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Wave Propagation in a Compressible Ionosphere. Part I

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In this paper, which consists of two parts, an extension of the magneto-ionic theory is systematically developed by employing the linearized, single fluid, continuum theory of plasma dynamics. The cubic equation specifying the square of the phase velocity of a plane wave, whose direction of propagation makes an arbitrary angle with that of the static external magnetic field, is derived. In the first approximation in which only the phase velocities of the order of the velocity of the electromagnetic waves in free space or higher are sought, the general cubic equation is found to degenerate into a variant of the well-known Appleton-Hartree equation. The regions of the normalized source frequency Ω and the normalized gyromagnetic frequency R, in which the ordinary and the extraordinary, purely transverse electromagnetic modes propagate are discussed with the help of a construction in the Ω^2-R^2 plane and with emphasis on the procedure used in the designation of the two modes. It is found that the dispersion curves for the arbitrary direction of propagation do not continuously go over to those for the direction parallel to that of the static magnetic field. An inconsistency in the designation procedure for the modes is found to exist in the magneto-ionic approximation.

1. Introduction

A vast majority of the investigations on the characteristics of wave propagation in the ionosphere are based on the application of the magneto-ionic theory. An important assumption of the magneto-ionic theory is that the plasma of the ionosphere is incompressible, with the result its macroscopic properties are adequately described by considering it to be a dielectric medium characterized by a tensor dielectric constant. The magneto-ionic theory has been successful in explaining the variety of wave phenomenon in the ionosphere, especially in situations where the effect of the alternating electric field predominates over that of the pressure gradient. This success is to be expected since, in the above circumstances, the dielectric description of the plasma is a reasonably good approximation. When the effect of the pressure gradient is not negligible, the magneto-ionic theory fails and consequently requires extension. The extension of the magneto-ionic theory by employing a linearized, single fluid continuum theory of plasma dynamics is systematically developed in this paper by considering the plasma of the ionosphere to be an unbounded, collision-free, and macroscopically neutral mixture of gas of mobile electrons and stationary ions.

At the outset, a brief discussion on the notation used in this paper is in order, especially since it differs from the conventional notation employed in the ionospheric literature [Ratcliffe, 1959; Budden, 1961]. In problems concerning the wave propagation in the ionosphere, three frequencies occur; namely, (i) the source frequency, ω, (ii) the plasma frequency, ω_p, and (iii) the gyromagnetic frequency, ω_c. In the ionospheric literature, the following normalized parameters are in common usage:

$$X=\frac{\omega_p^2}{\omega^2} \text{ and } Y=\frac{\omega_c}{\omega}.$$

In the analysis of the properties of wave propagation in the ionosphere, the determination of the propagation characteristics *explicitly* as a function of the source frequency, ω, is almost always desired. Such a determination is rendered unnecessarily cumbersome by adhering to the normalized parameters X and Y, since the source frequency is mixed in both of them. Therefore, instead of using the conventional notation, the following normalized frequencies are used:

$$\Omega=\frac{\omega}{\omega_p} \text{ and } R=\frac{\omega_c}{\omega_p}.$$

In this notation, Ω is the normalized source frequency and R is the normalized magnitude of the external magnetic field. The use of the normalized parameters, Ω and R, has the additional advantage of effecting considerable simplification of the coefficients in the dispersion equation, as will be seen in the next section.

There have been previous investigations of the nature of plane wave propagation in an unbounded, homogeneous, compressible plasma with an external magnetic field. These investigations are primarily restricted to two special directions of propagation; namely, those along and across that of the static magnetic field. An elegant presentation of the results for these two special cases may be found in the recent book by Allis, Buchsbaum, and Bers [1962]. In this paper, first, a brief derivation of the dispersion equation for the propagation of a plane wave at an arbitrary angle to the direction of the static magnetic field is given. The dispersion equation, which is essentially a cubic, is found to factor out into a linear and a quadratic equations for the special cases of propagation along and across the static magnetic field. These two special cases are briefly reviewed in this paper, partly for the sake of completeness and partly to emphasize the new construction in the Ω^2-R^2 space, which is used in elucidating the regions of Ω and R in which the two modes propagate. This new construction serves to localize the regions in the $\Omega-R$ space where the validity of the magneto-ionic theory is of suspect.

It is found that, except in a boundary layer about the direction of the static magnetic field, in the first approximation wherein the phase velocity of the order of the velocity of the electromagnetic waves in free space, c_0 or higher, is sought, the exact dispersion equation degenerates into a quadratic equation which is just a variant of the well-known Appleton-Hartree equation. The analysis of the above quadratic equation shows that the phase velocity is everywhere of the order c_0 or higher, except in the vicinities of the frequencies $\Omega=0$, Ω_3, and Ω_4, where it goes to zero. It is shown that Ω_3^2 and Ω_4^2 are, respectively, the smaller and the larger of the two roots of the equation $\Omega^4-\Omega^2(1+R^2)+R^2 l^2=0$, where $l=\cos\theta$ and θ is the angle which the propagation direction makes with that of the static magnetic field. Therefore, the results of the first approximation may not be valid in the neighborhood of the three frequencies, $\Omega=0$, Ω_3, and Ω_4.

A closer examination of the exact dispersion equation for $\Omega=0$ shows that the validity of the results of the first approximation is not impaired even though the phase velocity goes to zero at that frequency.

In the second approximation wherein the phase velocity of the order of the acoustic velocity a, in the electron gas or lower, is sought, the exact dispersion equation reduces to a linear equation. The study of the above linear equation shows that the dispersion curve has two branches. The first branch starts with a value of infinity for $\Omega=\Omega_3$, reduces progressively as Ω is increased, and goes to zero for $\Omega=Rl$. The second branch starts again with a value of infinity for $\Omega=\Omega_4$, reduces progressively as Ω is increased, and attains asymptotically the value a for $\Omega=\infty$. Obviously the results of the second approximation are also not valid in the neighborhood of the frequencies $\Omega=\Omega_3$ and $\Omega=\Omega_4$.

A third approximation wherein the phase velocity is of the order $\sqrt{ac_0}$ is, therefore, sought, and the exact dispersion equation degenerates then into another quadratic equation. The analysis of this quadratic equation yields finite, nonzero values for the phase velocity for $\Omega=\Omega_3$ and Ω_4, and these phase velocities are of the order $\sqrt{ac_0}$ and hence are an excellent approximation for the phase velocities at $\Omega=\Omega_3$ and Ω_4. Moreover, it is found that the phase velocities obtained in the third approximation merge with those given by the first and the second approximations, as the frequencies are sufficiently removed from the values Ω_3 and Ω_4, to the lower or the higher sides, respectively. From a synthesis of all the results given by the first, the second, and the third approximations, emerge the complete dispersion curves which are valid for all directions of propagation except very close to that of the static magnetic field. In all, there are three modes and these have been named as the modified plasma mode, the modified ordinary electromagnetic mode, and the modified extraordinary electromagnetic mode, respectively. Only the modified extraordinary electromagnetic mode has two branches.

