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Generation of an Electromagnetic Pulse by an Expanding Plasma in a Conducting Half-Space [1]

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The problem of the generation of an electromagnetic pulse by an expanding, infinitely conducting, spherical plasma under the earth is considered. The solution consists of the derivation of an appropriate tensor Green's function for the half-space which reduces the problem of determining the electric field at any point in space due to the current density generated by the interaction of the plasma with the earth's static magnetic field to evaluating an integral. The vertical component of the electric field at the earth's surface which is generated by the mechanism is calculated.

1. Introduction

The problems of determining the electromagnetic field resulting from a vertical and a horizontal dipole imbedded in a conducting half-space have been solved in the past essentially by following Sommerfeld's method for solving half-space boundary value problems. A similar problem such as the one of determining the field due to an arbitrary vector current distribution $\overline{J}(\overline{r}, t)$ imbedded in a conducting half-space with a time independent, uniform conductivity σ, is often of some interest. Such a problem, for example, may arise from the electromagnetic radiation of an expanding plasma which may result from a very intense underground explosion [Taylor, 1950]. The electromagnetic field generated by the current density $\overline{J}$ can be obtained by first evaluating the appropriate Green's function which satisfies the boundary conditions and then evaluating the volume integral of the product of $\overline{J}$ and the Green's function. In this paper, the Green's function for a time-varying source is derived by making use of Fourier transforms. The field expression at any point in space is then obtained. Numerical results are given for the vertical component of the electric field at the earth's surface.

2. Formulation of the Problem

From Maxwell's equations it is seen that the electric field $\overline{E}(\overline{r}, \omega)$ as a function of position $\overline{r}$ and angular frequency ω satisfies the equation

$$\nabla\times\nabla\times E(\overline{r}, \omega)-\beta^2 E(\overline{r}, \omega)=i\omega\mu_0\overline{J}(\overline{r}, \omega) \tag{1}$$

where $\overline{E}(\overline{r}, \omega)$ and $\overline{J}(,\overline{r}\ \omega)$ are respectively the Fourier transforms of the corresponding time-varying electric field $\overline{E}(\overline{r}, t)$ and the current $\overline{J}(\overline{r}, t)$, defined by

$$\overline{E}(\overline{r}, \omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E(r, t)e^{i\omega t}dt, \qquad \overline{J}(\overline{r}, \omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{J}(\overline{r}, t)e^{i\omega t}dt$$

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and β is the propagation constant defined by

$$\beta^2 = \begin{cases} \mu_0\epsilon_0\omega^2 \text{ in the nonconducting half-space} \\ \mu_0\epsilon_0\omega^2 + i\omega\mu\sigma \text{ in the conducting half-space.} \end{cases} \tag{2}$$

The symbols μ_0, ϵ_0, and σ denote respectively the free-space permeability, free-space dielectric constant and the conductivity of the conducting half-space.

Thus, if a dyad $\overline{\overline{\Gamma}}$ is defined which satisfies the equation

$$\nabla\times\nabla\times\overline{\overline{\Gamma}} - \beta^2\overline{\overline{\Gamma}} = \delta(\overline{r} - \overline{r}')\overline{\overline{I}} \tag{3}$$

where $\overline{\overline{I}}$ is a unit dyad and $\overline{\overline{\Gamma}}$ satisfies the appropriate boundary conditions at the interface of the finitely conducting and nonconducting half-spaces, the electric field intensity can be written as

$$\overline{E} = i\omega\mu_0 \int_V \overline{\overline{\Gamma}} \cdot \overline{J} dv', \tag{4}$$

the integration being taken over the region of space V where $\overline{J}$ does not vanish identically.

Let the conducting and nonconducting parts of the space media be respectively referred to as 1 and 2, as shown schematically in figure 1. Also let a cartesian coordinate system be chosen such that the z axis is normal to the interface between the media 1 and 2.

