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Propagation in Nonuniform Waveguides With Impedance Walls [1]

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Under certain conditions it is useful to exchange Maxwell's equations for an infinite set of coupled total differential equations; the set takes the form of generalized telegraphist's equations. This is done for a parallel-plate waveguide with impedance walls and varying plate separation. The characteristic modes of the waveguide are used in order that coupling between equations depends only on geometric perturbations of the guide walls. The utility of the technique is demonstrated by evaluating the mode conversion in an overmoded waveguide containing a geometric perturbation. A comparison with experimental work is presented for the perfectly conducting case.

1. Introduction

In recent years there has been a tendency to search for ways of obviating the difficulty which arises when Maxwell's equations are not separable in the chosen coordinates or when the boundary conditions are not simple. This is particularly true for the problem of propagation of electromagnetic energy in tubes. Schelkunoff [1937], perhaps more than any other person, has contributed considerably to that work, having shown that Maxwell's equations with appropriate boundary conditions can lead to equations analogous to telegraphist's equations. Other workers have contributed to the tendency to convert Maxwell's equations to ordinary differential equations [Marcuvitz and Schwinger, 1951; Stevenson, 1951; Schelkunoff, 1952; Reiter, 1959; Unger, 1958, 1961a, 1961b] and in the most general case, the equations found are of the form

$$\frac{dV_i}{dz} = -\sum_k Z_{ik} I_k - \sum_k T^V_{ik} V_k,$$

$$\frac{dI_i}{dz} = -\sum_k Y_{ik} V_k - \sum_k T^I_{ik} I_k. \tag{1}$$

These equations are more general than the classical telegraphist's equations for coupled transmission lines since they contain current and voltage transfer coefficients, in addition to the usual distributed impedance and admittance.

The voltages, V_k, and currents, I_k, are related to the amplitudes of the kth mode. To each subscript k there corresponds a certain electromagnetic field pattern in the transverse plane of the waveguide. To a large extent, the choice of these field patterns is arbitrary, the form being representative of the solutions of Maxwell's equations in media with variable dielectrics, dissipative walls, and other features which might contribute to mode coupling.

It is sometimes convenient to choose modes associated with the perfectly conducting waveguide [Schelkunoff, 1955]. The boundary conditions of the imperfectly conducting guide are then satisfied by the introduction of cross-coupling terms. While the complete set of modes is then immediately available, the difficulty of the problem comes about through the infinity of coupled equations which must result. The determination of the mode voltages and currents is made correspondingly more difficult by choosing simple mode functions. In contrast to this

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method, one may use characteristic modes or eigenmodes for the structure. Such functions individually satisfy the source-free Maxwell's equations and the desired boundary conditions. This choice leads to uncoupled transmission line equations and the mode voltages and currents can be found more readily. In this case the difficulty usually lies in finding the characteristic modes.

Reiter [1959] treats, in a very general way, the transformation of Maxwell's equations, with appropriate boundary conditions, to generalized telegraphist's equations. His work is restricted to the case of opaque boundaries, and he uses characteristic modes to reduce cross coupling to a minimum. Schelkunoff [1955] treats the imperfectly conducting case, but does not use characteristic modes. Unger [1958, 1961a, 1961b] has treated the helix waveguide using characteristic modes. The waveguide is made of closely wound insulated conducting wire covered with a lossy material and a conducting sheath. The resultant anisotropic sheath exhibits a nonzero surface impedance in the axial direction only.

The primary purpose of this paper is to develop the generalized telegraphist's equations for the parallel-plate waveguide. The walls are assumed lossy and may be characterized by a surface impedance. The equations found will represent the case of minimum coupling due to the use of characteristic modes to represent the field in the waveguide.