In the axial boundary layer, which is a narrow region in which the propagation direction is very close to that of the static magnetic field, the dispersion curves undergo very rapid changes as a result of the decoupling of the plasma mode. Consequently, dispersion curves which are obtained for the arbitrary direction of propagation to that of the magnetic field do not uniformly go over to those for the propagation direction along that of the static magnetic field. It is found that the dispersion in the axial boundary layer undergoes rapid changes in the neighborhood of the two frequencies $\Omega=R$ and $\Omega=1$. By using boundary layer techniques, the dispersion equations in the boundary layer in the close neighborhood of the two frequencies $\Omega=R$ and $\Omega=1$ are derived and analyzed. This analysis clarifies the manner of coupling between the longitudinal plasma waves and the transverse electromagnetic waves for the case of propagation in the directions other than that of the static magnetic field. When the propagation direction is changed from that of the magnetic field, for the case $R^2<1$, the ordinary electromagnetic mode couples into the plasma mode and vice versa, in the close neighborhood of the frequency $\Omega=1$, and when the phase velocity is approximately equal to c_0. For the case $R^2>1$, in addition to the above coupling, the plasma mode couples into the extraordinary electromagnetic mode (Whistler mode) and vice versa, in the close neighborhood of the frequency $\Omega=1$ and when the phase velocity is slightly lower than c_0. Also in the neighborhood of the frequency $\Omega=R$, when both the extraordinary electromagnetic mode and the plasma mode are slowed down to approximately the acoustic velocity a in the electron gas, they couple back into each other. The smaller and the larger of the two frequencies $\Omega=1$ and $\Omega=R$ are found to be respectively the approximations to the frequencies $\Omega=\Omega_3$ and $\Omega=\Omega_4$ in the axial boundary layer. As the propagation direction is changed so as to approach that perpendicular to the static magnetic field, the only essential changes in the dispersion are the decrease of Ω_3 and the increase of Ω_4.

The general dispersion equation is briefly derived in section 2. In sections 3 and 4, the two special cases corresponding to the direction of propagation along and across that of the static magnetic field are discussed. The first order approximation to the exact dispersion equation is treated in section 5. The higher order approximations for the region external to the axial boundary layer as well as the dispersion in the boundary layer are contained in part II of this paper.

2. Formulation of the Problem

Consider an unbounded, homogeneous plasma which, for the sake of simplicity, is assumed to be a lossless and macroscopically neutral mixture of gas of mobile electrons and immobile ions. It is proposed to restrict attention only to the linear, time-harmonic problem; the harmonic time dependence of the form $e^{-i\omega t}$ is implied for all the field components. It is further assumed that the drift velocity of the electrons is zero so that the plasma as a whole may be considered as stationary. A uniform magnetic field B_0 is assumed to be impressed externally throughout the plasma region in the z-direction, where x, y, and z form a right-handed rectangular coordinate system (fig. 1).

Let N_0 be the average number density, P the pressure deviation from the mean, and $\vec{V}$ the velocity of the electrons. Let $\vec{E}$ and $\vec{H}$ be the alternating electric and magnetic fields. The linearized time-harmonic hydrodynamic equation of motion for the electrons is

$$-i\omega m N_0 \vec{V} = N_0 e(\vec{E} + \vec{V} \times \hat{z} B_0) - \nabla P \tag{1}$$

where e is the charge and m is the mass of an electron. The equation of continuity after being linearized and combined with the equation of state is given by

$$a^2 m N_0 \nabla \cdot \vec{V} = i\omega P \tag{2}$$

where a is the velocity of sound in the electron gas. In addition, the electric and the magnetic fields satisfy the following time-harmonic Maxwell's equations

$$\nabla \times \vec{E} = i\omega\mu_0 \vec{H} \tag{3}$$

$$\nabla \times \vec{H} = -i\omega\epsilon_0 \vec{E} + N_0 e \vec{V} \tag{4}$$

where μ_0 and ϵ_0 are the permeability and the dielectric constant of free space.

It is desired to investigate the characteristics of a plane wave propagating in the plasma medium. Let $\vec{k}$ be the propagation vector such that $\hat{z} \times \vec{k}$ coincides with the y-axis and let θ be the angle that $\vec{k}$ makes with the direction of the static magnetic field, so that $k_x = k \sin\theta = kn$, $k_y = 0$ and $k_z = k \cos\theta = kl$. Therefore all the field components will have the spatial dependence of the form $e^{ik(nx+lz)}$ so that

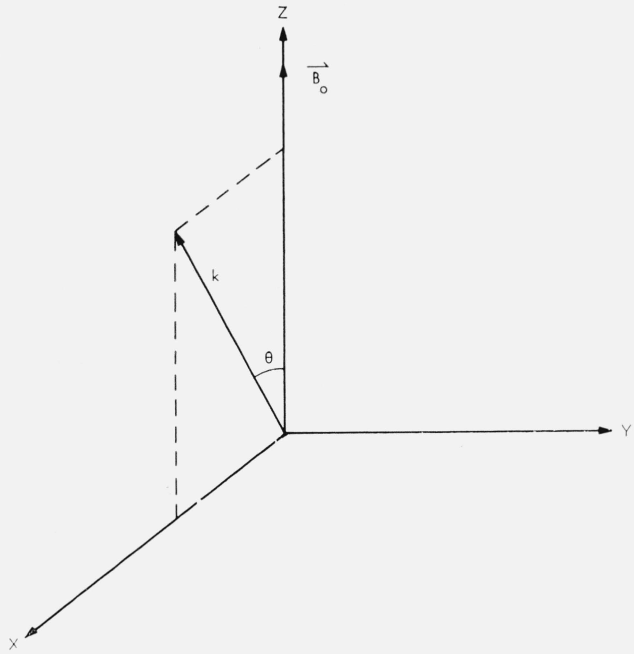

FIGURE 1. *Geometry of the problem.*

$$\frac{\partial}{\partial x} = ikn, \qquad \frac{\partial}{\partial y} = 0; \qquad \frac{\partial}{\partial z} = ikl. \tag{5}$$

The elimination of P from (1) with the help of (2) and the use of (5) leads, after some rearrangement, to the following simultaneous equations specifying V_x, V_y, and V_z in terms of E_x, E_y, and E_z:

$$\left(1 - \frac{k^2a^2n^2}{\omega^2\alpha}\right) V_x - \frac{k^2a^2nl}{\omega^2\alpha} V_z = \frac{ie}{\omega m\alpha} E_x + \frac{e\omega_c}{\omega^2 m\alpha} E_y$$

$$-\frac{i\omega_c}{\omega}\frac{k^2a^2n^2}{\omega^2\alpha} V_x + V_y - \frac{i\omega_c}{\omega}\frac{k^2a^2nl}{\omega^2\alpha} V_z = -\frac{e\omega_c}{\omega^2 m\alpha} E_x + \frac{ie}{\omega m\alpha} E_y$$

$$-\frac{k^2a^2nl}{\omega^2} V_x + \left(1 - \frac{k^2a^2l^2}{\omega^2}\right) V_z = \frac{ie}{\omega m} E_z, \tag{6a, b, c}$$

where

$$\alpha = 1 - \frac{\omega_c^2}{\omega^2} \tag{7}$$

and $\omega_c = -\frac{eB_0}{m}$ is the gyromagnetic frequency of the electrons. The substitution of V_x, V_y, and V_z obtained from the solution of (6) into (4) enables the right-hand side of (4) to be rewritten as $-i\omega\epsilon_0 \overleftrightarrow{\epsilon} \cdot \vec{E}$, where $\overleftrightarrow{\epsilon}$ is the dielectric tensor. Then, from (3) and (4), it is found that

$$\nabla \times \nabla \times \vec{E} = \frac{\omega^2}{C_0^2} \overleftrightarrow{\epsilon} \cdot \vec{E} \tag{8}$$

where $C_0=\frac{1}{\sqrt{\mu_0\epsilon_0}}$ is the velocity of electromagnetic waves in free space. On writing (8) in component form, the following set of equations specifying E_x, E_y, and E_z is obtained:

$$\begin{bmatrix} D_{11} & iD_{12} & D_{13} \\ -iD_{21} & D_{22} & -iD_{23} \\ D_{31} & iD_{32} & D_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = [D][E]=0 \tag{9}$$

