gamma_{nj} z}) \qquad (2)$$

where $j = 0$, 1 or 2 and where $B_{nj} = 0$. The orthonormal radial functions $R_p(\rho)$ are defined as

$$R_p(\rho) = N_p J_1(\lambda_p \rho) \qquad (3)$$

where $p = m$ or n and
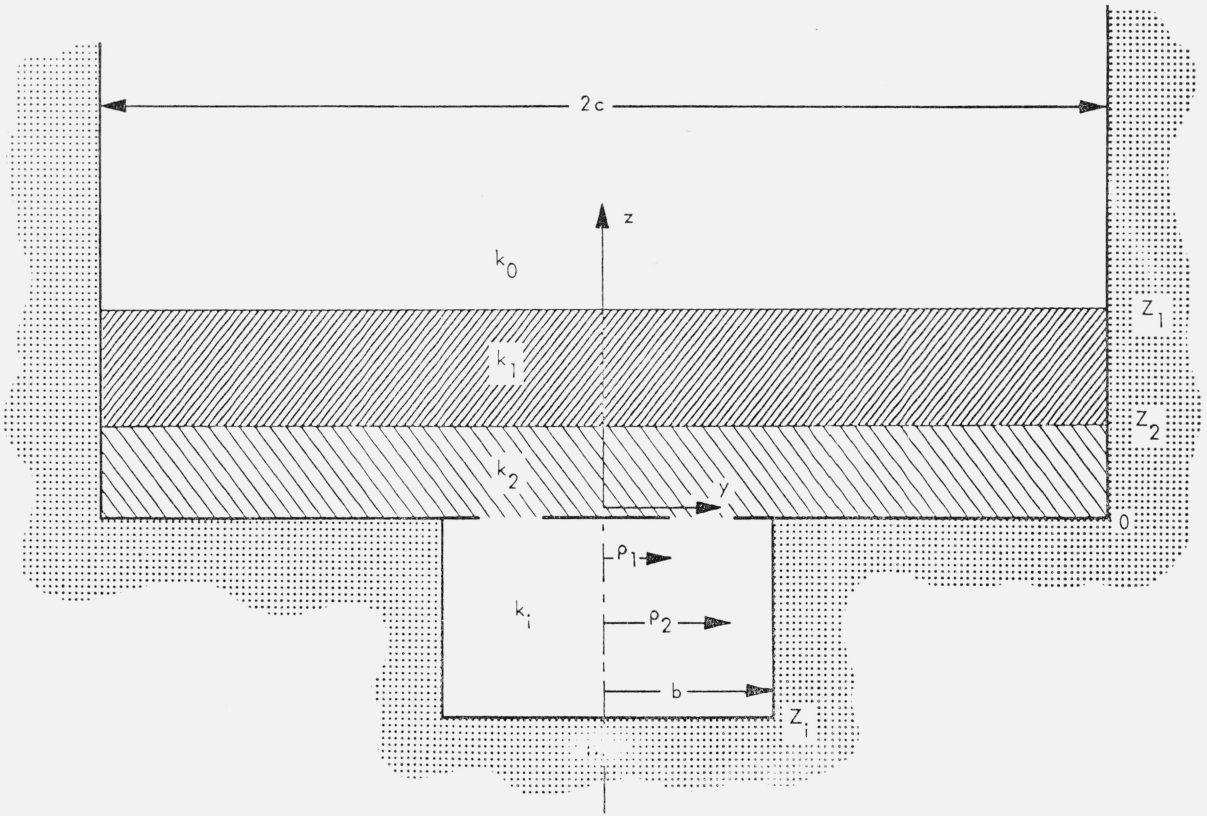
$$J_0(\lambda_m b) = J_0(\lambda_n c) = 0 \qquad (4)$$

FIGURE 1. *Slot geometry.*

$$N_m^{-1} = \frac{b}{\sqrt{2}} J_1(\lambda_m b). \tag{5}$$

$$N_n^{-1} = \frac{c}{\sqrt{2}} J_1(\lambda_n c) \tag{6}$$

The propagation constants γ_m and γ_{nj} are defined in the first quadrant of the complex plane by

$$\gamma_m = \sqrt{\lambda_m^2 - k_i^2} \tag{7}$$

$$\gamma_{nj} = \sqrt{\lambda_n^2 - k_j^2} \tag{8}$$

where

$$k_j^2 = \omega^2 \mu_0 \epsilon_j - j\omega\mu_0 \sigma_j \tag{9}$$

with a corresponding expression for k_i^2. The radial electric field component E_ρ is computed from (1) or (2) as

$$(\sigma_j + j\omega\epsilon_j)E_{\rho j}(\rho,z) = -\frac{\partial}{\partial t} H_{\phi j}(\rho,z) \tag{10}$$

where $j = 0, 1, 2,$ or i. The amplitudes A_q ($q = m$ or $n2$) are related to the excitation field of the cavity $E_\rho(\rho, 0) = \mathcal{E}(\rho)$. Multiplying $E_\rho(\rho,0)$ as computed from (10) with $R_p(\rho)\rho d\rho$ and integrating over the