Since (3) has coefficients which do not depend on x and y, one may introduce the Fourier transform

$$\overline{\overline{\Gamma}}(\xi, \eta, z) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\overline{\Gamma}}(x, y, z) e^{i\xi x + i\eta y} dx dy \tag{5}$$

into this equation. Then the operator ∇ is transformed into ∇_F where

$$\nabla_F = -i\xi\hat{x} - i\eta\hat{y} + \frac{\partial}{\partial z}\hat{z}. \tag{6}$$

In (6) and in the following, a caret is used to indicate a unit vector in the respective coordinate direction. Thus (3) can be written as

$$\nabla_F\times\nabla_F\times\overline{\overline{\Gamma}}(\xi, \eta, z) - \beta^2\overline{\overline{\Gamma}}(\xi, \eta, z) = \overline{\overline{I}}\,\frac{1}{2\pi}\,\delta(z - z')e^{i\xi x' + i\eta y'}. \tag{7}$$

Equation (7) represents a system of ordinary differential equations, in the variable z, for the components of $\overline{\overline{\Gamma}}$. Appropriate boundary conditions must be satisfied. The continuity of the tangential components of $\overline{E}$ for an arbitrary current source means that Γ_{rr}, Γ_{rz}, and $\Gamma_{\theta\theta}$ are continuous at $z=0$. Continuity of the tangential component of $\overline{H}$ (proportional to $\nabla \times \overline{\overline{\Gamma}}$) implies that the $\hat{r}\hat{\theta}$, $\hat{\theta}\hat{r}$, and $\hat{\theta}\hat{z}$ components of $\nabla \times \overline{\overline{\Gamma}}$ are continuous at $z=0$. The other tangential components of $\nabla \times \overline{\overline{\Gamma}}$ vanish identically.

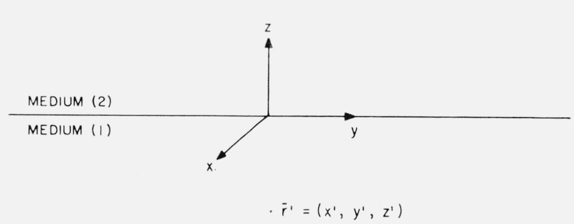

FIGURE 1. *Coordinate system used in deriving Green's function.*

The solution of (7) satisfying these conditions is readily obtained. If the source point is in medium 1 and the field point in medium 2, the Green's function $\overline{\overline{\Gamma}}$ which satisfies (3) and this boundary condition can now be expressed as

$$\overline{\overline{\Gamma}}(x,y,z,x',y',z')=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\xi d\eta e^{-\gamma_2 z+\gamma_1 z'-i\xi(x-x')-i\eta(y-y')}$$

$$\cdot\begin{bmatrix} \frac{1}{\gamma_1+\gamma_2}-\frac{\xi^2}{N} & \frac{-\eta\xi}{N} & \frac{i\gamma_2\xi}{N} \\ -\frac{\xi\eta}{N} & \frac{1}{\gamma_1+\gamma_2}-\frac{\eta^2}{N} & \frac{i\gamma_2\eta}{N} \\ \frac{i\gamma_1\xi}{N} & \frac{i\gamma_1\eta}{N} & \frac{\xi^2+\eta^2}{N} \end{bmatrix} \tag{8}$$

where

$$N=\gamma_1\beta_2^2+\gamma_2\beta_1^2$$

$$\gamma_{1,2}=\sqrt{\xi^2+\eta^2-\beta_{1,2}^2}$$

and the subscripts on the propagation constants refer to the conducting and nonconducting media.

3. Pulse Generated by an Expanding Plasma

This Green's function will now be used to solve the problem of the emission of an electromagnetic pulse by a very intense explosion under the earth. It will be assumed that the explosion creates an infinitely conducting, expanding spherical plasma which interacts with the earth's magnetic field. It is known that the magnetic field is frozen in the plasma as it expands [Cowling, 1957]. Thus, if the expansion is assumed to begin from a point, no magnetic flux will exist inside the volume containing the plasma.