where

$$D_{11}=\frac{k^2l^2C_0^2}{\omega^2}-1+\frac{\omega_p^2}{\omega^2\alpha D}\left(1-\frac{k^2a^2l^2}{\omega^2}\right)$$

$$D_{12}=D_{21}=-\frac{\omega_p^2}{\omega^2}\frac{\omega_c}{\omega\alpha D}\left(1-\frac{k^2a^2l^2}{\omega^2}\right)$$

$$D_{13}=D_{31}=-\frac{k^2lnC_0^2}{\omega^2}+\frac{\omega_p^2}{\omega^2\alpha D}\frac{k^2a^2nl}{\omega^2}$$

$$D_{22}=\frac{k^2C_0^2}{\omega^2}-1+\frac{\omega_p^2}{\omega^2\alpha D}\left(1-\frac{k^2a^2}{\omega^2}\right)$$

$$D_{23}=D_{32}=-\frac{\omega_p^2}{\omega^2}\frac{\omega_c}{\omega\alpha D}\frac{k^2a^2nl}{\omega^2}$$

$$D_{33}=\frac{k^2n^2C_0^2}{\omega^2}-1+\frac{\omega_p^2}{\omega^2 D}\left(1-\frac{k^2a^2n^2}{\omega^2\alpha}\right)$$

$$D=1-\frac{k^2a^2}{\omega^2\alpha}\left(1-\frac{\omega_c^2}{\omega^2}l^2\right) \tag{10}$$

and ω_p is the plasma frequency of the electron and is given by $\omega_p^2=\frac{N_0e^2}{m\epsilon_0}$. The determinant of D should be equal to zero in order that (9) may have a nontrivial solution and this condition, together with the help of (10), after considerable manipulations leads to the following cubic equations in λ^2:

$$\lambda^6+A_1\lambda^4+A_2\lambda^2+A_3=0 \tag{11}$$

where $\lambda=\frac{\omega}{k}$ is the phase velocity of the plane waves propagating in the medium. The expressions for the coefficients A_1, A_2, and A_3 in (11) may be simplified to yield the following:

$$A_1=-\frac{\Omega^2C_0^2}{A_0}[\{2\Omega^4-2\Omega^2(2+R^2)+2+R^2(1+l^2)\} + \delta\{\Omega^4-\Omega^2(2+l^2R^2)+1\}]=\frac{A_{10}}{A_0} \tag{12a}$$

$$A_2=\frac{\Omega^2C_0^4}{A_0}[\Omega^4-\Omega^2(1+R^2)+R^2l^2 + 2\Omega^2\delta\{\Omega^2-1-l^2R^2\}]=\frac{A_{20}}{A_0} \tag{12b}$$

$$A_3=-\frac{\Omega^4C_0^4a^2(\Omega^2-l^2R^2)}{A_0}=\frac{A_{30}}{A_0} \tag{12c}$$

$$A_0=(\Omega^2-1)[\Omega^4-\Omega^2(2+R^2)+1] \tag{12d}$$

where

$$\delta=\frac{a^2}{C_0^2},\ \Omega=\frac{\omega}{\omega_p},\ R=\frac{\omega_c}{\omega_p}. \tag{13}$$

3. Propagation Along the Static Magnetic Field

Before proceeding to obtain the characteristics of plane waves propagating at an arbitrary angle θ with respect to the direction of the external magnetic field, the characteristics of plane waves propagating along and across the static magnetic field will be briefly reviewed. For propagation along the magnetic field, $\theta=0$ and the dispersion equation (11) may be factored into the following two equations:

$$\lambda^2-\frac{a^2\Omega^2}{\Omega^2-1}=0 \tag{14a}$$

and

$$\lambda^4+B_1\lambda^2+B_2=0 \tag{14b}$$

where

$$B_1=-\frac{2\Omega^2C_0^2(\Omega^2-1-R^2)}{B_0}=\frac{B_{10}}{B_0} \tag{15a}$$

$$B_2=\frac{\Omega^2C_0^4(\Omega^2-R^2)}{B_0}=\frac{B_{20}}{B_0} \tag{15b}$$

$$B_0=[\Omega^4-\Omega^2(2+R^2)+1]. \tag{15c}$$

The phase velocity obtained from (14a) is given by

$$V_p=\frac{a\Omega}{\sqrt{\Omega^2-1}}. \tag{16}$$

The mode whose phase velocity is given by (16) propagates only for $\Omega>1$ and has a phase velocity equal to the acoustical velocity in the electron gas in the limit of infinite frequency. This mode is obviously the plasma mode. Note that (16) is independent of the static magnetic field and therefore the same as in an isotropic plasma. This is to be expected since the static magnetic field does not exert a force in a direction parallel to itself.

The dispersion equation (14b) is independent of a and hence is the same as in an incompressible plasma. It may be solved using a procedure to be described in connection with the case of propagation at an arbitrary angle and the resulting phase velocities of the ordinary and the extraordinary electromagnetic modes are given by

$$V_0 = C_0 \left[\frac{\Omega(\Omega+R)}{\Omega^2+\Omega R-1} \right]^{1/2} \tag{17a}$$

$$V_e = C_0 \left[\frac{\Omega(\Omega-R)}{\Omega^2-\Omega R-1} \right]^{1/2}. \tag{17b}$$

In figure 2a, the regions of propagation of the ordinary and the extraordinary modes are indicated. The ordinary electromagnetic mode is found to propagate for $\Omega > \Omega_1$ and the extraordinary electromagnetic mode for $\Omega < R$ and $\Omega > \Omega_2$, where

$$\Omega_{1,2} = \mp \frac{R}{2} + \sqrt{\left(\frac{R}{2}\right)^2 + 1}. \tag{18}$$

An examination of figure 2a reveals a fact which does not appear to have been emphasized previously in the literature. For $R^2 > 1/2$, there is at least one mode propagating in the entire frequency range. But for $R^2 < 1/2$, there is a frequency band which lies entirely below $\Omega^2 = 1$ and for which neither mode propagates. As R^2 is reduced from the value 1/2, this cutoff band increases progressively until finally for the isotropic case, this cutoff band extends from $\Omega = 0$ to $\Omega = 1$.

The phase velocities of the three modes, given respectively in (16) and (17a, b) are sketched in figure 2b for the case $R^2 < 1$ and in figure 2c for the case $R^2 > 1$. The ordinary electromagnetic mode is seen to have a phase velocity which starts with a value of infinity for $\Omega = \Omega_1$, becomes smaller as Ω is increased and attains asymptotically the free space value C_0 for infinite frequencies. The higher frequency branch of the extraordinary mode has a phase velocity which starts with infinity for $\Omega = \Omega_2$ and reduces asymptotically to the value C_0 for infinite frequencies. The low frequency branch of the extraordinary mode, usually referred to as the "whistler mode," has a phase velocity which starts

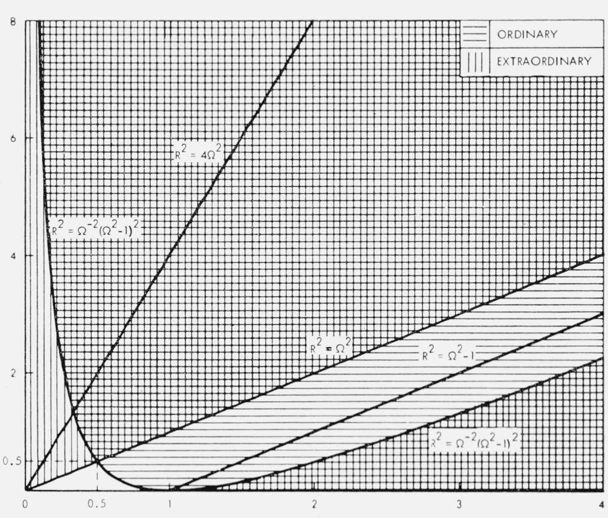

FIGURE 2a. *Regions of propagation of the ordinary and the extraordinary EM modes for propagation along the static magnetic field.*

