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Some Approximate Formulas Concerning the Reflection of Electromagnetic Waves From a Stratified Semi-infinite Medium

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Kane and Karp [1964] have given an approximate treatment of the reflection of electromagnetic waves at the plane interface between two homogeneous dielectrics. They introduced an approximate linear boundary condition which is a more accurate version of the well-known Leontovich boundary condition. It was noted that use of the new boundary condition is equivalent to writing the reflection coefficients of the interface in a certain approximate form. In the present paper some reflection coefficients for a continuously stratified semi-infinite medium are obtained in this approximate form by using known solutions for the field distributions. Linear and exponential profiles of the refractive index are considered, for both horizontally and vertically polarized waves.

1. Introduction

Kane and Karp [1964] have given an approximate treatment of the reflection of electromagnetic waves at the plane interface between two homogeneous dielectrics. In that work they introduced an approximate linear boundary condition, which is a more accurate version of the well-known Leontovich boundary condition [e.g., Brekhovskikh, 1960]. It was noted that use of the new boundary condition is equivalent to writing the reflection coefficients of the interface in the approximate form

$$R(\theta)=\frac{C-(A-BS^2)}{C+(A-BS^2)} \tag{1}$$

where $C=\cos\theta$ and $S=\sin\theta$, θ being the angle of incidence.

Kane and Karp [1964] then obtained the Fresnel reflection coefficients of the interface between two homogeneous media in the approximate form (1) by making suitable choices of A and B.

It was mentioned by Kane and Karp [1964] that their approximate procedure need not be restricted to the case of two homogeneous dielectrics, but may be extended to other cases in which the reflection coefficient of some medium is known.

The purpose of the present paper is to consider the reflection of waves from a continuously stratified semi-infinite medium. Linear and exponential profiles of the refractive index are considered, for both horizontally and vertically polarized waves. Expressions for the surface admittance and surface impedance of the half-space are obtained from the known solutions for the field distributions. The asymptotic expansions of these results are used to obtain expressions for the coefficients A and B in (1). Thus the approximate procedure of Kane and Karp [1964] is extended to the case of reflection from a continuously stratified semi-infinite medium.

2. Basic Theory

In a horizontally stratified medium, horizontally and vertically polarized electromagnetic fields can be represented by a magnetic Hertz vector, $\mathbf{\Pi}^m$, and an electric Hertz vector, $\mathbf{\Pi}^e$, respectively, these two vectors having vertical or z components only [Bremmer, 1958].

It is found after separation of variables that the z variations of the magnetic and electric Hertz vectors are given by functions $f^m(z)$ and $f^e(z)$, respectively, where these satisfy [Bremmer, 1958]

$$\frac{d^2f^m}{dz^2}+[k^2(z)-\gamma^2]f^m=0 \tag{2}$$

and

$$\frac{d^2f^e}{dz^2}+[k_{\text{eff}}^2-\gamma^2]f^e=0 \tag{3}$$

where

$$k_{\text{eff}}^2=k^2(z)-k(z)\frac{d^2}{dz^2}\left\{\frac{1}{k(z)}\right\}. \tag{4}$$

In these equations $k(z)=k_0n(z)$, where $n(z)$ is the refractive index and $k_0=\omega/c$, ω being the angular frequency of the fields and c the speed of light in free space. Also, $\gamma=k_0S$ is a propagation constant of the waves.

In the case of horizontal polarization, the electric and magnetic fields **E** and **H** are obtained from

$$\mathbf{E}=i\omega\mu_0\nabla\times\mathbf{\Pi}^m \tag{5}$$

and

$$\mathbf{H}=\nabla\times\nabla\times\mathbf{\Pi}^m. \tag{6}$$

A time factor $e^{-i\omega t}$ is assumed and the permeability, μ_0, of the medium is taken to have the free space value everywhere. For plane waves propagating in the x-direction with $\partial/\partial y\equiv 0$, the wave admittance Y is defined by

$$Y=-\frac{H_x}{E_y} \tag{7}$$

$$=\frac{1}{i\omega\mu_0}\frac{1}{f^m}\frac{df^m}{dz}. \tag{8}$$

For vertically polarized fields, **E** and **H** are given by

$$\mathbf{E}=\frac{k_0}{k^2(z)}\nabla\times\nabla\times\{k(z)\mathbf{\Pi}^e\} \tag{9}$$

and

$$\mathbf{H}=\frac{-ik_0}{\omega\mu_0}\nabla\times\{k(z)\mathbf{\Pi}^e\}. \tag{10}$$

Then, the wave impedance Z is defined by

$$Z=\frac{E_x}{H_y} \tag{11}$$

$$=-\frac{i\omega\mu_0}{k^2}\frac{1}{kf^e}\frac{d(kf^e)}{dz}. \tag{12}$$

The waves are considered to be incident from free space onto a stratified half space which stretches from $z=0$ to $z=\infty$. The surface admittance Y_s and surface impedance Z_s of the interface are defined to be Y and Z evaluated at $z=0$.