The physical mechanism for canceling the earth's magnetic field, $\overline{B}_0$, inside the plasma is a surface current which is induced by $\overline{B}_0$. Hence, the plasma may be replaced, for computational purposes, by an equivalent current density $\overline{J}$ in the earth. This is more convenient than solving directly the difficult moving boundary value problem which requires, in addition to the vanishing of the normal component of the total magnetic field at the plasma surface, the vanishing of the tangential component of [3]

$$\overline{E}+\frac{d\overline{a}}{dt}\times(\overline{B}_0+\overline{B})$$

at the plasma surface where $\overline{E}=\overline{E}(\overline{r},t)$ and $\overline{B}=\overline{B}(\overline{r},t)$ are respectively the electric and magnetic fields generated by the plasma. Here $\overline{a}=a(t)\hat{r}$ is the radius of the plasma.

To obtain an expression for the equivalent current density generated by the expanding plasma, we first remark that if the explosion takes place at a sufficiently great depth, the effect of the earth-air boundary on the field near the plasma will be negligible. In fact, even neglecting attenuation in propagation through the earth and assuming that the earth-air boundary is a perfectly reflecting plane, the ratio of the reflected field to the primary field at the surface

[3] To see that this boundary condition must be satisfied instead of the usual vanishing of the tangential component of $\overline{E}$, one can consider, in addition to the fixed coordinate system used above, a coordinate system which momentarily is moving with the velocity of some point on the surface of the sphere. In this coordinate system, a portion of the plasma surface is stationary and the boundary condition is that the tangential component of $\overline{E}$ vanishes. Transforming back to the fixed coordinate system, we find that the boundary condition is that stated above.

of the plasma is of the order of

$$\left(\frac{R_{\max}}{2d}\right)^3$$

for low frequency waves where d is the depth of explosion origin and $R_{\max}$ is the maximum radius which the plasma achieves. For example, the effect of the earth-air boundary is less than 0.1 percent for a plasma which achieves a final radius of 25 meters if $d=150$ meters.

Thus, to calculate the current density at the surface of the plasma which will cancel the earth's magnetic field inside, it can be assumed that the plasma is located in an infinitely extended medium with conductivity σ. Let r, θ', ϕ' be the spherical coordinates of the point $\bar{r}$ in a coordinate system using $\overline{B}_0$ as the polar axis and the center of the plasma as the origin. In this case, the symmetry of the problem dictates that the current density be in the $\hat{\phi}'$ direction and proportional to sin θ'. For, it is clear that any vector associated with the electromagnetic field is independent of the angle ϕ'. Furthermore, if a reflection in any plane containing $\overline{B}_0$ followed by multiplication by minus one is performed, it is seen that $\overline{B}_0$ (which transforms as a pseudovector) and the geometry remain invariant. Therefore, all vectors associated with the EM field must be invariant under these operations. This implies that vectors can have only a $\hat{\phi}'$ component and pseudovectors $\hat{r}$ and $\hat{\theta}'$ components. The fact that the angular dependence of $\overline{J}$ is given by sin θ' can be seen by recalling that a plasma moving in a radial direction interacts with the external field $\overline{B}_0$ by a term proportional to $\bar{v}\times\overline{B}_0$ (where $\bar{v}$ is the radial velocity vector). By substituting a driving term of this form into Maxwell's equations and using the above results, it is immediately found that $\overline{J}$ is proportional to sin θ'.

Hence, $\overline{J}$ can be written as

$$\overline{J}=K(t)\sin\theta'\delta(r-a)\hat{\phi}' \tag{9}$$

where the δ function has been inserted because the current is confined to the surface of the plasma. Here $a=a(t)$ is the radius of the plasma and $K(t)$, which may depend on time, is to be found.