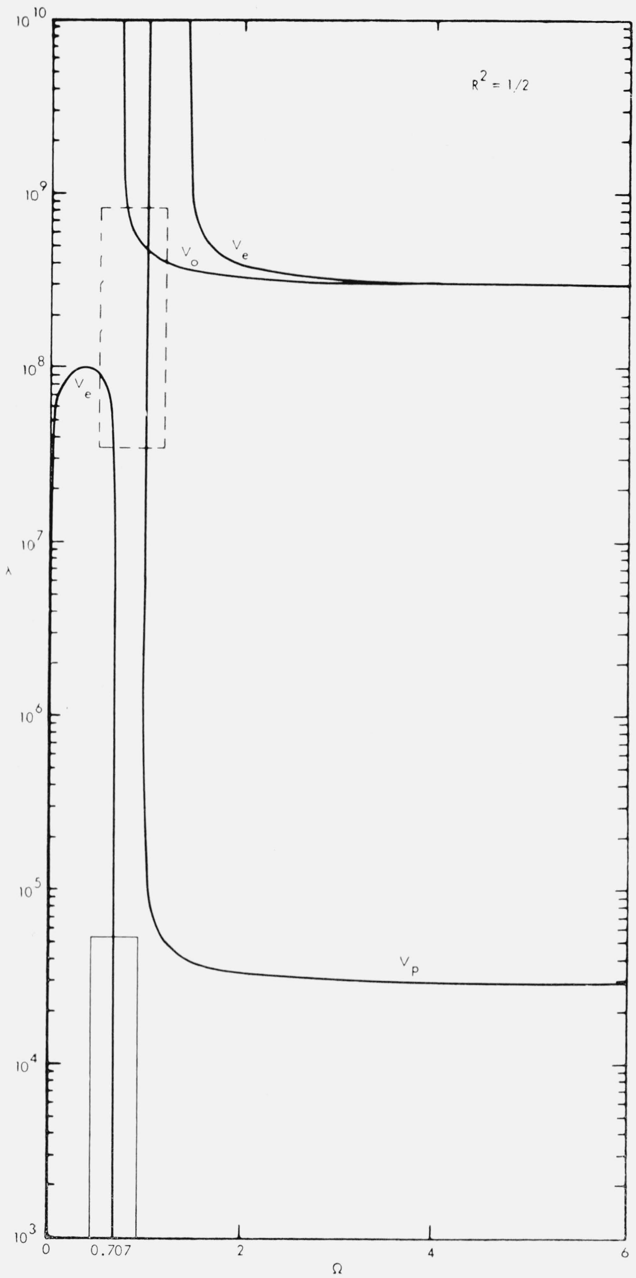

FIGURE 2b. *Dispersion curves for propagation along the direction of the static magnetic field* ($R^2 = \frac{1}{2}$).

with a value zero for $\Omega = 0$, increases and reaches a maximum value for $\Omega = R/2$ and for further increase in Ω, the phase velocity reduces and finally becomes zero for $\Omega = R$. The maximum value of the phase velocity which is attained for $\Omega = R/2$ is equal to $C_0 R(R^2+4)^{-1/2}$ which is always less than C_0 but approaches that value as R is increased. Only in the frequency range $0 \leq \Omega \leq R/2$, the phase velocity increases with frequency. In figure 2a, the region to the left of the critical line $R^2 = 4\Omega^2$, is the region of the "whistler mode," where the phase velocity increases with the frequency.

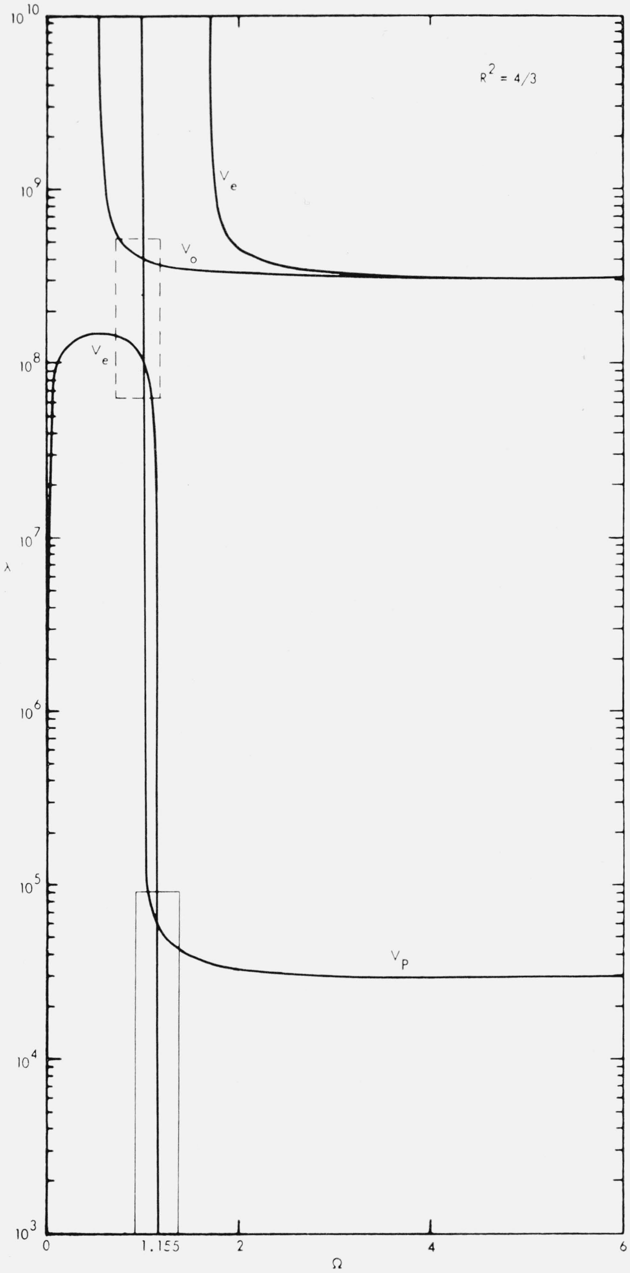

FIGURE 2c. *Dispersion curves for propagation along the direction of the static magnetic field* ($R^2=4/3$).

4. Propagation Across the Static Magnetic Field

For propagation across the magnetic field, $\theta=\Pi/2$ and the general dispersion equation (11) may be factored into the following two equations:

$$\lambda^2-\frac{C_0^2\Omega^2}{\Omega^2-1}=0 \tag{19a}$$

and

$$\lambda^4+B_1\lambda^2+B_2=0 \tag{19b}$$

where

$$B_1=\frac{B_{10}}{B_0}=-\frac{[\Omega^2C_0^2(\Omega^2-1-R^2)+\Omega^2a^2(\Omega^2-1)]}{B_0} \tag{20a}$$

$$B_2=\frac{B_{20}}{B_0}=\frac{a^2C_0^2\Omega^4}{B_0} \tag{20b}$$

$$B_0=\Omega^4-\Omega^2(2+R^2)+1. \tag{20c}$$

The phase velocity obtained from (19a) is given by

$$V_0=\frac{C_0\Omega}{\sqrt{\Omega^2-1}}. \tag{21}$$

The mode whose phase velocity is given by (21) propagates only for $\Omega>1$ and has its phase velocity asymptotically approach the free space electromagnetic wave velocity. Also this mode is the same as in an isotropic plasma. Evidently this mode is the ordinary electromagnetic mode with its electric vector parallel to the direction of the static magnetic field.