The reflection coefficients R_h and R_v for horizontally and vertically polarized waves, respectively, can be written [Wait, 1962]

$$R^h=\frac{C-\eta_0 Y_s}{C+\eta_0 Y_s} \tag{13}$$

and

$$R^v=\frac{\eta_0 C-Z_s}{\eta_0 C+Z_s} \tag{14}$$

where $\eta_0=(\mu_0/\epsilon_0)^{1/2}=\omega\mu_0/k_0$, ϵ_0 being the permittivity of free space.

3. Linear Profile

In this section the continuously stratified semi-infinite medium is taken to have a refractive index profile given by

$$k(z)=a(1+bz),\qquad z>0 \tag{15}$$

where a and b are independent of z. Then [Gould and Burman, 1964; Försterling and Wüster, 1952]

$$f^m(z)=(1+bz)^{-1/2}W_{\pm p,1/4}[\pm ia(1+bz)^2/b] \tag{16}$$

and

$$f^e(z)=(1+bz)^{-1/2}W_{\pm p,3/4}[\pm ia(1+bz)^2/b] \tag{17}$$

where

$$p=i\gamma^2/4ab. \tag{18}$$

The W's are Whittaker functions, the notation having its usual meaning [Magnus and Oberhettinger, 1949; Slater, 1960].

The requirement that the fields must not become infinite as $z\to\infty$ must be satisfied. The Whittaker functions have the asymptotic expansion [Magnus and Oberhettinger, 1949; Slater, 1960]

$$W_{p,m}(q)=q^pe^{-q/2}\left[1+\frac{m^2-(p-\frac{1}{2})^2}{q}+\dots\right] \tag{19}$$

which is applicable for large values of $|q|$, provided $|\arg q|<3\pi/2$. For large z the arguments of the Whittaker functions in (16) and (17) are approximately $\pm iabz^2$. Thus it is seen from (19) that if the imaginary part of ab is positive, the lower signs in (16) and (17) must be taken. If the imaginary part of ab is negative the upper signs must be taken.

From (16) and (17) it is seen that

$$Y=\frac{1}{i\omega\mu_0}\left[\frac{-b}{2(1+bz)}\pm 2ia(1+bz)\frac{W'_{\pm p,1/4}\{\pm ia(1+bz)^2/b\}}{W_{\pm p,1/4}\{\pm ia(1+bz)^2/b\}}\right] \tag{20}$$

and [Burman and Gould, 1964]

$$Z=-i\omega\mu_0\left[\frac{b}{2a^2(1+bz)^3}\pm\frac{2i}{a(1+bz)}\cdot\frac{W'_{\pm p,3/4}\{\pm ia(1+bz)^2/b\}}{W_{\pm p,3/4}\{\pm ia(1+bz)^2/b\}}\right] \tag{21}$$

where a dash denotes differentiation with respect to the argument. From these

$$Y_s=\frac{a}{\omega\mu_0}\left[\frac{ib}{2a}\pm 2\frac{W'_{\pm p,1/4}(\pm ia/b)}{W_{\pm p,1/4}(\pm ia/b)}\right] \tag{22}$$

and

$$Z_s=\frac{\omega\mu_0}{a}\left[\frac{-ib}{2a}\pm 2\frac{W'_{\pm p,3/4}(\pm ia/b)}{W_{\pm p,3/4}(\pm ia/b)}\right]. \tag{23}$$

The asymptotic approximation (19) gives

$$\frac{W'_{p,m}(q)}{W_{p,m}(q)}=-\frac{1}{2}+\frac{p}{q}-\frac{m^2-(p-\frac{1}{2})^2}{q^2}+\dots. \tag{24}$$

