RADIO SCIENCE Journal of Research NBS/USNC–URSI
Vol. 68D, No. 2, February 1964

Geometrical Optics Convergence Coefficient for the Whistler Case

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(Received July 25, 1963)

In a previous report [Crary, 1962] the field strength, direction of arrival, and apparent polarization of whistler signals was calculated by the use of ray theory (or geometrical optics). The convergence coefficient is a factor in the ray theory equations which expresses the net convergence or divergence of the rays caused by reflection from the curved earth and ionosphere.

Intuitive reasoning led to the assumption of unity for this coefficient in the whistler case, where there are an equal number of reflections from the concave ionosphere and convex earth. This is contrasted with the convergence coefficient for the case of ground-to-ground transmission; this coefficient contains singularities at critical distances.

The derivation and evaluation of the expression for the coefficient for the whistler case confirms the accuracy of the assumption of unity; this greatly simplifies whistler calculations.

The author has shown the methods for and the results of calculations of the field strength and apparent polarization and direction of arrival of whistler signals [Crary, 1961, 1964]. These calculations utilized the methods of geometrical optics to express the field as the sum of a series of rays. The assumption is made that the ionosphere may be represented by a sharply-bounded homogeneous slab with a constant vertical magnetic field.

One of the factors in the ray equation is the convergence coefficient, which is a purely geometrical expression for the focusing or defocusing of the rays confined between the spherical earth and ionosphere.

The convergence coefficient for the case of ground-to-ground transmission (α_{gg}) was derived by Bremmer [1949] and discussed by Wait and Murphy [1957]. The geometry of this case for an n-reflection ray is illustrated in figure 1. The corresponding convergence coefficient is given by (1).

$$\alpha_{gg}=(1+h/a)\left[\frac{2n\sin\left(\frac{D}{2an}\right)}{\sin\left(\frac{D}{a}\right)}\right]^{1/2}\left[\frac{(1+h/a)-\cos\left(\frac{D}{2an}\right)}{(1+h/a)\cos\left(\frac{D}{2an}\right)-1}\right]^{1/2}. \tag{1}$$

This coefficient increases rapidly near the critical distance and becomes infinite at this point. The critical distance is that where the vertical angle of takeoff-arrival at the ground is 90°. This point is called a caustic in the language of geometrical optics. The expression is not valid near the caustic, where higher-order approximations are necessary to calculate a value for the field. Wait [1961] derives expressions which are valid near and beyond the caustic, and discusses the region of validity of (1).