The dispersion equation (19b) contains both a and C_0 and hence pertains to modes obtained as a result of mixing of the extraordinary electromagnetic wave and the plasma wave. These modes will be designated as the modified extraordinary electromagnetic mode (MEM[x]) and the modified plasma mode (MP) and will be denoted by the subscripts mx and mp respectively. On solving (19b) and simplifying the resulting expressions by exploiting the fact that $a^2/C_0^2<<1$, the following expressions for the phase velocities of the modified extraordinary electromagnetic mode and the modified plasma mode are easily obtained:

$$V_{mx}=\left[\frac{C_0^2\Omega^2(\Omega^2-\Omega_3^2)}{B_0}\right]^{1/2} \text{ for } \Omega_2<\Omega<\infty \tag{22a}$$

where

$$\Omega_3^2=1+\frac{R^2}{1+\dfrac{a^2}{C_0^2}}\approx 1+R^2. \tag{22b}$$

It may be easily established that $\Omega_1^2<\Omega_3^2<\Omega_2^2$. Also,

$$V_{mp}=\left[\frac{C_0^2\Omega^2(\Omega^2-\Omega_3^2)}{B_0}\right]^{1/2} \text{ for } \Omega_1<\Omega<\Omega_3 \tag{23a}$$

$$=\Omega\sqrt{ac_0}\left[\frac{C_0}{2a}(\Omega^2-\Omega_3^2)+(-B_0)^{1/2}\right]^{-1/2}$$
$$\text{for } |\Omega-\Omega_3|\leq 0(10^{-4}) \tag{23b}$$

$$V_{mp}=\left[\frac{a^2\Omega^2}{(\Omega^2-\Omega_3^2)}\right]^{1/2} \text{ for } \Omega_3<\Omega<\infty. \tag{23c}$$

The phase velocity of the modified extraordinary electromagnetic mode starts with the value infinity for $\Omega=\Omega_2$, becomes smaller as Ω is increased and asymptotically approaches the free space electromagnetic wave velocity C_0 in the limit of infinite frequency. The phase velocity of the modified plasma mode starts with the value of infinity for $\Omega=\Omega_1$, becomes smaller as Ω is increased and reaches the value C_0 approximately for $\Omega=1$. At about $\Omega=\Omega_3$, the phase velocity rapidly falls and reaches the value of the acoustic velocity in the electron gas in the limit of infinite frequency. It is not difficult to verify that the modified plasma mode is a predominantly transverse wave for $\Omega_1<\Omega<\Omega_3$ and a predominantly longitudinal wave for $\Omega_3<\Omega<\infty$. Obviously the coupling between the transverse and the longitudinal waves takes place in the close neighborhood of $\Omega=\Omega_3$.

In figure 3a, the regions of propagation of the modified extraordinary electromagnetic mode and the modified plasma mode are indicated. Also the phase velocities given by (21), (22a), and (23) are sketched in figure 3b for the case $R^2=1/2$.

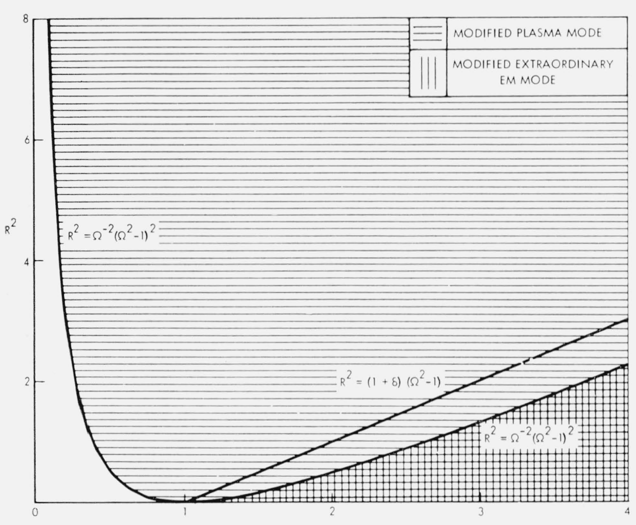

FIGURE 3a. *Regions of propagation of the modified plasma and the modified extraordinary EM modes for propagation across the static magnetic field.*

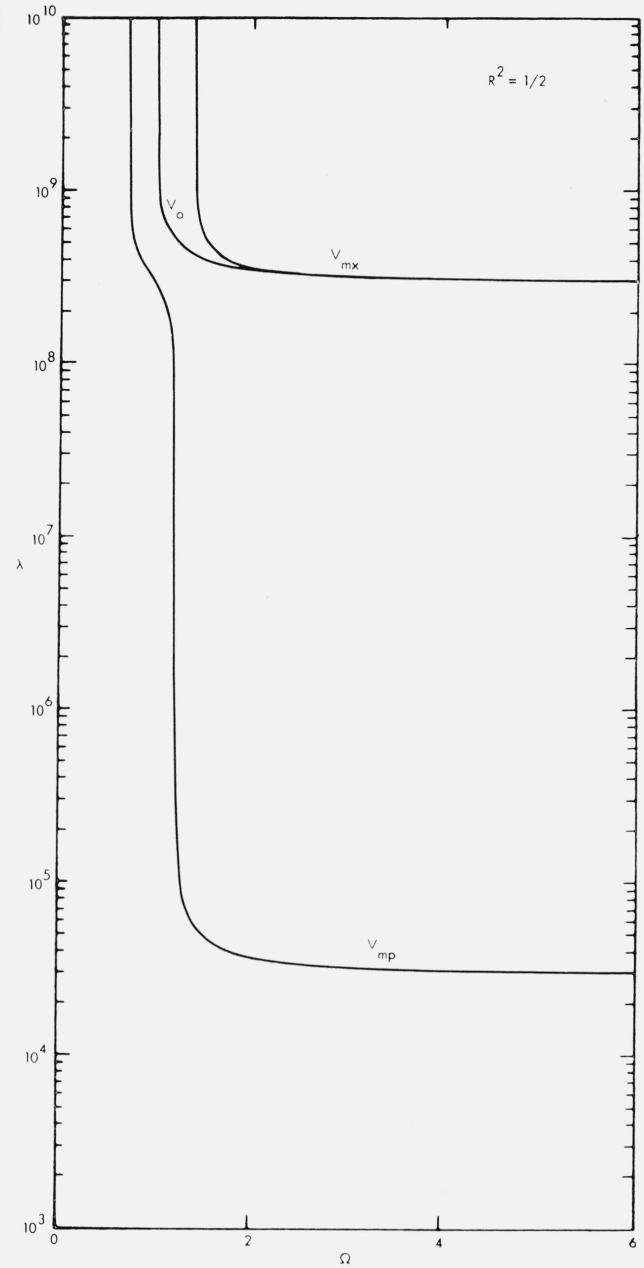

FIGURE 3b. *Dispersion curves for propagation across the direction of the static magnetic field* ($R^2=\frac{1}{2}$).

5. Propagation at an Arbitrary Angle to the Direction of the Static Magnetic Field (First Approximation)

Except for the cases of propagation along and across the static magnetic field, the dispersion equation (11) which is a cubic in λ^2, does not exactly factor out into two simpler equations. Nevertheless, it is possible to exploit the fact that $a^2/C_0^2<<1$ and perform a perturbation analysis of (11), which in essence leads to a separation of (11) into simpler equations which are either linear or quadratic in λ^2.