An important problem which seems well suited to study via the generalized telegraphist's equations and which has taken on added significance in recent years is the problem of VLF propagation. While the mechanics of VLF propagation have been understood quite well for many years, it was only recently that advantage was taken of the waveguide character of the region between the earth and ionosphere [Budden, 1952, 1957a, 1957b; Wait, 1957, 1960]. Thus for waves that travel a considerable distance, it is convenient to turn to the mode theory of propagation; for it is the mode theory of propagation at VLF which has proved quite successful in treating many of the problems of major interest. It is, in fact, the mode theory that causes the problem to be put in that class of problems adaptable to the generalized telegraphist's equations. For it is precisely the characteristic modes which are taken for the modal functions. We will restrict ourselves to the case of a flat earth and a sharply bounded homogeneous ionosphere. While this model is rather rough by today's standards [Johler, 1962; Johler and Harper, 1962], it serves well the purpose of this initial study on this method of attack.

The concept of surface impedance, so necessary to the successful completion of this method, is likewise well adapted to the earth-ionosphere waveguide problem [Wait, 1957, 1960, 1961a].

2. Eigenfunction Problem

The main concern in this study is the detailed description of the fields in a parallel-plate waveguide of varying plate separation. To facilitate this description it will be convenient to employ a field representation which is characteristic of the waveguide region. At any cross section of the waveguide, the electromagnetic field can be represented as a superposition of the characteristic modes in question.

It is convenient, in treating the eigenfunction problem, to begin with a consideration of a cylindrical waveguide of arbitrary cross section as shown schematically in figure 1. It consists of a tube bounded by a surface such that any plane perpendicular to the z-axis cuts the surface

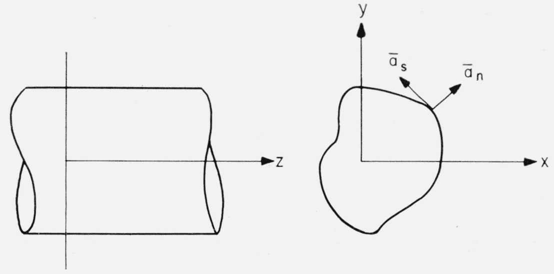

FIGURE 1. *Cylindrical waveguide model.*

in a smooth closed curve L. The area interior to any cross section is denoted by A, which is independent of z. The unit vector $\bar{a}_n$ is normal to L and perpendicular to $\bar{a}_z$, the unit vector in the axial direction. The unit tangent to L is designated $\bar{a}_s$ and is in a direction such that $\bar{a}_z \times \bar{a}_n = \bar{a}_s$.

In the case of time steady state with the time variation exp $(j\omega t)$, the field in a source free region must satisfy Maxwell's equations:

$$\nabla \times \bar{E} = -j\omega\mu\bar{H}, \tag{2a}$$

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E}. \tag{2b}$$

The $\bar{E}$ and $\bar{H}$ fields and the del operator may be resolved into longitudinal and transverse components:

$$\bar{E} = \bar{E}_t + \bar{a}_z E_z, \tag{3a}$$

$$\bar{H} = \bar{H}_t + \bar{a}_z H_z, \tag{3b}$$

$$\nabla = \nabla_t + \bar{a}_z \frac{\partial}{\partial z}, \tag{3c}$$

where the subscript t denotes components transverse to the z-axis. On the guide walls the fields are to satisfy the boundary condition

$$\bar{E} = \bar{\bar{Z}} \cdot \bar{H} \times \bar{a}_n, \tag{4}$$

where $\bar{\bar{Z}}$ is an impedance dyadic. The choice in (4) is made in a manner which is compatible with the definition of the positive sense of $\bar{a}_s$. For the purposes of this study it is convenient to rewrite Maxwell's equations as

$$j\omega\mu H_z = \nabla_t \cdot (\bar{a}_z \times \bar{E}_t), \tag{5a}$$

$$j\omega\epsilon E_z = \nabla_t \cdot (\bar{H}_t \times \bar{a}_z), \tag{5b}$$

$$-\frac{\partial \bar{E}_t}{\partial z} = j\omega\mu(\bar{H}_t \times \bar{a}_z) - \frac{1}{j\omega\epsilon} \nabla_t \nabla_t \cdot (\bar{H}_t \times \bar{a}_z), \tag{6a}$$

$$-\frac{\partial \bar{H}_t}{\partial z} = j\omega\epsilon(\bar{a}_z \times \bar{E}_t) - \frac{1}{j\omega\mu} \nabla_t \nabla_t \cdot (a_z \times \bar{E}_t). \tag{6b}$$

