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A Note on the Insulated Loop Antenna Immersed in a Conducting Medium

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The impedance of a circular loop in an insulated cavity immersed in a conducting medium is discussed. The results given in an earlier paper are extended.

In an earlier paper [Wait, 1957],[1] an analysis was given for the problem of an insulated circular loop immersed in a conducting medium. It is the purpose of this note to present some generalized formulas for the self impedance of the circular loop. At the same time, certain corrections in the original equations are made.

For convenience of the reader, the model is briefly described. With respect to a spherical coordinate system (r, θ, φ), the loop is at $\theta=\beta$ and $r=b$. The insulating cavity which surrounds the loop is bounded by $r=a$, where $a>b$. The electrical constants of the insulating material are ϵ_0 and μ, whereas the surrounding medium has constants ϵ, σ, and μ.

The impedance of the loop in free space is denoted Z_0, which is assumed known. On the other hand, the impedance of the loop inside the cavity is given by $Z=Z_0+\Delta Z$, where ΔZ is the incremental change resulting from the finite size of the cavity. The expression for ΔZ is given by

$$\Delta Z=i\mu\omega A^s 2\pi b(\sin\beta)/I \tag{1}$$

where I is the current (assumed constant) in the loop while A^s is the secondary part of the vector potential given by (3) of the previous paper [Wait, 1957]. Under the assumption that the diameter of the cavity is small compared with the free-space wavelength, the corrected version of (13) from the previous paper reads

$$\Delta Z=i\mu\omega(\sin^2\beta)\pi b\sum_{n=1}^{\infty}\frac{s_n}{n(n+1)}\left(\frac{b}{a}\right)^{2n+1}[P_n^1(\cos\beta)]^2, \tag{2}$$

where

$$s_n=\frac{n+\alpha_n}{(n+1)-\alpha_n}, \tag{3}$$

$$\alpha_n=\left[z\frac{d}{dz}\log k_n(z)\right]_{z=\gamma a}, \tag{4}$$

and $\gamma=[i\mu\omega(\sigma+i\epsilon\omega)]^{\frac{1}{2}}$. In the above, P_n^1 is the associated Legendre polynomial of order n, and $k_n(z)$ is a modified spherical Hankel function. The latter is defined by

$$k_n(z)=e^{-z}\sum_{m=0}^{n}\frac{(n+m)!}{m!(n-m)!(2z)^m}. \tag{5}$$

To gain some insight into these results, the function α_n is expressed as an ascending series in powers of (γa) which itself is complex for conducting media. Thus, we require that

$$\alpha_n=\sum_{k=0}^{\infty}A_{kn}(\gamma a)^k \tag{6}$$

be identical with

$$\alpha_n=\left[-n-\frac{1}{2}\frac{\sum_{m=0}^{n-1}\frac{(n-1+m)!}{m!(n-1-m)!}(2z)^{n-m+1}}{\sum_{m=0}^{n}\frac{(n+m)!}{m!(n-m)!}(2z)^{n-m}}\right]_{z=\gamma a}. \tag{7}$$

To obtain the desired series, each of the polynomials in the quotient in the above expression is written in ascending powers of $(2z)$, which is followed by a tedious long division. This leads to the following results:

$$A_{0n}=-n \qquad (n=1, 2, 3\ldots),$$

$$A_{1n}=0 \qquad (n=1, 2, 3\ldots),$$

$$A_{2n}=-\frac{1}{2n-1} \qquad (n=1, 2, 3\ldots),$$

$$A_{3n}=\begin{cases}0 & (n=2, 3, 4\ldots)\\ 1 & (n=1),\end{cases}$$

$$A_{4n}=\frac{1}{(2n-1)^2(2n-3)} \qquad (n=1, 2, 3\ldots)$$

$$A_{5n}=\begin{cases}0 & (n=3, 4, 5\ldots)\\ -1/9 & (n=2)\\ 1 & (n=1).\end{cases}$$

[1] Wait, J. R. (Aug. 1957), Insulated loop antenna immersed in a conducting medium, J. Res. NBS **59**, No. 2, 133–137.