The free space electromagnetic wave velocity C_0 is approximately equal to 3×10^8 m/sec and the acoustic velocity in the electron gas is of the order 10^4 m/sec and may be taken for convenience to be equal to 3×10^4 m/sec. Hence $\delta=a^2/C_0^2=10^{-8}$ which is negligible in comparison with unity. It is convenient to rewrite (11) together with (12) as follows:

$$\lambda^6+\lambda^4\left[A_1^{(1)}+\frac{a^2}{C_0^2}A_1^{(2)}\right]+\lambda^2\left[A_2^{(1)}+\frac{a^2}{C_0^2}A_2^{(2)}\right]+A_3=0 \tag{24a}$$

where

$$A_1^{(1)}=-\frac{\Omega^2C_0^2}{A_0}[2\Omega^4-2\Omega^2(2+R^2)+2+R^2(1+l^2)]=\frac{A_{10}^{(1)}}{A_0} \tag{24b}$$

$$A_2^{(1)}=\frac{\Omega^2C_0^4}{A_0}[\Omega^4-\Omega^2(1+R^2)+R^2l^2]=\frac{A_{20}^{(1)}}{A_0} \tag{24c}$$

$$A_3^{(1)}=A_3=-\frac{\Omega^4C_0^4a^2(\Omega^2-l^2R^2)}{A_0}=\frac{A_{30}^{(1)}}{A_0} \tag{24d}$$

$$A_1^{(2)}=-\frac{\Omega^2C_0^2}{A_0}[\Omega^4-\Omega^2(2+R^2l^2)+1] \tag{24e}$$

and

$$A_2^{(2)}=\frac{\Omega^2C_0^4}{A_0}[2\Omega^2(\Omega^2-1-l^2R^2)]. \tag{24f}$$

If the phase velocity is of the order C_0, it is seen from (24a) that there are three terms of the order C_0^6 and the other three are of the order $C_0^6(a^2/C_0^2)$. Hence only the terms of the order C_0^6 in (24c) need be retained resulting in the following quadratic equation in λ^2:

$$\lambda^4+A_1^{(1)}\lambda^2+A_2^{(1)}=0. \tag{25}$$

It is to be noted that on setting a equal to zero in (11) and (12), (25) results. This is equivalent to saying that had the magneto-ionic theory been used from the outset, the dispersion equation (25) will be obtained. It is therefore obvious that (25) is a variant of the well-known Appleton-Hartree equation. It is proposed to analyze the dispersion equation in a more systematic way than has hitherto been done and then proceed to determine the perturbation caused by taking into account the compressibility of the medium.

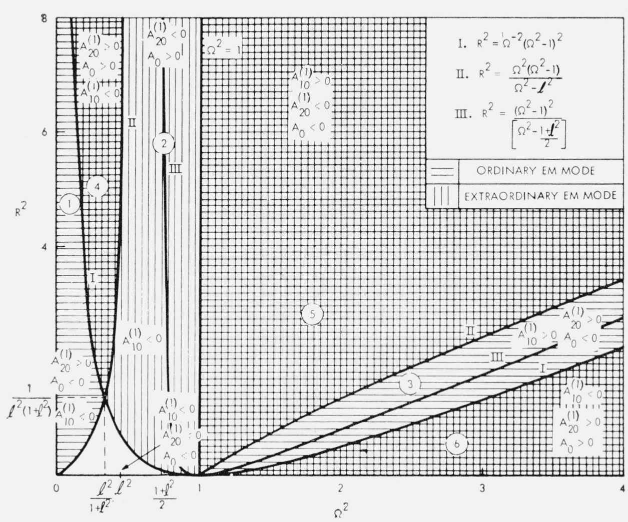

FIGURE 4. *Regions of propagation of the ordinary and the extraordinary EM modes for the arbitrary direction of propagation (first approximation).*

The solutions of (25) are given by

$$\lambda_{e,0}^2=-\frac{A_{10}^{(1)}}{2A_0}\pm\frac{1}{A_0}\sqrt{\left(\frac{A_{10}^{(1)}}{2}\right)^2-A_{20}^{(1)}A_0}. \tag{26}$$

The lower and the upper signs in (26) correspond respectively to the ordinary (0) and the extraordinary (e) electromagnetic modes. When

$$A_{20}^{(1)}A_0<0,\left|\frac{1}{A_0}\sqrt{\left(\frac{A_{10}^{(1)}}{2}\right)^2-A_{20}^{(1)}A_0}\right|>\left|\frac{A_{10}^{(1)}}{2A_0}\right|$$

and hence the solution (26) which corresponds to the positive sign in front of the radical will yield a positive value for λ^2 resulting in the propagation of the corresponding mode. In order to find out the region in the Ω^2-R^2 space in which $A_{20}^{(1)}A_0<0$, the critical lines corresponding to $A_0=0$ and $A_{20}^{(1)}=0$ are drawn in figure 4. The critical lines corresponding to $A_0=0$ are given by $\Omega^2=1$ and $R^2=\Omega^{-2}(\Omega^2-1)^2$ and that corresponding to $A_{20}^{(1)}$ are given by $\Omega^2=0$ and $R^2=\Omega^2(\Omega^2-1)/(\Omega^2-l^2)$. It may be easily shown that the line $R^2=\Omega^2(\Omega^2-1)/(\Omega^2-l^2)$ is above the line $R^2=\Omega^{-2}(\Omega^2-1)^2$ for $\Omega>1$. The signs of A_0 and $A_{20}^{(1)}$ in the various regions into which the Ω^2-R^2 space is divided by the above mentioned critical lines are indicated in figure 4. In the regions marked 1, 2, and 3, $A_{20}^{(1)}A_0<0$. In the regions 1 and 3, $A_0<0$ and hence, the propagating mode will correspond to the lower sign in (26) and is the ordinary electromagnetic mode. This is indicated by shading with the horizontal lines. In the region 2, $A_0>0$ and the propagating mode will correspond to the upper sign in (26) and is the extraordinary electromagnetic mode which is indicated by shading with the vertical lines.

Using (24a, b) and (12d), it can be shown that

$$\left(\frac{A_{10}^{(1)}}{2}\right)^2-A_{20}^{(1)}A_0=C_0^4\Omega^2R^2\left[(\Omega^2-1)^2l^2+\frac{\Omega^2R^2}{4}(1-l^2)^2\right]>0 \tag{27}$$

and hence $\lambda_{0,e}^2$ are always real. If $A_{20}^{(1)}A_0>0$,

$$\left|\frac{1}{A_0}\sqrt{\left(\frac{A_{10}^{(1)}}{2}\right)^2-A_{20}^{(1)}}\right|<\left|\frac{A_{10}^{(1)}}{2A_0}\right|$$

and hence the signs of both λ_e^2 and λ_0^2 will be the same as $A_{10}^{(1)}/A_0$. It is therefore evident that both the ordinary and the extraordinary electromagnetic modes will propagate if $A_{10}^{(1)}/A_0<0$. In figure 4, the critical lines corresponding to $A_0=0$ and $A_{20}^{(1)}=0$, and also the critical lines $\Omega^2=0$ and $R^2=(\Omega^2-1)/\left(\Omega^2-\frac{1+l^2}{2}\right)$ corresponding to $A_{10}^{(1)}=0$ are drawn. For $R^2>0$ the line $R^2=\Omega^2(\Omega^2-1)^2/\left(\Omega^2-\frac{1+l^2}{2}\right)$ may be shown to be always above the line $R^2=\Omega^{-2}(\Omega^2-1)^2$. Also the line $R^2=\Omega^2(\Omega^2-1)/(\Omega^2-l^2)$ is found to be above the line $R^2=(\Omega^2-1)^2/\left(\Omega^2-\frac{1+l^2}{2}\right)$ for $\Omega^2>1$. Furthermore, $l^2<\frac{1+l^2}{2}<1$. In the regions for which $A_{20}^{(1)}A_0>0$, the signs of both $A_{10}^{(1)}$ and A_0 are indicated in figure 4. It is found that $A_{10}^{(1)}/A_0<0$ in the regions marked 4, 5, and 6 and, therefore, both modes propagate in those regions. The line R^2=constant intersects the line $R^2=\Omega^2(\Omega^2-1)/(\Omega^2-l^2)$ for two values of Ω^2 given by

$$\Omega_{3,4}^2=\frac{1+R^2}{2}\mp\sqrt{\left(\frac{1+R^2}{2}\right)^2-R^2l^2}. \tag{28}$$