The transverse fields may now be expressed in terms of a complete set of vector functions as

$$\bar{E}_t(x, y, z) = \sum_k V_k(z) \bar{e}_k(x, y), \tag{7a}$$

$$\bar{H}_t(x, y, z) = \sum_k I_k(z) \bar{h}_k(x, y), \tag{7b}$$

where the voltages, V_k, and currents, I_k, are related to the amplitudes of the field quantities associated with each mode. We adopt the characteristic vector mode functions for the cylindrical guide as the vector functions $\bar{e}_k$ and $\bar{h}_k$. They are required to satisfy Maxwell's equations and the appropriate boundary conditions. This requirement will prove to be useful when considering the waveguide with varying cross section. The separability of the functional dependence in the manner shown is a result of the assumed uniformity of the guide in the z-direction. Inserting (7) into (6) and using standard separation techniques leads to the results

$$\frac{dV_i}{dz} + j\Gamma_i Z_i I_i = 0, \tag{8a}$$

$$\frac{dI_i}{dz} + j\Gamma_i Y_i V_i = 0, \tag{8b}$$

$$\Gamma_i Z_i e_i = \omega\mu(\bar{h}_i \times \bar{a}_z) + \frac{1}{\omega\epsilon} \nabla_t \nabla_t \cdot (\bar{h}_i \times \bar{a}_z), \tag{9a}$$

$$\Gamma_i Y_i \bar{h}_i = \omega\epsilon(\bar{a}_z \times \bar{e}_i) + \frac{1}{\omega\mu} \nabla_t \nabla_t \cdot (\bar{a}_z \times \bar{e}_i), \tag{9b}$$

where $j\Gamma_i Z_i$ and $j\Gamma_i Y_i$ are separation constants.

For a cylindrical waveguide, the impedance dyadic is diagonal in an $\bar{a}_s$, $\bar{a}_z$ sense and (4) implies

$$\nabla_t \cdot (\bar{H}_t \times \bar{a}_z) = -j\omega\epsilon Z_z \bar{a}_n \cdot \bar{H}_t \times \bar{a}_z, \tag{10a}$$

$$\nabla_t \cdot (\bar{a}_z \times \bar{E}_t) = -\frac{j\omega\mu}{Z_s} \bar{a}_n \cdot \bar{a}_z \times \bar{E}_t, \tag{10b}$$

where Z_z is the surface impedance[2] for the z-polarized field and Z_s is the surface impedance for the tangentially polarized field. Since each modal function is required to be capable of existing independently in the waveguide, we require

$$\nabla_t \cdot (\bar{h}_i \times \bar{a}_z) = -j\omega\epsilon Z_z [\bar{a}_n \cdot \bar{h}_i \times \bar{a}_z], \tag{11a}$$

$$\nabla_t \cdot (\bar{a}_z \times \bar{e}_i) = -\frac{j\omega\mu}{Z_s} [\bar{a}_n \cdot \bar{a}_z \times \bar{e}_i] \tag{11b}$$

on the boundary.

Equations (9) and (11) represent the complete characterization of the vector eigenfunction problem. The solution is a mathematical problem which can be realized only in a few restricted cases.

An important special case for which the vector eigenfunction problem separates and the solution is tractable is the case wherein the cross section is unbounded in one transverse direction and there is no field variation in that direction (here taken to be the x-direction). In that case, the orthonormality condition

$$\iint_A \bar{e}_n \cdot \bar{h}_m \times \bar{a}_z dA = \iint_A \bar{h}_n \cdot \bar{a}_z \times \bar{e}_m dA = \delta_{mn} \tag{12}$$

holds [Gallawa, 1964] provided that the appropriate boundary condition holds on the parallel plates. Here δ_{mn} is the Kronecker delta and A is the area bounded by the parallel plates and by imaginary planes at $x = \pm b$. The value of b is arbitrary, affecting only the normalization constant.