Thus, if the medium does not vary too rapidly so that $|a/b|$ is large, the expansion (24) can be used to give

$$Y_s = \frac{a}{\omega\mu_0}\left[\mp 1 + \frac{ib}{2a} \pm \frac{\gamma^2}{2a^2} \mp \frac{3b^2}{8a^2} \pm \frac{\gamma^4}{8a^4} + \frac{ib\gamma^2}{2a^3} + \dots\right] \quad (25)$$

and

$$Z_s = \frac{\omega\mu_0}{a}\left[\mp 1 \pm \frac{\gamma^2}{2a^2} - \frac{ib}{2a} \pm \frac{5b^2}{8a^2} \pm \frac{\gamma^4}{8a^4} + \frac{ib\gamma^2}{2a^3} + \dots\right]. \quad (26)$$

4. Exponential Profile

The situation is the same as that considered in the last section except that the continuously stratified semi-infinite medium has a refractive index profile given by

$$k(z) = ae^{bz}, \qquad z > 0 \quad (27)$$

where a and b are independent of z, and b is real and positive. From the work of Brekhovskikh [1960], Galejs [1961], Wait [1962], and others, it follows that

$$f^m(z) = H_\alpha^{(1,2)}\left(\frac{a}{b}\,e^{bz}\right) \quad (28)$$

where

$$\alpha^2 = \gamma^2/b^2, \quad (29)$$

and

$$f^e(z) = H_\beta^{(1,2)}\left(\frac{a}{b}\,e^{bz}\right) \quad (30)$$

where

$$\beta^2 = (\gamma^2 + b^2)/b^2. \quad (31)$$

The H's are Hankel functions, the notation having its usual meaning [Magnus and Oberhettinger, 1949].

The requirement that the fields must not become infinite as $z \to \infty$ must be satisfied. The Hankel functions have the asymptotic expansions [Magnus and Oberhettinger, 1949]

$$H_\nu^{(1)}(\rho) = \left(\frac{2}{\pi\rho}\right)^{1/2} \exp\left\{i\left(\rho - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right\} \left[1 + \frac{i(4\nu^2 - 1)}{8\rho} + \dots\right] \quad (32)$$

and

$$H_\nu^{(2)}(\rho) = \left(\frac{2}{\pi\rho}\right)^{1/2} \exp\left\{-i\left(\rho - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right\} \left[1 - \frac{i(4\nu^2 - 1)}{8\rho} + \dots\right] \quad (33)$$

valid for $|\rho| >> 1$ and $|\rho| >> |\nu|$. Thus, since $b > 0$ it is seen that when Re $(ia) < 0$ the Hankel function of the first kind must be chosen in (28) and (30). When Re $(ia) > 0$, the Hankel functions of the second kind must be chosen.

From (28) and (30) it is seen that

$$Y = \frac{ae^{bz}}{i\omega\mu_0} \frac{H_\alpha^{(1,2)\prime}\left(\frac{a}{b}\,e^{bz}\right)}{H_\alpha^{(1,2)}\left(\frac{a}{b}\,e^{bz}\right)} \quad (34)$$

and [Burman and Gould, 1964]

$$Z = -\frac{i\omega\mu_0}{a^2}\left[be^{-2bz} + ae^{-bz} \frac{H_\beta^{(1,2)\prime}\left(\frac{a}{b}\,e^{bz}\right)}{H_\beta^{(1,2)}\left(\frac{a}{b}\,e^{bz}\right)}\right], \quad (35)$$

where a dash denotes differentiation with respect to the argument. From these

$$Y_s = \frac{a}{i\omega\mu_0} \frac{H_\alpha^{(1,2)\prime}(a/b)}{H_\alpha^{(1,2)}(a/b)} \quad (36)$$

and

$$Z_s = \frac{\omega\mu_0}{ia}\left[\frac{b}{a} + \frac{H_\beta^{(1,2)\prime}(a/b)}{H_\beta^{(1,2)}(a/b)}\right]. \quad (37)$$

The asymptotic approximations (32) and (33) give

$$\frac{H_\nu^{\binom{2}{1}\prime}(\rho)}{H_\nu^{\binom{2}{1}}(\rho)} = \mp i - \frac{1}{2\rho} \pm \frac{i(4\nu^2 - 1)}{8\rho^2} + \dots, \quad (38)$$

where the upper and lower signs apply when the Hankel functions of the second and first kinds, respectively, are taken.