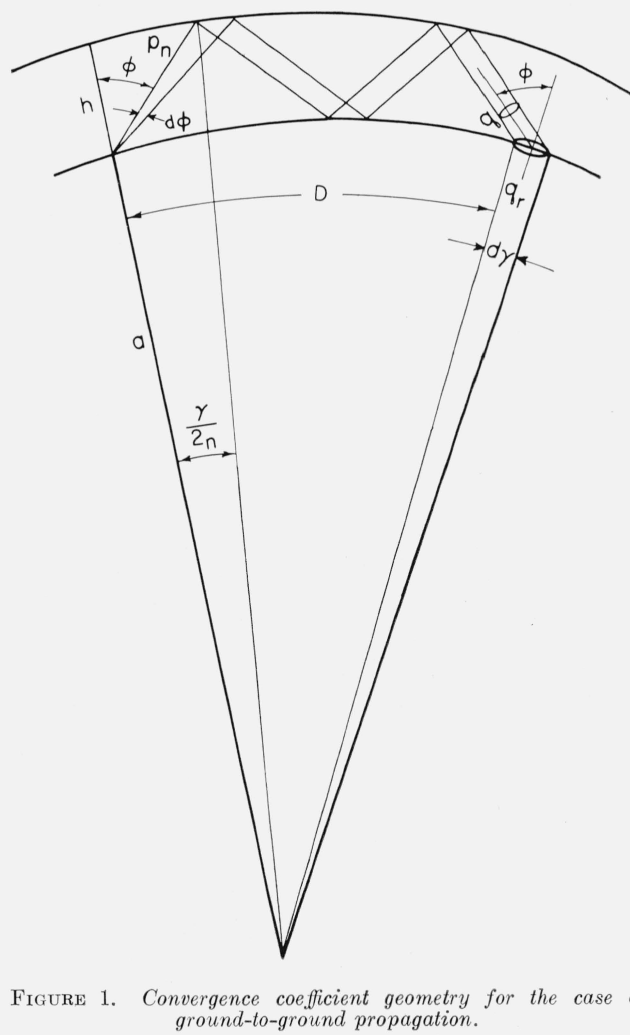

FIGURE 1. *Convergence coefficient geometry for the case of ground-to-ground propagation.*

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A similar, but significantly different, geometrical situation occurs for the whistler case. The geometry of this case is shown in figure 2. The equation for the convergence coefficient (α_{wh}) for this case may be derived as follows, using a method of analysis similar to that of Bremmer [1949]:

$$q=q_r \cos \theta_n \tag{2}$$

$$q_r=(a+h)^2 \sin \gamma d\gamma d\delta \tag{3}$$

$$q'=R_n^2 \sin \Phi d\Phi d\delta \tag{4}$$

$$\alpha=\left[\frac{q'}{q}\right]^{1/2} \tag{5}$$

$$\alpha_{wh}=\frac{R_n}{(a+h)}\left[\frac{\sin \Phi}{\cos \theta_n \sin \gamma}\left(\frac{d\Phi}{d\gamma}\right)\right]^{1/2} \tag{6}$$

$$R_n=(2n+1)P_n=\text{total path distance} \tag{7}$$

$$P_n=\frac{a}{\sin \theta_n} \sin\left[\frac{\gamma}{2n+1}\right]=\frac{(a+h)}{\sin \Phi} \sin\left[\frac{\gamma}{2n+1}\right] \tag{8}$$

(7) and (8) yield:

$$R_n=\frac{(2n+1)a}{\sin \theta_n} \sin\left[\frac{\gamma}{2n+1}\right]. \tag{9}$$

The Law of cosines yields:

$$P_n^2=a^2+(a+h)^2-2a(a+h) \cos\left[\frac{\gamma}{2n+1}\right] \tag{10}$$

(8) and (10) may be combined to yield:

$$(a+h)^2 \csc^2 \Phi \sin^2\left(\frac{\gamma}{2n+1}\right)$$
$$=a^2+(a+h)^2-2a(a+h) \cos\left(\frac{\gamma}{2n+1}\right). \tag{11}$$

Differentiation then yields:

$$(a+h) \csc^2 \Phi \cos\left(\frac{\gamma}{2n+1}\right) d\gamma-(a+h)$$
$$\csc^2 \Phi \cot \Phi \sin\left[\frac{\gamma}{2n+1}\right] d\Phi=\frac{a d\gamma}{2n+1} \tag{12}$$

and

$$\frac{d\Phi}{d\gamma}=\frac{(a+h) \csc^2 \Phi \cos\left(\frac{\gamma}{2n+1}\right)-a}{(2n+1)(a+h) \csc^2 \Phi \cot \Phi \cot \Phi \sin\left(\frac{\gamma}{2n+1}\right)}. \tag{13}$$