3. Parallel-Plate Waveguide

We consider a waveguide which consists simply of two parallel plates. The plates are located at the planes $y = \pm a$ and are characterized by a surface impedance dyadic. The plates are unbounded in the x- and z-directions. We seek a set of characteristic modes which are independent of x with propagation assumed to be in the z-direction.

When $\frac{\partial}{\partial x} = 0$, (9) reduces to the scalar equations

$$\Gamma_i Z_i e_{xi} = \omega\mu h_{yi}, \tag{13}$$

$$\Gamma_i Z_i e_{yi} = -\omega\mu h_{xi} - \frac{1}{\omega\epsilon} \frac{\partial^2 h_{xi}}{\partial y^2}, \tag{14}$$

$$\Gamma_i Y_i h_{xi} = -\omega\epsilon e_{yi}, \tag{15}$$

$$\Gamma_i Y_i h_{yi} = \omega\epsilon e_{xi} + \frac{1}{\omega\mu} \frac{\partial^2 e_{xi}}{\partial y^2}. \tag{16}$$

[2] No confusion should arise from the various uses of the letter Z. A subscript in each case distinguishes the meaning intended.

Inspection of these equations indicates the existence of two subsets of modes which are transverse electric and transverse magnetic to the x-direction. If we consider the first and the last of these four equations and then the remaining two equations, the nature of the modes becomes evident. Consider first the H-type mode designated by a prime and characterized by the identity

$$e'_{xi} \equiv 0. \tag{17}$$

Then we have the combination

$$\frac{\partial^2 h'_{xi}}{\partial y^2} + (k^2 - (\Gamma'_i)^2 Z'_i Y'_i) h'_{xi} = 0, \tag{18}$$

$$\frac{\partial h'_{xi}}{\partial y} = \mp j\omega\epsilon Z_z h'_{xi} \text{ at } y = \pm a, \tag{19}$$

where

$$k^2 = \omega^2 \mu\epsilon. \tag{20}$$

The E-type modes, designated by a double prime, can be determined in a similar manner. Begin by requiring

$$h''_{xi} = 0. \tag{21}$$

Then there evolves the equation

$$\frac{\partial^2 e''_{xi}}{\partial y^2} + (k^2 - (\Gamma''_i)^2 Z''_i Y''_i) e''_{xi} = 0. \tag{22}$$

subject to

$$\frac{\partial e''_{xi}}{\partial y} = \mp \frac{j\omega\mu}{Z_s} e''_{xi} \text{ at } y = \pm a. \tag{23}$$

It is convenient and entirely compatible to let

$$Y'_i = \frac{1}{Z'_i} = \frac{\omega\epsilon}{\Gamma'_i}, \tag{24}$$

$$Y''_i = \frac{1}{Z''_i} = \frac{\Gamma''_i}{\omega\mu}. \tag{25}$$

Then (18) and (22) become, respectively,

$$\frac{\partial^2 h'_{xi}}{\partial y^2} + (k'_{ci})^2 h'_{xi} = 0, \tag{18a}$$

$$\frac{\partial^2 e''_{xi}}{\partial y^2} + (k''_{ci})^2 e''_{xi} = 0, \tag{22a}$$

where

$$(k'_{ci})^2 = k^2 - (\Gamma'_i)^2, \tag{26}$$

$$(k''_{ci})^2 = k^2 - (\Gamma''_i)^2. \tag{27}$$

4. Nonuniform Parallel-Plate Waveguide

Having developed the eigenvalue-eigenfunction problem for the parallel-plate waveguide, it is appropriate to consider the parallel-plate waveguide with variable plate separation (2a). To do so it is necessary to define the flare angle θ. If $\bar{a}_h$ is taken to be the unit normal to the guide surface, θ is the angle between $\bar{a}_h$ and $\bar{a}_n$. The sense of θ is such that $\bar{a}_n \times \bar{a}_h$ is in the direction of $\bar{a}_s$. Thus θ is positive if the cross section flares outward as we move in the positive z-direction. We also require

$$|\theta| < \frac{\pi}{2}. \tag{28}$$

For varying plate separation the boundary conditions become

$$E_z = \frac{-Z_z}{\cos\theta}(\overline{H}_t \times \overline{a}_z \cdot \overline{a}_n) - \overline{E}_t \cdot \overline{a}_n \tan\theta, \tag{29}$$

$$H_z = \frac{-1}{Z_s \cos\theta}(\overline{a}_z \times \overline{E}_t \cdot \overline{a}_n) - \overline{H}_t \cdot {}_n\overline{a} \tan\theta \tag{30}$$

on the bounding impedance surfaces.