$z = 0$ plane gives

$$A_q = \frac{(\sigma_j + j\omega\epsilon_j)}{(1 - B_q)\gamma_q} \int_{\rho_1}^{\rho_2} \mathcal{E}(\rho) R_p(\rho) \rho d\rho \tag{11}$$

with $j = i$ or 2. The tangential magnetic fields $H_{\phi i}(\rho, 0)$ and $H_{\phi 2}(\rho, 0)$ differ by the linear source current density $J_s(\rho)$ across the slot, where $J_s(\rho)$ is assumed to be of the form

$$J_s(\rho) = \frac{I}{2\pi\rho}. \tag{12}$$

After substituting (11) in (1) and (2) one obtains an integral equation

$$H_{\phi i}(\rho,0) = H_{\phi 2}(\rho,0) + J_s(\rho) \tag{13}$$

in the unknown field distribution $\mathcal{E}(\rho)$. A variational principle for determining $\mathcal{E}(\rho)$ is constructed by multiplying (13) with $\rho\mathcal{E}(\rho)$ and then integrating from $\rho = \rho_1$ to ρ_2. After defining the voltage across the slot V as

$$V = -\int_{\rho_1}^{\rho_2} \mathcal{E}(\rho) d\rho, \tag{14}$$

it follows that

$$\frac{IV}{2\pi}=\sum_{n=1}^{\infty}\frac{\sigma_2+j\omega\epsilon_2}{\gamma_{n2}}\frac{1+B_{n2}}{1-B_{n2}}\left[\int_{\rho_1}^{\rho_2}\mathcal{E}(\rho)R_n(\rho)\rho d\rho\right]^2$$

$$-\sum_{m=1}^{\infty}\frac{\sigma_i+j\omega\epsilon_i}{\gamma_m}\frac{1+B_m}{1-B_m}\left[\int_{\rho_1}^{\rho_2}\mathcal{E}(\rho)R_m(\rho)\rho d\rho\right]^2. \quad (15)$$

The integrations can be carried out for a general power series representation of $\mathcal{E}(\rho)$ and it is formally possible to determine all the coefficients in this power series. [Levine and Papas, 1951; Galejs and Thompson, 1962.] However, the subsequent numerical work appears to be tractable only for the simplest form of the trial function

$$\mathcal{E}(\rho)=\frac{a_0}{\rho}. \quad (16)$$

The integrations are straightforward for this form of $\mathcal{E}(\rho)$ and the slot admittance $Y=I/V$ follows as:

$$Y=\frac{4\pi}{\left(\log\frac{\rho_2}{\rho_1}\right)^2}\left\{\sum_{n=1}^{\infty}\frac{\sigma_2+j\omega\epsilon_2}{\gamma_{n2}}\frac{1+B_{n2}}{1-B_{n2}}\right.$$

$$\left[\frac{J_0(\lambda_n\rho_1)-J_0(\lambda_n\rho_2)}{\lambda_n c J_1(\lambda_n c)}\right]^2-\sum_{m=1}^{\infty}\frac{\sigma_i+j\omega\epsilon_i}{\gamma_m}\frac{1+B_m}{1-B_m}$$

$$\left.\left[\frac{J_0(\lambda_m\rho_1)-J_0(\lambda_m\rho_2)}{\lambda_m b J_1(\lambda_m b)}\right]^2\right\}. \quad (17)$$

The reflection coefficient B_m is determined from the requirement that $E_\rho(\rho,z_0)=0$. This gives

$$B_m=e^{-2\gamma_m z_i}. \quad (18)$$

The reflection coefficient B_{nj} is determined after matching the E_ρ and H_ϕ components across the dielectric interfaces at $z=z_1$ and $z=z_2$. A computation results in

$$B_{nj}=e^{-2\gamma_{ni}z_i}\quad\frac{S_{nj}-D_{nj}}{S_{nj}+D_{nj}} \quad (19)$$

where

$$S_{nj}=\left(\frac{k_{j-1}}{k_j}\right)^2[e^{-2\gamma_{n(j-1)}z_i}+B_{n(j-1)}] \quad (20)$$

$$D_{nj}=\frac{\gamma_{n(j-1)}}{\gamma_{nj}}[e^{-2\gamma_{n(j-1)}z_i}-B_{n(j-1)}] \quad (21)$$

and where $j=1$ or 2. The computation starts by determining B_{n1} with $B_{n0}=0$ (there are only outgoing waves for $z>z_1$). This value of B_{n1} is then used for computing B_{n2}, which completes the determination of the slot admittance.