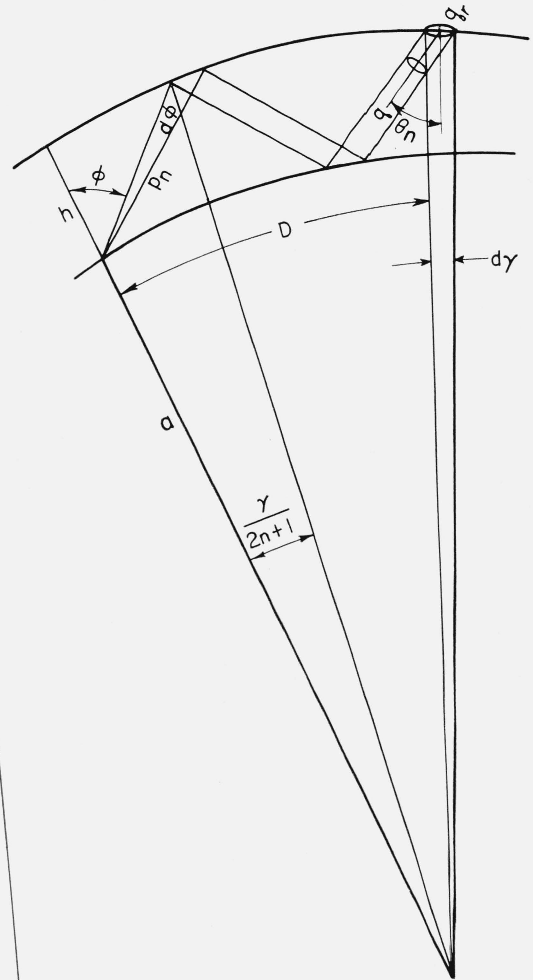

FIGURE 2. *Convergence coefficient geometry for the case of whistler propagation.*

The relation between the angles is expressed by:

$$\theta_n=\tan^{-1}\left\{\frac{\sin\left(\frac{\gamma}{2n+1}\right)}{1+h/a-\cos\left(\frac{\gamma}{2n+1}\right)}\right\}. \tag{14}$$