In order to convert Maxwell's equations to the form desired, (6a) and (6b) may be scalar post-multiplied by $(\overline{h}_i^\alpha \times \overline{a}_z)$ and $(\overline{a}_z \times \overline{e}_i^\alpha)$, respectively, and integrated over the cross section. In so doing, α is first taken to be prime and then double prime. The result is

$$\iint_A -\frac{\partial \overline{E}_t}{\partial z} \cdot (\overline{h}_i^\alpha \times \overline{a}_z) dA = j\omega\mu \iint_A (\overline{H}_t \times \overline{a}_z) \cdot (\overline{h}_i^\alpha \times a_z) dA - \frac{1}{j\omega\epsilon} \iint_A [\nabla_t \nabla_t \cdot (\overline{H}_t \times \overline{a}_z)] \cdot [\overline{h}_i^\alpha \times \overline{a}_z] dA, \tag{31}$$

$$\iint_A -\frac{\partial \overline{H}_t}{\partial z} \cdot (\overline{a}_z \times \overline{e}_i^\alpha) dA = j\omega\epsilon \iint_A (\overline{a}_z \times \overline{E}_t) \cdot (\overline{a}_z \times \overline{e}_i^\alpha) dA - \frac{1}{j\omega\mu} \iint [\nabla_t \nabla_t \cdot (\overline{a}_z \times \overline{E}_t)] \cdot [\overline{a}_z \times \overline{e}_i^\alpha] dA. \tag{32}$$

After considerable ado, these equations may be expressed as

$$-\frac{d}{dz} \iint_A \overline{E}_t \cdot \overline{e}_i' dA = \left[j\omega\mu + \frac{(k_{ci}')^2}{j\omega\epsilon} \right] \iint_A \overline{H}_t \cdot \overline{h}_i' dA - \iint_A \overline{E}_t \cdot \frac{\partial \overline{e}_i'}{\partial z} dA$$
$$+ \int_C \left[\frac{Z_z}{\cos\theta} - Z_z \right] [(\overline{H}_t \cdot \overline{a}_s)(\overline{h}_i' \cdot \overline{a}_s)] dl - \int_C (\overline{E}_t \cdot \overline{a}_s)(\overline{e}_i' \cdot \overline{a}_s) \tan\theta \, dl, \tag{33}$$

$$-\frac{d}{dz} \iint_A \overline{H}_t \cdot \overline{h}_i' dA = j\omega\epsilon \iint_A \overline{E}_t \cdot \overline{e}_i' dA - \iint_A \overline{H}_t \cdot \frac{\partial \overline{h}_i'}{\partial z} dA$$
$$+ \int_C \frac{1}{Z_s \cos\theta} (\overline{E}_t \cdot \overline{a}_s)(\overline{e}_i' \cdot \overline{a}_s) dl - \int_C (\overline{H}_t \cdot \overline{a}_s)(\overline{h}_i' \cdot \overline{a}_s) \tan\theta \, dl, \tag{34}$$

$$-\frac{d}{dz} \iint_A \overline{E}_t \cdot \overline{e}_i'' dA = j\omega\mu \iint_A \overline{H}_t \cdot \overline{h}_i'' dA - \iint_A \overline{E}_t \cdot \frac{\partial \overline{e}_i''}{\partial z} dA$$
$$+ \int_C \frac{Z_z}{\cos\theta} (\overline{H}_t \cdot \overline{a}_s)(\overline{h}_i'' \cdot \overline{a}_s) dl - \int_C (\overline{e}_i'' \cdot \overline{a}_s)(\overline{E}_t \cdot \overline{a}_s) \tan\theta \, dl, \tag{35}$$