To determine K, we find the magnetic field which is generated in a medium of conductivity σ by a current density of the form indicated in (9). It is convenient to Fourier transform into the frequency domain. Since the electric field Green's function in an infinite medium can be expressed as [Morse and Feshbach, 1953a]

$$\left[\overline{\overline{I}}+\frac{\nabla\nabla}{\beta^2}\right]\frac{e^{i\beta|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}=i\beta\sum_{\substack{\sigma=\text{odd,}\\ \text{even}}}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{2l+1}{l(l+1)}\epsilon_m\frac{(l-m)!}{(l+m)!}\cdot\{\overline{M}^{(1)}_{\sigma lm}(\beta r_>)\overline{M}^{(3)}_{\sigma ln}(\beta r_<)+\overline{N}^{(1)}(\beta r_>)\overline{N}^{(3)}(\beta r_<)\}$$

where $\overline{M}$ and $\overline{N}$ are vector spherical harmonics, $\epsilon_m=\begin{cases}1 \text{ if } m=0\\ 2 \text{ if } m\neq 0\end{cases}$ and the symbols $r_>$ and $r_<$ indicate respectively the larger and smaller of r, r', it is found, as a result of the orthogonality relations among the vector spherical harmonics, that

$$\overline{B}(r,\omega)=\frac{1}{i\omega}\nabla\times\overline{E}=i\mu_0\beta^2\overline{N}^{(1)}_{\text{even},1,0}(\beta r)\int_r^{\infty}h_1(\beta r')J(r',\omega)r'^2dr'+\overline{N}^{(3)}_{\text{even},1,0}\int_0^r j_1(\beta r')J(r',\omega)r'^2dr'$$

where h_1 and j_1 are spherical Bessel functions and $J(r',\omega)$ is the Fourier transform of $K(t)\delta(r-a)$. Fourier transforming to the time domain, it is found, after taking (9) into account, that

$$\overline{B}(\bar{r},t)=\frac{1}{(\sqrt{2\pi})^2}\int_0^{\infty}dt'K(t')\int_{-\infty}^{\infty}d\omega e^{-i\omega(t-t')}i\mu_0\beta^2a^2$$
$$\{\overline{N}^{(1)}_{\text{even},1,0}(\beta r)h_1(\beta a)H(a-r)+\overline{N}^{(3)}_{\text{even},1,0}(\beta r)j_1(\beta a)H(r-a)\} \tag{10}$$

where the argument of a is t'.

Now, it is readily seen from dimensional arguments [Cowling, 1957] that if a non-uniform spatial distribution of the magnetic field exists inside the sphere, the time for the non-uniformity to decay to zero is of the order of $\mu_0\sigma r^2$. For a sphere with a radius of 25 meters and an earth conductivity of 10^{-2} mho/m, this time is about 10^{-5} seconds which is much smaller than any practical expansion time for the plasma. Thus, the field at any point within the radius $a(t)$ can be considered to be quasi-static and equal to the field at $r=0$. Hence, to evaluate $K(t)$, we may set $r=0$ in the following.

In this case, (10) reduces to

$$\overline{B}(0, t)=\frac{1}{2\pi}\int_0^\infty dt' K(t') i\mu_0\beta^2 a^2 \overline{N}^{(1)}_{\text{even},\,1,\,0}(0)h_1(\beta a)$$
$$=-\frac{i\mu_0}{2\pi}\frac{2}{3}(\cos\theta'\hat{r}-\sin\theta'\hat{\theta}')\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}[\beta a+i]e^{-i\beta a}. \quad (11)$$

Since βa is small for all frequencies of interest, the term $e^{i\beta a}$ may be expanded in a power series. Keeping terms of order $\beta^2 a^2$, it is found that

$$\overline{B}(0, t)=\frac{\mu_0}{2\pi}\frac{2}{3}(\cos\theta'\hat{r}-\sin\theta'\hat{\theta}')\int_0^\infty dt' K(t')\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}\left[1+\frac{\beta^2 a^2}{2}\right]$$
$$=\mu_0\frac{2}{3}(\cos\theta'\hat{r}-\sin\theta'\hat{\theta}')\int_0^\infty dt' K(t')\left[\delta(t-t')+\frac{a^2}{2}\left\{-\mu_0\sigma\frac{d}{dt}-\mu_0\epsilon\frac{d^2}{dt^2}\right\}\delta(t-t')\right]$$
$$=\mu_0\frac{2}{3}(\cos\theta'\hat{r}-\sin\theta'\hat{\theta}')\left[K(t)+\frac{\mu_0\sigma}{2}\frac{d}{dt}(Ka^2)-\frac{\mu_0\epsilon}{2}\frac{d^2}{dt^2}(Ka^2)\right].$$