In (28), Ω_3^2 and Ω_4^2 correspond to the upper and the lower signs respectively and are always real. Also the line R^2=constant intersects the line $R^2=\Omega^{-2}(\Omega^2-1)^2$ for two values of Ω^2, namely Ω_1^2 and Ω_2^2 whose expressions are given in (18). From figure 4, it is clear that the ordinary electromagnetic mode propagates in the two frequency ranges $0<\Omega^2<\Omega_3^2$ and $1<\Omega^2<\infty$, and the extraordinary electromagnetic mode in the ranges $\Omega_1^2<\Omega^2<\Omega_4^2$ and $\Omega_2^2<\Omega^2<\infty$. For the sake of convenience in the redesignation of these modes after the modification resulting from the inclusion of the effect of the compressibility of the medium, the ordinary electromagnetic modes propagating in the frequency ranges $1<\Omega^2<\infty$ and $0<\Omega^2<\Omega_3^2$ are denoted by O^{I} and O^{II}, respectively, and the extraordinary electromagnetic modes propagating in the frequency ranges $\Omega_2^2<\Omega^2<\infty$ and $\Omega_1^2<\Omega^2<\Omega_4^2$ are denoted by e^{I} and e^{II}, respectively.

With the help of figure 4, at least one mode is seen to propagate in any frequency range, for $R^2>1/l^2(1+l^2)$. When $R^2<1/l^2(1+l^2)$, there is a frequency band which lies entirely below $\Omega^2=1$ and in which there is no propagation. As R^2 is decreased from the value $1/l^2(1+l^2)$, this cutoff band increases progressively in width until finally for the isotropic plasma this cutoff band extends from $\Omega^2=0$ to $\Omega^2=1$.

As the direction of propagation of the plane wave is changed so as to approach the direction normal to the static magnetic field, l^2 becomes smaller and as a consequence Ω_3^2 becomes smaller and Ω_4^2 becomes larger. In the limiting case of the propagation across the static magnetic field, $l^2=0$ and therefore, $\Omega_3^2=0$ and $\Omega_4^2=1+R^2$. Note that Ω_1^2 and Ω_2^2 do not depend on l^2 and therefore, on the direction of propagation of the plane waves. Therefore, as the propagation direction of the plane waves approaches the direction perpendicular to the static magnetic field, the frequency range of propagation of the ordinary electromagnetic mode O^{II} decreases and that of the extraordinary electromagnetic mode e^{II} increases progressively until for the limiting case of propagation across the static magnetic field, the O^{II} mode disappears completely and the propagation range of the e^{II} mode extends from $\Omega_1^2<\Omega^2<1+R^2$. It is to be noted that the frequency bands of propagation of both the O^{I} mode and the e^{I} mode remain unaltered as the propagation direction is changed.

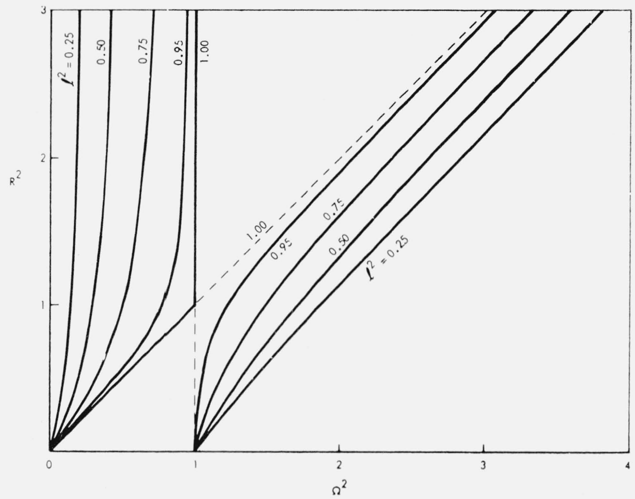

FIGURE 5. *Variation of the critical frequencies Ω_3 and Ω_4 with l.*

As the direction of propagation of the plane wave is changed so as to approach the direction of the static magnetic field, l^2 becomes progressively larger and reaches the limiting value of unity. As l^2 becomes larger, Ω_3^2 increases for all values of R^2 in such a way as to approach the value which is equal to the smaller of the two quantities, R^2 and 1, as shown in figure 5. Also, Ω_4^2 decreases continuously for all values of R^2 in such a way as to approach the value which is equal to the larger of the two quantities, R^2 and 1. In the limiting case of propagation along the static magnetic field, the critical line corresponding to Ω_3^2 for $R^2\leq 1$ and that corresponding to Ω_4^2 for $R^2\geq 1$ join to form the critical line $\Omega^2=R^2$ whereas the critical line $\Omega^2=1$ corresponding to that of Ω_4^2 for $R^2\leq 1$ and of Ω_3^2 for $R^2\geq 1$ disappears. It is found with the help of (12d) and (26) that when the critical line $\Omega^2=1$ disappears, the sign in front of the radical in (26) will be reversed for $\Omega^2<1$, with the result in the limiting case of propagation along the static magnetic field, the O^{II} and the e^{II} mode will become the extraordinary and the ordinary electromagnetic modes respectively. Finally the situation depicted in figure 2a is obtained and the ordinary electromagnetic mode propagates in the range $\Omega_1<\Omega<\infty$ and the extraordinary electromagnetic mode in the two frequency ranges $0<\Omega<R$ and $\Omega_2<\Omega<\infty$. The reason for the discontinuous disappearance of the critical line $\Omega^2=1$ for the case of propagation along the direction of the static magnetic field is clarified when the compressibility of the plasma is taken into account and is essentially due to the decoupling of the plasma mode, which remains coupled to the electromagnetic mode for all directions of propagation other than that of the static magnetic field.

The expressions for the phase velocities of the ordinary and the extraordinary electromagnetic modes may be written down explicitly with the help of (12d), (24b, c), (26), and (27) as follows:

$$\lambda^2_{e,0} = \frac{\Omega^2 C_0^2}{2(\Omega^2-1)[\Omega^4-\Omega^2(R^2+2)+1]}$$

$$\left[2\Omega^4-2\Omega^2(2+R^2)+2+R^2(1+l^2) \pm 2\frac{R}{\Omega}\left\{(\Omega^2-1)^2 l^2+\frac{\Omega^2R^2}{4}(1-l^2)^2\right\}^{\frac{1}{2}}\right]. \quad (29)$$

The upper sign in (29) corresponds to propagating extraordinary modes in the frequency ranges $\Omega_1<\Omega<\Omega_4$ and $\Omega_2<\Omega<\infty$ and the lower sign in (29), to the ordinary electromagnetic mode in the frequency ranges $0<\Omega<\Omega_3$ and $1<\Omega<\infty$. The phase velocities of the two modes as obtained from (29) are plotted (solid line) in figure 6a and b for for $l^2=1/2$ and for two values of R^2, namely (i) $R^2=1/2$ and (ii) $R^2=4/3$. The phase velocities of both the O^{I} and the e^{I} modes are seen to start with a value of infinity for $\Omega=1$ and $\Omega=\Omega_2$, respectively, decrease continuously as Ω is increased and asymptotically approach the velocity C_0 and hence are always higher than C_0. The phase velocities given by (29) are obtained from (25) which is an approximation