As long as σ_i, σ_1, or $\sigma_2\neq0$, the real part of Y will differ from the radiation conductance G_r. However, G_r may be determined after computing the power flow for $z>z_1$. The radiated power is given by

$$P_r=G_r V^2$$

$$=\mathrm{Re}\iint E_{\rho0}\times H_{\phi0}^*\rho d\rho d\phi$$

$$=2\pi\,\mathrm{Re}\sum_n\frac{\gamma_{n0}}{j\omega\epsilon_0}|e^{-\gamma_{n0}z}A_{n0}|^2. \quad (22)$$

The only values of n which contribute to P_r are those with imaginary γ_{n0}. Hence $|e^{-\gamma_{n0}z}|=1$ and

$$G_r=2\pi\,\mathrm{Re}\sum_n\frac{\gamma_{n0}}{j\omega\epsilon_0}\left|\frac{A_{n0}}{V}\right|^2, \quad (23)$$

A_{n0} is related to A_{n2} by matching either H_ϕ or E_ρ across the dielectric boundaries at $z=z_1$ and z_2. It follows that

$$\frac{A_{n0}}{A_{n2}}=\frac{e^{-\gamma_{n1}z_1}+B_{n1}e^{\gamma_{n1}z_1}}{e^{-\gamma_{n0}z_1}}\frac{e^{-\gamma_{n2}z_2}+B_{n2}e^{\gamma_{n2}z_2}}{e^{-\gamma_{n1}z_2}+B_{n1}e^{\gamma_{n1}z_2}} \quad (24)$$

where B_{n1} and B_{n2} are obtained from (19) to (21). The amplitude A_{n2} is related to V from (11), (14), and (16) as

$$\frac{A_{n2}}{V}=-\frac{\sqrt{2}(\sigma_2+j\omega\epsilon_2)[J_0(\lambda_n\rho_1)-J_0(\lambda_n\rho_2)]}{\left(\log\frac{\rho_2}{\rho_1}\right)(1-B_{n2})\gamma_{n2}\lambda_n c J_1(\lambda_n c)}. \quad (25)$$

The substitution of (24) and (25) in (23) completes the specification of G_r.