It is seen, then, that the magnetic field generated by the current density $\overline{J}$ will cancel the earth's magnetic field

$$B_0(\cos\theta'\hat{r}-\sin\theta'\hat{\theta}')$$

if $K(t)$ satisfies the differential equation

$$-\frac{3}{2}\frac{B_0}{\mu_0}=K(t)+\frac{\mu_0\sigma}{2}\frac{d}{dt}(Ka^2)-\frac{\mu_0\epsilon}{2}\frac{d^2}{dt^2}(Ka^2). \quad (12)$$

In particular, for slowly expanding plasmas, $K(t)$ is nearly a constant:

$$K(t)\approx-\frac{3}{2}\frac{B_0}{\mu_0}.$$

This will be true in cases of practical interest, for the term $\frac{\mu_0\sigma}{2}\frac{d}{dt}(Ka^2)$ is of order $\frac{\mu_0\sigma a^2}{\Delta t}$ smaller than the leading term in (12). This is very small. The third term of (12) is of even smaller order.

Thus, the current density

$$\overline{J}(\bar{r}, t)=-\frac{3}{2}\frac{B_0}{\mu_0}\sin\theta'\delta(r'-a)\hat{\phi}' \quad (13)$$

will approximately cancel the earth's magnetic field inside the plasma. It can be readily checked by expanding the integrand in (10) in powers of βr that this expression for the current density will lead to a field $\overline{B}(a, t)$ at the boundary of the plasma whose normal component differs from $-\cos\theta' B_0$ by a term in the integrand of order $(\beta a)^2$ smaller than the leading term. A similar calculation shows that the second boundary condition is also satisfied to the same order of magnitude. For frequencies near 1 kc/s, an earth conductivity of 10^{-2} mho/m, and a maximum sphere radius of 25 meters, the error of order $(\beta a)^2$ in the integrand is about three percent of the leading term.

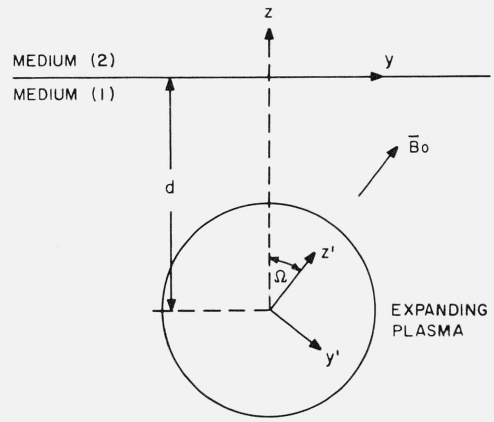

FIGURE 2. *Illustrating geometry for electric field calculation for an expanding plasma.*

To calculate the electric field at the earth's surface produced by the expanding plasma, it is necessary to write (13) in terms of the coordinate system in which the Green's tensor is expressed. Let the earth's magnetic field be inclined at an angle Ω to the vertical axis, as is illustrated in figure 2. The primed system of coordinates is also illustrated with the x and x' axes oriented so that they coincide. Then the y' axis is in the $y-z$ plane. The following equations hold:

$$\hat{x}'=\hat{x}$$
$$\hat{y}'=\cos\,\Omega\hat{x}-\sin\,\Omega\hat{z}.$$

Furthermore, the vector $\hat{\phi}'$ which occurs in the expression for the current density is given by

$$\hat{\phi}'=-\sin\,\phi'\hat{x}'+\cos\,\phi'\hat{y}'=-\sin\,\phi'\hat{x}+\cos\,\phi'\,\cos\,\Omega\hat{y}-\cos\,\phi'\,\sin\,\Omega\hat{z}.$$