Where

$$\Phi=\theta_n+\left(\frac{\gamma}{2n+1}\right). \tag{15}$$

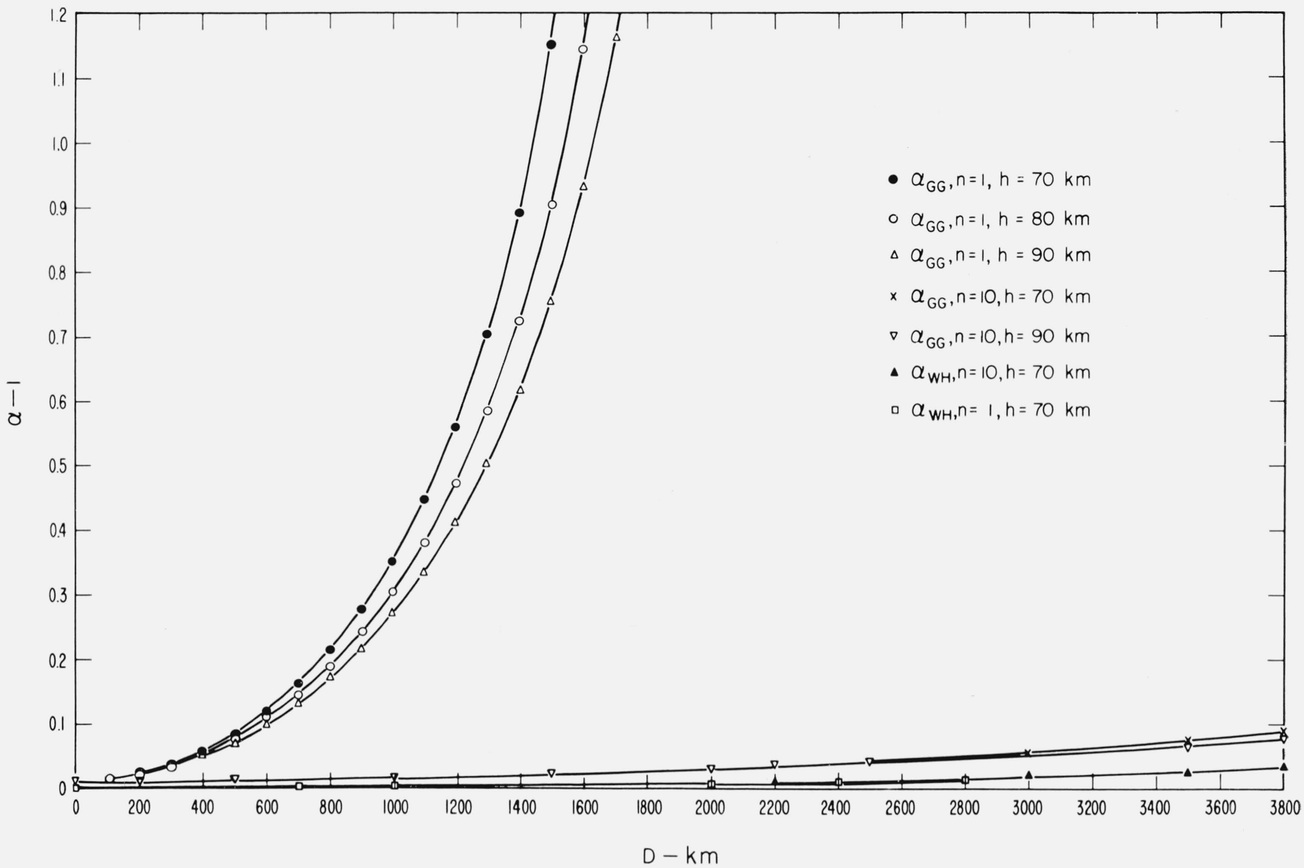

FIGURE 3. *Convergence coefficients for the cases of ground-to-ground and whistler propagation as a function of ground distance.*

The combination of (6), (7), and (13) then yields

$$\alpha_{wh}=\frac{a(2n+1)}{(a+h)\sin\theta_n}\sin\left(\frac{\gamma}{2n+1}\right)\left\{\frac{\sin\Phi\left[(a+h)\csc^2\Phi\cos\left(\frac{\gamma}{2n+1}\right)-a\right]}{(a+h)(2n+1)\cos\theta_n\csc^2\Phi\cot\Phi\sin\left(\frac{\gamma}{2n+1}\right)}\right\}^{1/2} \tag{16}$$

$$\alpha_{wh}=\frac{a}{(a+h)\sin\theta_n}\left\{\frac{(2n+1)\sin\Phi\tan\Phi\sin\left(\frac{\gamma}{2n+1}\right)\left[(a+h)\cos\left(\frac{\gamma}{2n+1}\right)-a\sin^2\Phi\right]}{(a+h)\sin\gamma\cos\theta_n}\right\}^{1/2} \tag{17}$$

The values of the coefficients from (1) and (17) are shown in figure 3. The rapid variation of the coefficient in the ground-to-ground case is quite evident as the critical distance is approached. The critical distance is, of course, dependent on the reflection height ($D_c\cong\sqrt{8nha}$ [Wait and Murphy, 1957]).

The results, therefore, substantiate the intuitive assumption of unity for the convergence coefficient for the whistler case. The maximum deviation is seen to be of the order of 5 percent and this occurs for large n, at distances greater than those which are normally of interest in whistler propagation. This allows a substantial reduction in the complexity of the calculations in most cases of interest in whistler propagation.

The coefficients are plotted for several values of n and h. The curves for the ground-to-ground case are shown first for $n=1$, $h=70$, and 90 km, where the critical distance is in the range of 1900 to 2200 km. The coefficients are also plotted for $n=10$, $h=70$ and 90 km. These are almost constant out to 3800 km since the critical distance is the $\sqrt{n}$ times that for $n=1$.

In contrast to the ground-to-ground case, the curves for the whistler case show that the coefficient is nearly constant for $n=1$ to 10 for distances up to 3800 km. An examination of (17) shows that there are no infinities of the expression at large distances

References

Bremmer, H. [1949], Terrestrial Radio Waves, p. 168 (Elsevier Publishing Co., New York, N.Y.).

Crary, J. H. [1961], The effects of the earth-ionsphere waveguide on whistlers, Technical report, No. 9, Contract AF 18 (603)—126, Radioscience laboratory, Stanford Electronic Laboratories, Stanford University, Stanford, Calif., (AFOSR–1092) July 17, 1961.

Crary, J. H. [1964], The effect of the earth-ionosphere waveguide on whistlers. Parts I and II (to be published).

Wait, J. R., and A. Murphy [1957], The geometrical optics of VLF sky wave propagation, Proc. IRE **45,** No. 6, 754–60.

Wait, J. R. [1961], A diffraction theory for LF sky wave propagation, J. Geophys. Res. No. 6, 1713–1724.

(Paper 68D2–334)