In particular

$$\alpha_1 = -1-(\gamma a)^2+(\gamma a)^3-(\gamma a)^4+(\gamma a)^5- \ldots,$$

which may be verified to be identical to

$$\alpha_1 = \left[z\frac{d}{dz}\log k_1(z)\right]_{z=\gamma a} \text{ where } k_1(z)=e^{-z}\left(1+\frac{1}{z}\right)$$

$$= -\frac{1+(\gamma a)+(\gamma a)^2}{1+(\gamma a)}.$$

It is also worth noting that

$$\alpha_2 = -2-\frac{(\gamma a)^2}{3}+\frac{(\gamma a)^4}{9}-\frac{(\gamma a)^5}{9}+\ldots.$$

To obtain the corresponding series expansion for s_n, the expansion for α_n is inserted into (3). Then, after carrying out another tedious long division, one finds that

$$s_n = \sum_{k=0}^{\infty} B_{kn}(\gamma a)^k, \tag{8}$$

where

$$B_{0n}=B_{1n}=0 \qquad (n=1,2,3\ldots),$$

$$B_{2n} = -\frac{1}{(2n+1)(2n-1)} \qquad (n=1,2,3\ldots),$$

$$B_{3n} = \begin{cases} 0 & (n=2,3,4\ldots) \\ \frac{1}{3} & (n=1), \end{cases}$$

$$B_{4n} = \frac{2}{(2n+1)^2(2n-1)(2n-3)} \qquad (n=1,2,3\ldots),$$

$$B_{5n} = \begin{cases} 0 & (n=3,4,5\ldots) \\ -1/45 & (n=2) \\ 1/9 & (n=1). \end{cases}$$

In particular

$$s_1 = -\frac{(\gamma a)^2}{3}+\frac{(\gamma a)^3}{3}-\frac{2}{9}(\gamma a)^4+\frac{(\gamma a)^5}{9}-\ldots$$

$$s_2 = -\frac{(\gamma a)^2}{15}+\frac{2}{75}(\gamma a)^4-\frac{(\gamma a)^5}{45}+\ldots.$$

Neglecting terms above $k=2$ in (8), it follows that

$$s_n \simeq -\frac{(\gamma a)^2}{(2n+1)(2n-1)},$$

which is certainly valid when $|\gamma a| << 1$. Using this result and specializing to the centrally located loop (i.e., $\theta=\beta=\pi/2$), it follows from (2) that

$$\Delta Z = \frac{(\mu\omega)^2(\sigma+i\epsilon\omega)S^2}{\pi a} \sum_{n=1,3,5,\ldots}^{\infty} \frac{[P_n^1(0)]^2}{n(n+1)(2n+1)(2n-1)}\left(\frac{b}{a}\right)^{2n-2}, \tag{9}$$

where $S=\pi b^2$ is the area of the loop. It should be noted that even terms in the expansion vanish for this case. Retaining just the first two terms of (9), it is seen that

$$\Delta Z \simeq \frac{(\mu\omega)^2(\sigma+i\epsilon\omega)S^2}{6\pi a}\left[1+\frac{9}{280}\left(\frac{b}{a}\right)^4\right], \tag{10}$$

where the first neglected term is of order $(b/a)^8$.

The real part of ΔZ is the input resistance increment ΔR in ohms. Thus the power P supplied to the surrounding conducting medium is $I^2\Delta R/2$ W, where $I/\sqrt{2}$ is the rms current in amperes. Under the assumptions that displacement currents are small (i.e., $\epsilon\omega << \sigma$) and $(b/a)^4 << 1$), it is seen from (10) that

$$P \simeq \frac{(\mu\omega)^2\sigma I^2 S^2}{12\pi a}, \tag{11}$$

which agrees with a result [Wait, 1952][2] derived from energy considerations. It confirms the conclusion that the power radiated from a small loop varies inversely as the first power of the radius of the cavity. The finite size of the loop does not essentially modify this conclusion.

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[2] Wait, J. R. (Oct. 1952), The magnetic dipole antenna immersed in a conducting medium, Proc. IRE **40**, 1244–1246.

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