We want to express the quantity $\sin\,\theta'\phi'$ in terms of the unprimed variables. Now,

$$\sin\,\theta'\,\sin\,\phi'=\hat{r}'\cdot\hat{y}'=\sin\,\theta\,\sin\,\phi\,\cos\,\Omega-\cos\,\theta\,\sin\,\Omega$$

and

$$\sin\,\theta'\,\cos\,\phi'=\hat{r}'\cdot\hat{x}'=\sin\,\theta\,\cos\,\phi.$$

Thus,

$$\sin\,\theta'\hat{\phi}'=[-\sin\,\theta\,\sin\,\phi\,\cos\,\Omega+\cos\,\theta\,\sin\,\Omega]\hat{x}+\sin\,\theta\,\cos\,\phi\,\cos\,\Omega\hat{y}-\sin\,\theta\,\cos\,\phi\,\sin\,\Omega\hat{z}. \quad (14)$$

The vertical component of the electric field at the earth's surface can be found from (4), (8), (13), and (14). It is convenient to perform the volume integral indicated in (4) before attempting the ξ, η integrals in the expression for $\overline{\overline{\Gamma}}$. Thus, we write

$$E_z=-\frac{\omega\mu_0}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\gamma_1\xi p_x+\gamma_1\eta p_y-\{\xi^2+\eta^2\}p_z]\cdot\frac{e^{-\gamma_1 d}}{N}\,e^{-i\xi x-i\eta y}d\xi d\eta \quad (15)$$

where

$$p_\alpha=\int_{\text{sphere}}J_\alpha(\omega)e^{i\xi x+i\eta y+\gamma_1(z+d)}dV. \quad (16)$$

In this equation $\alpha=x$, y, or z and $J_\alpha(\omega)$ is the Fourier transform of $J_\alpha(t)$. We have also set $z=0$ in the equation for E_z and have assumed that the center of the expanding plasma is at a depth d below the interface.

The integrals in (16) are to be evaluated using (13) and (14). Then p_x, for example, is given by

$$p_x=-\frac{3}{2}\frac{B_0}{\mu_0}\int F[\delta(r-a)]r^2dr\int_0^{\pi} d\theta \sin\theta e^{\gamma_1 r\cos\theta}$$

$$\cdot\int_0^{2\pi} d\phi[-\sin\theta\sin\phi\cos\Omega+\cos\theta\sin\Omega]\cdot e^{i\lambda r\sin\theta\cos(\phi-\alpha)}$$

where $F[\delta(r-a)]$ is the time Fourier transform of $\delta(r-a(t))$ and we have introduced the spherical coordinates r, θ, ϕ centered at the plasma center (z axis vertical). The parameters λ and α are defined by the equations

$$\xi=\lambda\cos\alpha$$

$$\eta=\lambda\sin\alpha.$$

The ϕ integration is readily performed and we obtain

$$p_x=-\frac{3}{2}\frac{B_0}{\mu_0}\int F[\delta(r-a)]r^2dr\int_0^{\pi} d\theta e^{\gamma_1 r\cos\theta}$$

$$\cdot[-\sin^2\theta\cos\Omega 2\pi i\sin\alpha J_1(\lambda r\sin\theta)+\sin\theta\cos\theta\sin\Omega 2\pi J_0(\lambda\sin\theta)]$$

where J_0 and J_1 denote Bessel functions of the zeroth and first order. To perform the θ integration, we use the following result listed in Morse and Feshbach [1953b]:

$$\int_0^{\pi} e^{iz\cos\theta\cos u}J_m(z\sin\theta\sin u)P_n^m(\cos u)\sin u du=i^{n-m}\sqrt{\frac{2\pi}{z}}P_n^m(\cos\theta)J_{n+1/2}(z)$$