Hence, if the medium does not vary too rapidly so that $|b/a| << 1$ and $|a/b| >> |\alpha|$, the expansion (38) can be used to give

$$Y_s = \frac{a}{\omega\mu_0}\left[\mp 1 + \frac{ib}{2a} \pm \frac{\gamma^2}{2a^2} \mp \frac{b^2}{8a^2} + \dots\right]. \quad (39)$$

Similarly if $|b/a| << 1$ and $|a/b| >> |\beta|$, then

$$Z_s = \frac{\omega\mu_0}{a}\left[\mp 1 \pm \frac{\gamma^2}{2a^2} - \frac{ib}{2a} \pm \frac{3b^2}{8a^2} + \dots\right]. \quad (40)$$

The expressions (39) and (40) are seen to be very similar to the corresponding expressions (25) and (26) for a linear profile of refractive index.

5. The Resulting Expansions

From (13) and (14) in section 2, it is seen that writing R^h and R^v in the form given by (1) is equivalent to writing

$$Y_s \doteq \frac{k_0}{\omega\mu_0}(A - BS^2) \quad (41)$$

and

$$Z_s \doteq \frac{\omega\mu_0}{k_0}(A - BS^2). \quad (42)$$

In the case of a linear profile of refractive index, (25) shows that

$$A_h = \frac{a}{k_0}\left[\mp 1 + \frac{ib}{2a} \mp \frac{3b^2}{8a^2} + \dots\right] \quad (43)$$

and

$$B_h = \frac{a}{k_0}\left[\mp \frac{k_0^2}{2a^2} - \frac{ibk_0^2}{2a^3} + \dots\right] \quad (44)$$

where the subscript h denotes horizontal polarization. In obtaining (43) and (44) the term in $\gamma^4/(8a^4)$ in (25) has been neglected. This term would contribute to terms in S^4 in (1), corresponding to the use of a higher order boundary condition [Kane and Karp, 1964].

Equation (26) shows that

$$A_v=\frac{k_0}{a}\left[\mp 1-\frac{ib}{2a}\pm\frac{5b^2}{8a^2}+\ldots\right] \tag{45}$$

and

$$B_v=\frac{k_0}{a}\left[\mp\frac{k_0^2}{2a^2}-\frac{ik_0^2 b}{2a^3}+\ldots\right] \tag{46}$$

where the subscript v denotes vertical polarization. In these equations the term in $\gamma^4/(8a^4)$ in (26) has been neglected.

In the case of an exponential variation of refractive index, (39) shows that for horizontal polarization

$$A_h=\frac{a}{k_0}\left[\mp 1+\frac{ib}{2a}\mp\frac{b^2}{8a^2}+\ldots\right] \tag{47}$$

and

$$B_h=\frac{a}{k_0}\left[\mp\frac{k_0^2}{2a^2}+\ldots\right]. \tag{48}$$

For vertical polarization:

$$A_v=\frac{k_0}{a}\left[\mp 1-\frac{ib}{2a}\pm\frac{3b^2}{8a^2}+\ldots\right] \tag{49}$$

and

$$B_v=\frac{k_0}{a}\left[\mp\frac{k_0^2}{2a^2}+\ldots\right] \tag{50}$$

When $b=0$, the half-space $z>0$ is homogeneous with refractive index $\overline{n}=a/k_0$. Putting $b=0$ in (43) to (50) gives:

$$A_h=\mp\overline{n}, \tag{51}$$

$$B_h=\mp 1/2\overline{n}, \tag{52}$$

$$A_v=\mp 1/\overline{n}, \tag{53}$$

and

$$B_v=\mp 1/2\overline{n}^3. \tag{54}$$

These may be compared with the results obtained by Kane and Karp [1964] for reflection from an interface separating two homogeneous media.

6. Conclusion

Some approximate formulas relating to the reflection of waves from a continuously stratified semi-infinite medium have been obtained. These correspond to the use of an approximate linear boundary condition [Kane and Karp, 1964] which is a more accurate version of the well-known Leontovich boundary condition. Linear and exponential profiles of the refractive index of the semi-infinite medium have been considered, for both horizontally and vertically polarized waves.

Thus, the approximate procedure of Kane and Karp [1964] has been extended to the case of reflection from a continuously stratified semi-infinite medium. The results obtained reduce to those given by Kane and Karp [1964] when appropriate special cases are taken.

7. References

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