3. Discussion

The slot admittance is examined first for radiation into a homogeneous large guide of increasing radius c. The plot of the slot conductance in figure 2 is discontinuous with discontinuities occurring at those points where an additional mode starts propagating in the large guide. The slot susceptance, not shown in figure 2, exhibits discontinuities at the same points and large susceptances occur below the cut-off radius of the guide when γ_{no} is real and approaches zero. (This behavior of the susceptance does not follow from the elementary considerations of sec. 1.) The slot admittance for the radiation into a homogeneous half space is shown dashed in figure 2. The conductance curves for the waveguide cross periodically the half-space conductance curve. The changes of the waveguide admittance tend to become less for increasing guide radius c, and a randomly selected guide radius will give a fair approximation to the correct admittance if c is of the order of 15 to 30λ. However, the relatively slow convergence of the sum representing the slot susceptance restricts the choice of c to smaller values. For smaller guides
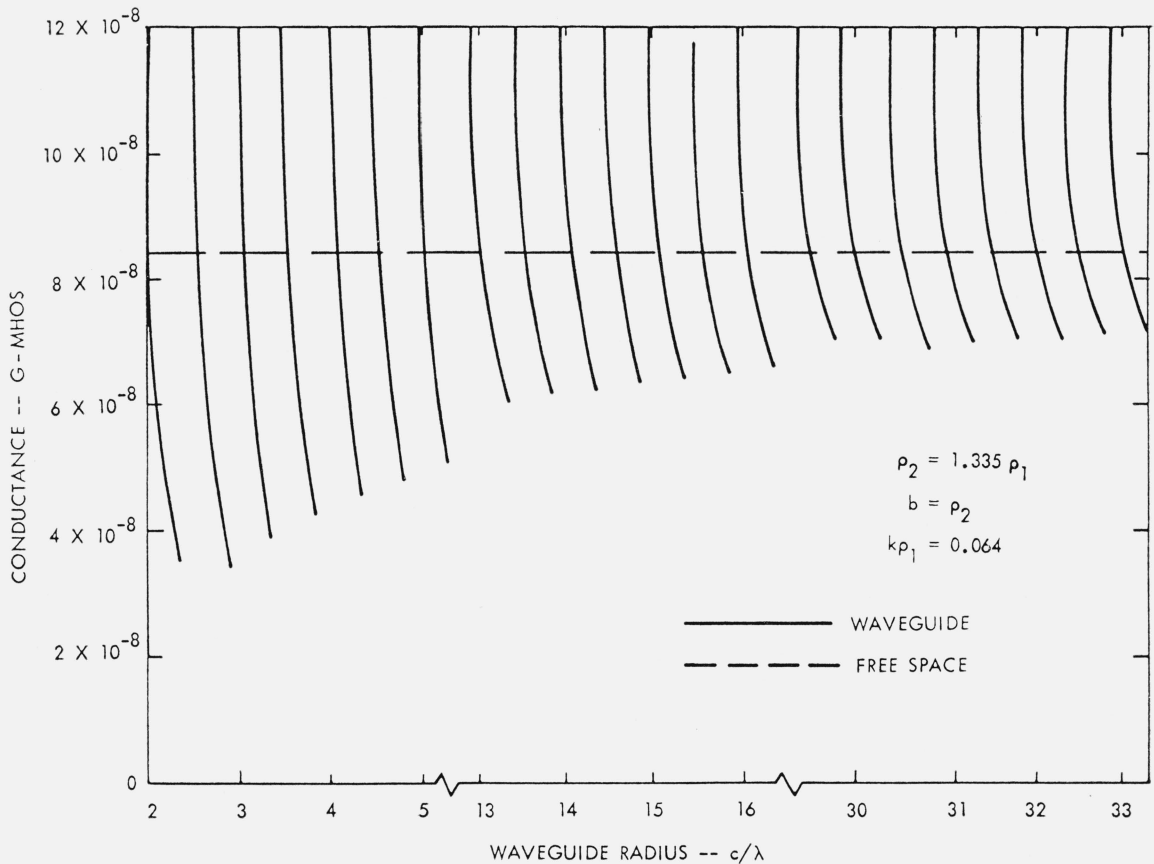
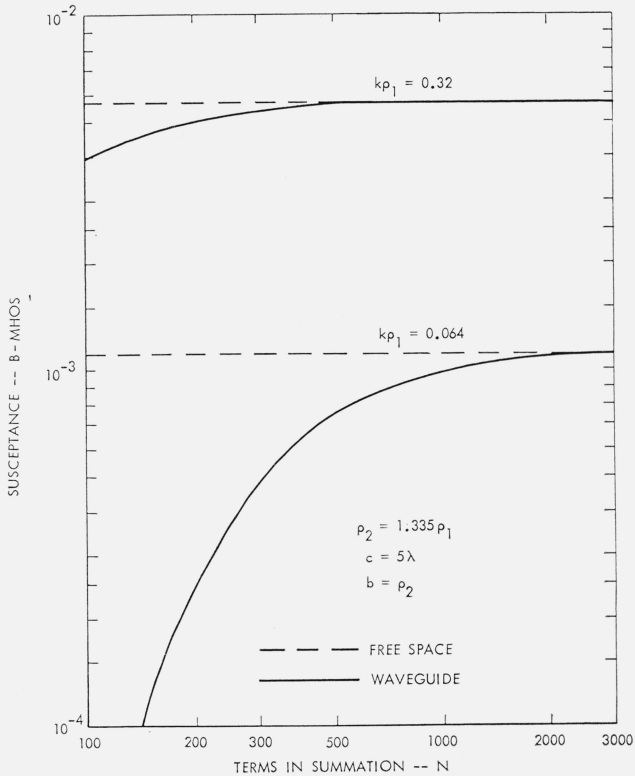FIGURE 2. *Conductance of the annular slot.*

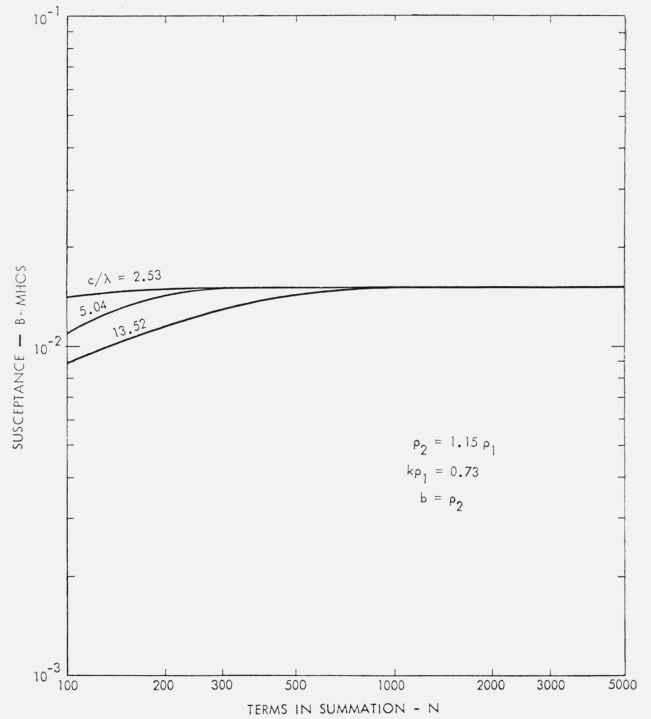FIGURE 3. *Susceptance of the annular slot.*

FIGURE 4. *Susceptance of the annular slot.*

the radius may be always selected such that the waveguide admittance is equal to the slot admittance for a homogeneous half space. The susceptance calculations are illustrated in figure 3 for two values of slot radius ρ_1. For small guide radii the series converges slowly, and a larger number of terms will be required in the summation. Similarly, a larger number of terms is required for guides of large radii c, as is evident from figure 4.