where P_n^m is an associated Legendre polynomial. By using this result, one finds for p_x, and in the same manner, p_y and p_z

$$p_x=-\frac{4\pi}{\beta_1}[-i\lambda\sin\alpha\cos\Omega+\gamma_1\sin\Omega]\frac{3}{2}\frac{B_0}{\mu_0}\int F[\delta(r-a)]j_1(\beta_1 r)r^2dr$$

$$p_y=-\frac{4\pi i\lambda}{\beta_1}\cos\alpha\cos\Omega\frac{3}{2}\frac{B_0}{\mu_0}\int F[\delta(r-a)]j_1(\beta_1 r)r^2dr$$

$$p_z=\frac{4\pi i\lambda}{\beta_1}\sin\Omega\cos\alpha\frac{3}{2}\frac{B_0}{\mu_0}\int F[\delta(r-a)]j_1(\beta_1 r)r^2dr$$

where j_1 is the spherical Bessel function of order one.

Substituting these results into (15), we find, after performing the α integration and making the substitutions $\rho^2=x^2+y^2$, $\tan\theta=\frac{y}{x}$ and using the fact that $\frac{\partial}{\partial\rho}J_0(\lambda\rho)=-J_1(\lambda\rho)\lambda$ that

$$E_z(\rho,\theta,d,\omega)=-2i\omega\mu_0\beta_1\sin\Omega\cos\theta\frac{3}{2}\frac{B_0}{\mu_0}\cdot\int F[\delta(r-a)]j_1(\beta_1 r)r^2dr\frac{\partial}{\partial\rho}\int_0^{\infty}\frac{e^{-\gamma_1 d}}{\gamma_1^2\beta_2^2+\gamma_2\beta_1^2}J_0(\lambda\rho)\lambda d\lambda \quad (17)$$

The integral over λ in (17) represents the effect of propagation over the earth and has occurred in many studies of radio propagation over a conducting half-space. For distances $\rho\leq 50$ km (but much larger than the skin depth Im β_1^{-1}) and for frequencies $f=\frac{\omega}{2\pi}<10^3$ cps, one can show that [Space-General Corporation, 1960]

$$\frac{\partial}{\partial\rho}\int_0^{\infty}\frac{e^{-\gamma_1 d}}{\gamma_1\beta_2^2+\gamma_2\beta_1^2}J_0(\lambda\rho)\lambda d\lambda\approx\frac{e^{i\beta_1 d}}{\beta_1^2}\left[-i\beta_2+\frac{1}{\rho}\right]\frac{e^{i\beta_2\rho}}{\rho} \quad (18)$$

correct to within 15 percent.

To evaluate the remaining integral in (17), it is necessary to be more specific about the nature of the source. According to Taylor [1950], the radius of the plasma increases with time (at least during the initial instants) according to the formula

$$a(t)=At^{2/5} \qquad (0\leq t) \tag{19}$$

where A is a constant. We will assume that this expansion continues until a time $t_{\max}$ at which time the plasma radius is R and then stops abruptly.

With this expression for $a(t)$, it is convenient to perform the r integration before performing the Fourier transform in the source integral in (17). Thus,

$$\begin{aligned}\int_0^R F[\delta(r-a)]j_1(\beta_1 r)r^2dr &= \frac{1}{\sqrt{2\pi}}\int_0^\infty dt e^{i\omega t}\int_0^R dr r^2 j_1(\beta_1 r)\delta(r-a(t)) \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty a^2(t)j_1(\beta_1 a(t))e^{i\omega t}dt \\ &= \frac{1}{\sqrt{2\pi}}\left\{\frac{iR^2 j_1(\beta_1 R)e^{i\omega t_{\max}}}{\omega+i\epsilon}+\int_0^{t_{\max}} a^2 j_1(\beta_1 a)e^{i\omega t}dt\right\}.\end{aligned}$$

The last form results as a consequence of our assumption that $a(t)=R$ for $t\geq t_{\max}$. The quantity ϵ is a small positive number whose purpose is to shift the pole off of the axis of integration when $\frac{1}{\omega+i\epsilon}$ is multiplied by a function and integrated over ω. Because of the multiplying factor ω in (17), no improper integrals will occur in our problem and we can set $\epsilon=0$ in all that follows.