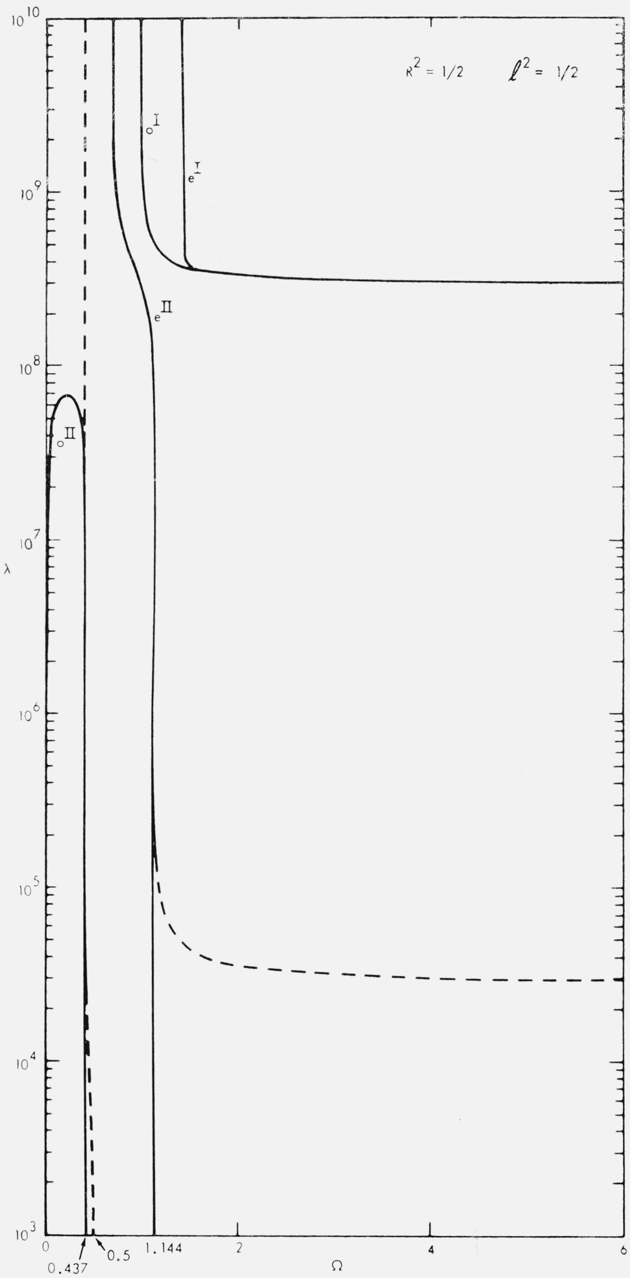

FIGURE 6a. *Dispersion curves for the arbitrary direction of propagation in the first two approximations* ($R^2=½$).

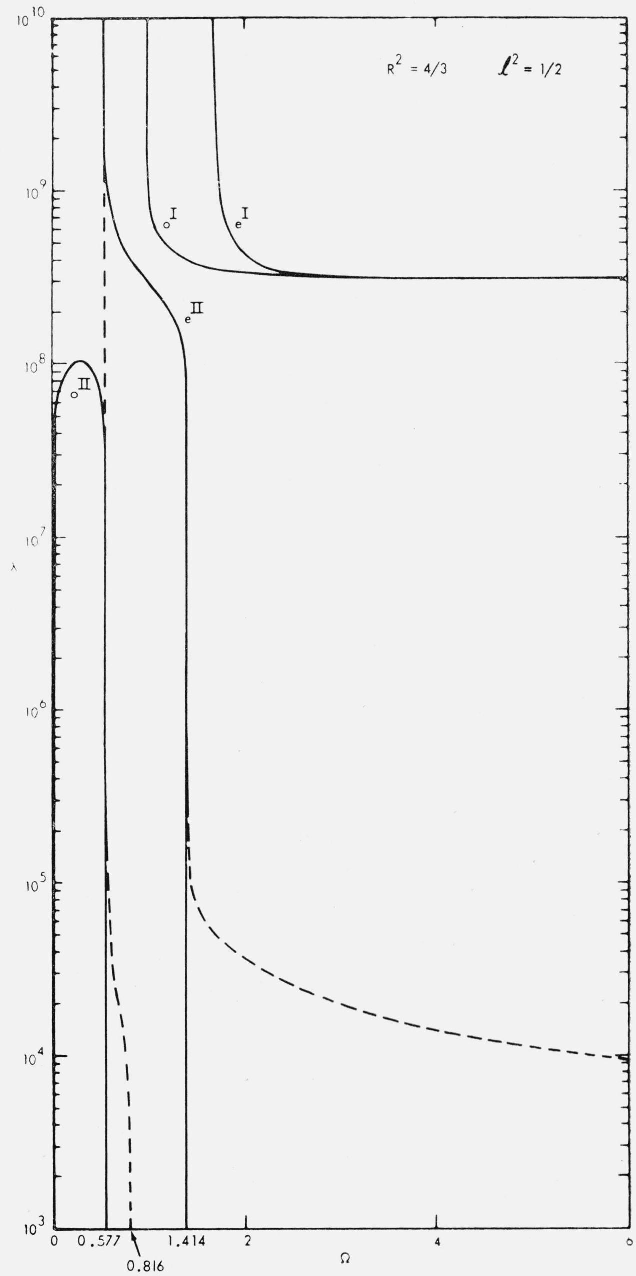

FIGURE 6b. *Dispersion curves for the arbitrary direction of propagation in the first two approximations* ($R^2=⅘$).

to the exact dispersion equation (11) for the case in which the phase velocity is of the order C_0 or higher. Consequently the phase velocities of the O^{I} and the e^{I} modes are excellent approximations to the actual values obtained from the solution of (11).

The phase velocity of the O^{II} mode starts with a value zero for $\Omega=0$, increases as Ω is increased till a maximum value of the order of C_0 is reached and thereafter decreases for further increase in Ω and reaches the value zero for $\Omega=\Omega_3$. Note that in the first part of the frequency range $0<\Omega<\Omega_3$, the phase velocity of the O^{II} mode increases with frequency. The phase velocity of the e^{II} mode starts with a value of infinity for $\Omega=\Omega_1$ (which is always less than 1), decreases as Ω is increased, attains the value C_0 approximately for $\Omega=1$ and rapidly decreases to the value of zero at $\Omega=\Omega_4$. Since the phase velocity of the O^{II} mode in the vicinity of $\Omega=0$ and $\Omega=\Omega_3$ and that of e^{II} mode in the vicinity of $\Omega=\Omega_4$ are considerably below C_0 and since these values are derived from (25) which is a valid approximation to the exact dispersion equation (11) only if the phase velocity is of the order C_0 or higher, it follows that the phase velocity obtained from (25) is of doubtful validity in the vicinity of the frequencies $\Omega=0$, Ω_3, and Ω_4 and therefore a second approximation to (11) is needed to examine and obtain the phase velocities in the neighborhood of $\Omega=0$, Ω_3, and Ω_4. This is carried out in part II of this paper.

If a uniform procedure is used in designating the modes, it is found that due to the discontinuous disappearance of the critical line $\Omega^2=1$ for the case of propagation in the direction of the static magnetic field, the "whistler mode" has its nomenclature changed from extraordinary to ordinary as soon as the direction of propagation differs from that of the static magnetic field. This inconsistency appears only in the first approximation and therefore characteristic only of the magneto-ionic theory. With the higher order approximations to the exact dispersion relation and with the effect of the compressibility of the plasma thus taken into account, the three independent modes which propagate in an electron plasma are capable of being designated using a procedure which is uniformly valid for all directions of propagation. The discontinuous disappearance of the critical line $\Omega^2=1$ is shown in part II to be the consequence of the decoupling of the purely longitudinal plasma wave for propagation in the direction of the static magnetic field.

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