Admittance calculations for plasma layers of various thicknesses h are illustrated in figures 5 and 6. The slot conductance G and the radiation conductance G_r decrease with increasing layer thicknesses h, while the slot susceptance is almost constant for $h/\lambda > 0.1$. For plasma layer of $h/\lambda > 0.3$, the conductance depends on the guide radius c. The computed conductance is more accurate for the larger

c values when it approaches G of the homogeneous half space [1] for $h/\lambda > 1$. A similar effect of increasing waveguide size was also observed for rectangular plasma covered slots [Galejs, 1963b]. For slots covered by thin plasma layers the admittance is principally determined by the nearby air-to-plasma interface, and is hardly affected by the relatively distant guide walls. The guide walls have an effect for large plasma thicknesses, particularly in cases where ϵ_p/ϵ_0 is near unity, or where the electrical discontinuity at the air-to-plasma interface is small.

Only the admittance reflected into the slot plane by the outside space $(z > 0)$ has been indicated in figures 5 and 6. The admittance reflected by the cavity or Y_i has been discussed earlier by Galejs and Thompson [1962], and is not included in the above admittance figures.

Slot admittance calculations will be also made for the parameters of a typical reentry plasma. An example of the electron density N_i and collision frequency ν profiles near a hypersonic reentry vehicle is given in figure 19 of Rotman and Meltz [1961]. These data have been replotted in figure 7 along with
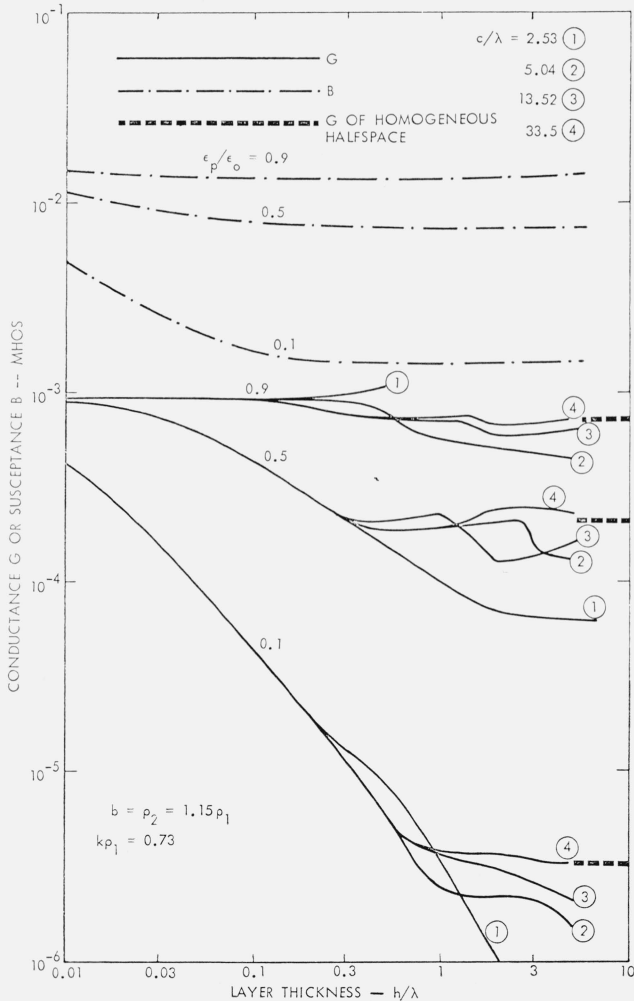

FIGURE 5. *Slot admittance for loss less plasma layers.*

[1] For $k\rho_1 \ll 1$ and $\rho_2/\rho_1 \approx 1$ and for a lossless dielectric half space

$$G_0 = \sqrt{\frac{\epsilon}{\epsilon_0}} \frac{(k\rho)^4}{360} \left[1 - \frac{(k\rho)^2}{5} + \frac{(k\rho)^4}{56} + \cdots \right]$$