Because of the attenuation factor $e^{-\mathrm{Im}\,\beta_1 d}$ which occurs in (18), high frequencies will be attenuated very rapidly in the earth. Furthermore, the maximum radius R of the plasma

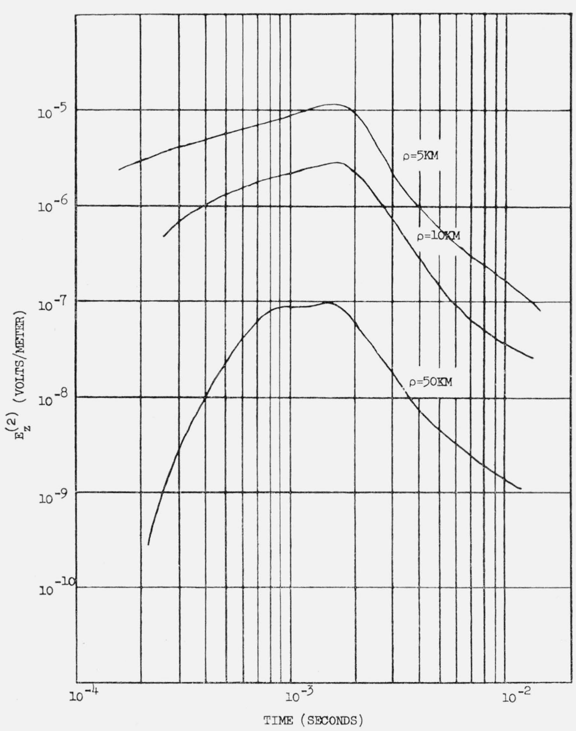

FIGURE 3. *Electric field strength vs time after beginning of expansion.*

will not be large in most conceivable cases. Thus $\beta_1 R<1$ and we can expand the spherical Bessel functions $j_1(\beta_1 a)$ and $j_1(\beta_1 R)$ in a power series, keeping only the first term. When this is done, one can write, after collecting the above results

$$E_z(\rho, \theta, d, \omega)=\frac{1}{\sqrt{2\pi}} B_0 R^3 \sin \Omega e^{i\beta_1 d} \frac{e^{-\beta_2 \rho}}{\rho}\left[-i\beta_2+\frac{1}{\rho}\right] A(\omega) \cos \theta$$

where

$$A(\omega)=ie^{i\omega t_{\max}}+\omega \int_0^{t_{\max}} \left(\frac{a}{R}\right)^3 e^{i\omega t} dt. \tag{20}$$

Finally, the time dependent field is

$$E_z(\rho, \theta, d, t)=\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_z(\rho, \theta, d, \omega) e^{-i\omega t} d\omega. \tag{21}$$

Figure 3 illustrates some results obtained by use of (20) and (21). The conductivity of the earth was set to $\sigma=10^{-2}$ mho/m. B_0 was taken equal to 0.5×10^{-4} webers/m^2 and $\sin \Omega=0.707$. The depth of the plasma was taken as $d=165$ meters and the plasma was assumed to expand to a radius $R=26$ meters in a time $t_{\max}=1.8\times 10^{-3}$ sec while obeying the law expressed by (19). The time integral was evaluated numerically on an IBM 704 computer. Figure 3 presents the vertical component of the electric field pulse at the earth's surface for several distances ρ evaluated at $\theta=0$.

4. Conclusion

The tensor Green's function for the electric field in a conducting half-space has been obtained. With the aid of this function the calculation of the electric field generated by an arbitrary current distribution imbedded in a conducting half-space is reduced to evaluating an integral. The formalism was illustrated by an example which indicated the usefulness of this approach.

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