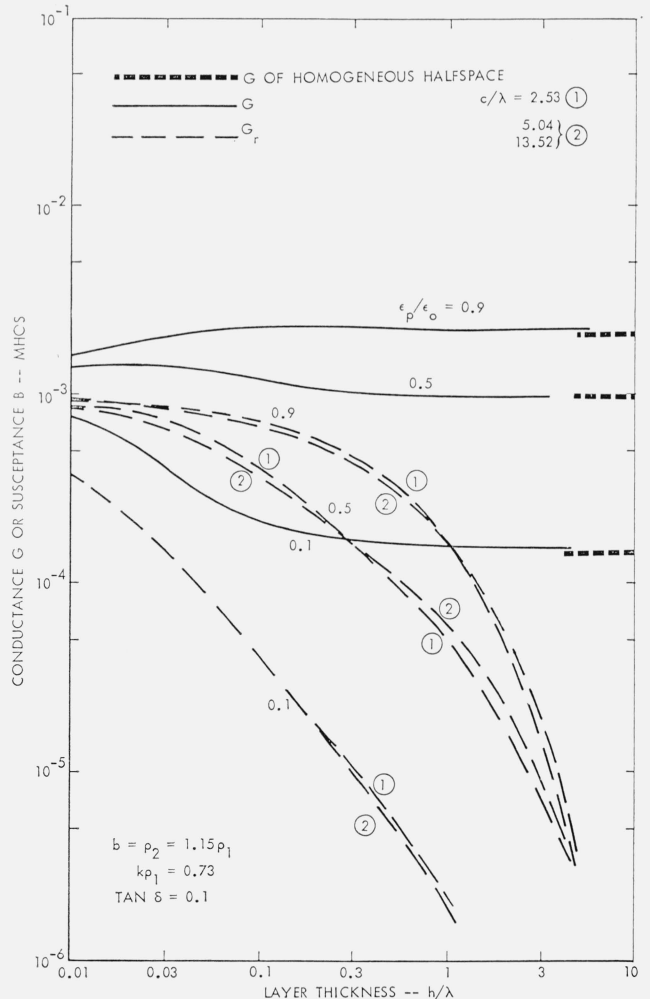where $\rho = (\rho_1 + \rho_2)/2$. For a lossy dielectric half space $G_1 = G_0 + B \tan \delta$.
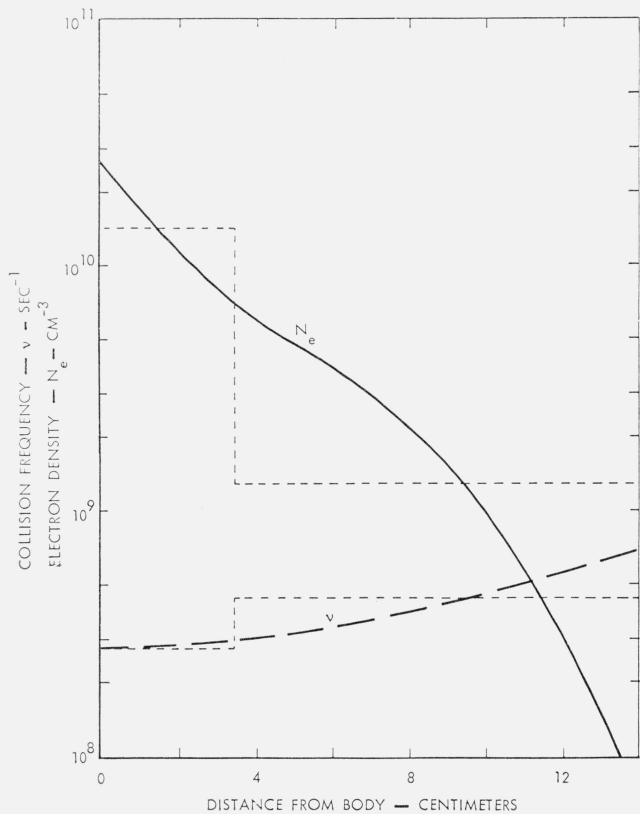
FIGURE 6. *Slot admittance for lossy plasma layers.*

FIGURE 7. *Profile of electron density and collision frequency and the two layer approximation.*

Velocity of 17.5 K fps and Altitude of 200 K ft.

FIGURE 8. *Slot admittance for a two layer approximation of a reentry plasma profile.*

the two-layer approximation that is suited for the present calculation. The calculated slot admittance is shown in figure 8 for various operating frequencies while the slot and the waveguide dimension remain constant in relation to the operating wavelength. For frequencies above the plasma frequency $(f > f_p)$ the radiation conductance is nearly the same as the total slot conductance. For frequencies below the plasma frequency the slot susceptance B is inductive and there are high losses and a low radiation conductance. The radiation conductance exhibits dips near the plasma frequencies of the two layers, which are at $f_p = 3.22 \times 10^8$ and 1.06×10^9 c/s. Obviously, this example does not cover all the plasma and slot parameters of interest in reentry problems. However, the above calculations are straightforward and further examples will not be considered at present.

The present model is also suited for determining the slot admittance in presence of lossy dielectric layers. In an example the layer near the slot is assumed to be of low losses (tan $\delta_2 = 0.01$), while the outer layer is highly lossy $(1 \leq \tan \delta_1 \leq 100)$.
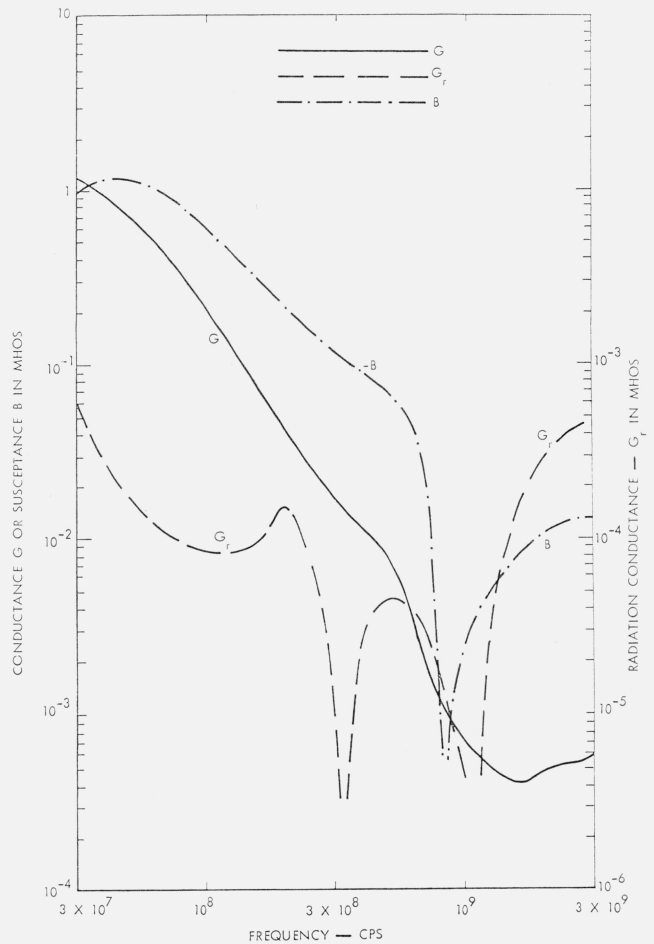
The calculated slot admittance is shown in figures 9 to 11, for various distances z_2 between the slot and the lossy layer. For a given thickness $(z_1 - z_2)$ of the lossy layer the losses and G are highest for z_2 small, but the radiation conductance G_r is essentially constant over a range of z_2 and $(z_1 - z_2)$ values. For very thin lossy layers the capacitive susceptance B approaches the free space value B of figure 4 for z_2 small. For thick highly lossy layers (tan $\delta = 100$) B is highest for z_2 small.

The above waveguide model provided convenient means for analyzing slot and plasma or dielectric configurations that would appear untractable in more exact geometries. The susceptance and loss computations can be justified either by reasoning that distant guide walls do not perturb near fields of the slot or by comparing the admittance in the limit of very thick or thin plasma layers with known results for a homogeneous half space or free space. The dimensions of the large guide may be also selected in such a way that the calculated radiation conductance agrees in the same limiting cases with known
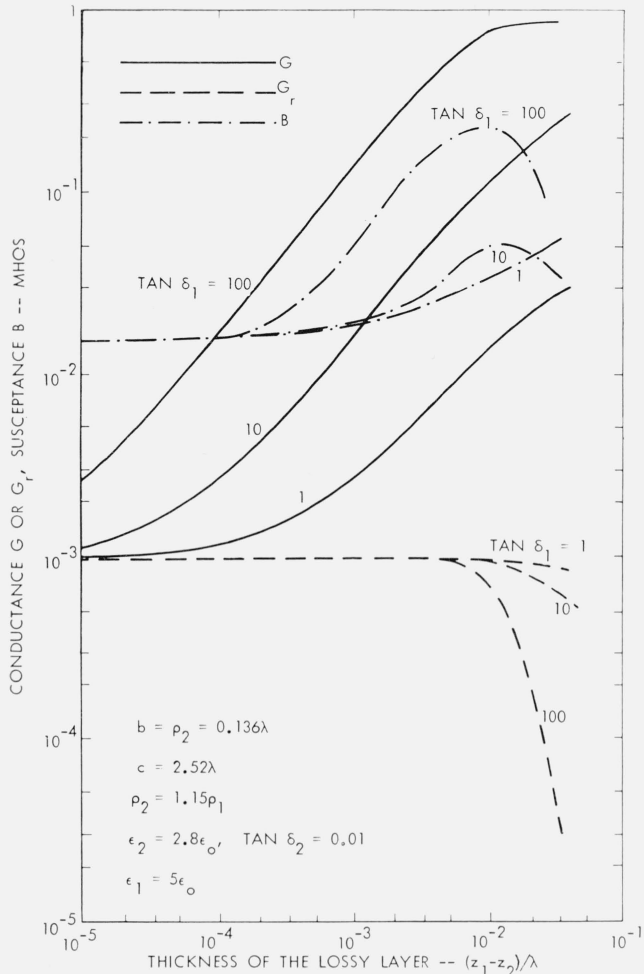
FIGURE 9. *Admittance of a slot covered by lossy dielectric at a distance* $Z_2 = 3 \times 10^{-4} \lambda$.

FIGURE 10. *Admittance of a slot covered by lossy dielectric at a distance* $Z_2 = 3 \times 10^{-3} \lambda$.

results. There are no other solutions for the intermediate range of layer thicknesses for the annular slot. However, for rectangular slots there are comparable solutions for waveguides and for inhomogeneous half space [Galejs, 1963a, b].

4. References

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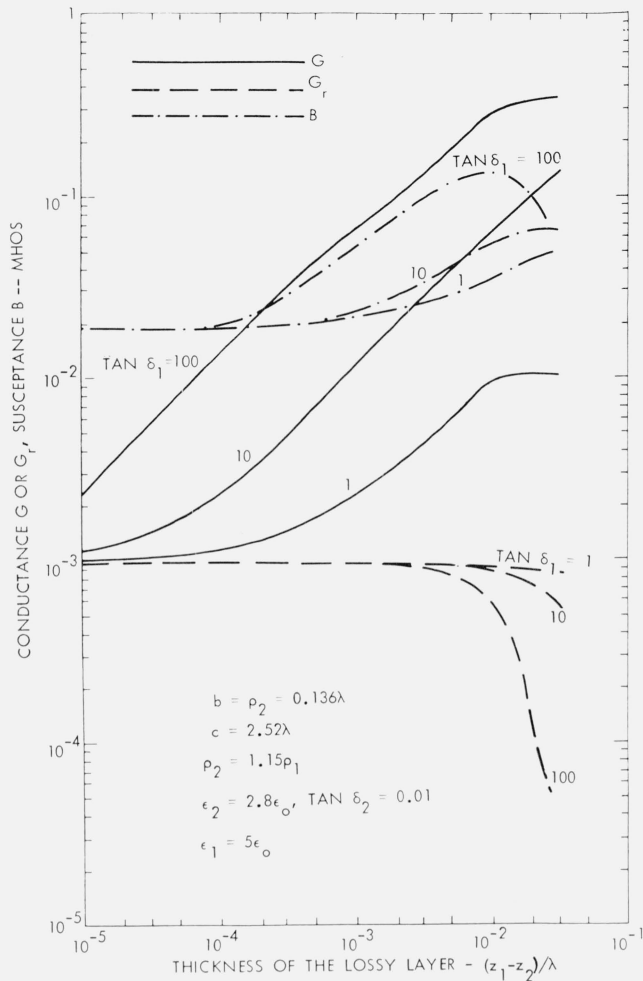
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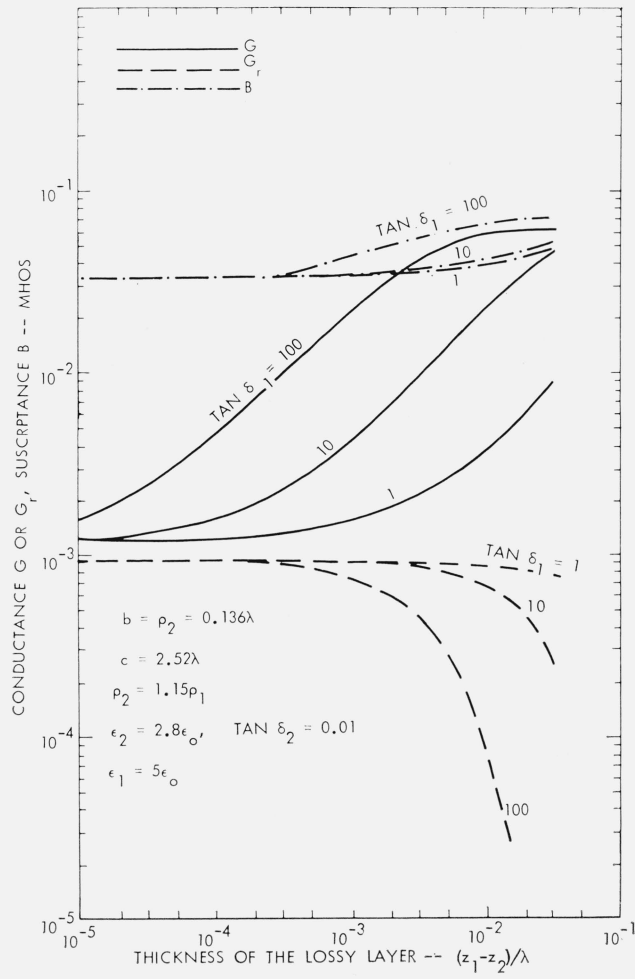
FIGURE 11. *Admittance of a slot covered by lossy dielectric at a distance* $Z_2 = 3 \times 10^{-2} \lambda$.

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