$$-\frac{d}{dz} \iint_A \overline{H}_t \cdot \overline{h}_i'' dA = \left[j\omega\epsilon + \frac{(k_{ci}'')^2}{j\omega\mu} \right] \iint_A \overline{E}_t \cdot \overline{e}_i'' dA - \iint_A \overline{H}_t \cdot \frac{\partial}{\partial z} \overline{h}_i'' dA$$
$$+ \int_C \left[\frac{1}{Z_s \cos\theta} - \frac{1}{Z_s} \right] [(\overline{E}_t \cdot \overline{a}_s)(\overline{e}_i'' \cdot \overline{a}_s)] dl - \int_C (\overline{H}_t \cdot \overline{a}_s)(\overline{h}_i'' \cdot \overline{a}_s) \tan\theta \, dl. \tag{36}$$

Using L to designate the path enclosing A, c is that portion of L which is represented by the y-boundaries (the impedance planes). The remainder of the path need not be considered because the field quantities are independent of x.

The last four equations may be reduced to their final form by expanding the transverse field quantities as a sum of the vector modal functions,

$$\overline{E}_t = \sum_k V_k \overline{e}_k, \tag{37a}$$

$$\overline{H}_t = \sum_k I_k \overline{h}_k. \tag{37b}$$

The summation is over all possible modal functions. This permits the following for (33)–(36).

$$-\frac{dV_i'}{dz} = j\Gamma_i' Z_i' I_i' - \sum_k P_{ki}' V_k + \sum_k R_{ki}' I_k, \tag{38}$$

$$-\frac{dI_i'}{dz} = j\Gamma_i' Y_i' V_i' - \sum_k S_{ki}' I_k, \tag{39}$$

$$-\frac{dV_i''}{dz} = j\Gamma_i'' Z_i'' I_i'' - \sum_k P_{ki}'' V_k, \tag{40}$$

$$-\frac{dI_i''}{dz} = j\Gamma_i'' Y_i'' V_i'' - \sum_k S_{ki}'' I_k + \sum_k G_{ki}'' V_k, \tag{41}$$

where

$$P_{ki}' = \iint_A \overline{e}_k \cdot \frac{\partial \overline{e}_i'}{\partial z} \, dA, \tag{42}$$

$$R_{ki}' = \int_C \left[\frac{Z_z}{\cos\theta} - Z_z \right] [(\overline{h}_k \cdot \overline{a}_s)(\overline{h}_i' \cdot \overline{a}_s)] \, dl, \tag{43}$$

$$S_{ki}' = \iint_A \overline{e}_k \cdot \frac{\partial \overline{e}_i'}{\partial z} \, dA + \int_C (\overline{h}_k \cdot \overline{a}_s)(\overline{h}_i' \cdot \overline{a}_s) \tan\theta \, dl, \tag{44}$$

$$P_{ki}'' = \iint_A \overline{e}_k \cdot \frac{\partial \overline{e}_i''}{\partial z} \, dA + \int_C (\overline{e}_k \cdot a_s)(\overline{e}_i'' \cdot \overline{a}_s) \tan\theta \, dl, \tag{45}$$

$$S_{ki}'' = \iint_A \overline{e}_k \cdot \frac{\partial \overline{e}_i''}{\partial z} \, dA, \tag{46}$$

$$G_{ki}'' = \int \left[\frac{1}{Z_s \cos\theta} - \frac{1}{Z_s} \right] (\overline{e}_k \cdot \overline{a}_s)(\overline{e}_i'' \cdot \overline{a}_s) dl. \tag{47}$$

Equations (38)–(47) represent the desired telegraphist's equations for the waveguide under consideration. The form that they take justifies their designation as generalized telegraphist's equations with the V_i and I_i terms representing mode voltage and current terms, designating the magnitude of a particular characteristic field pattern. Each of the differential equations may be associated with a transmission line, the characteristic wave number and characteristic impedance being determined by the waveguide cross section. There is coupling between the various transmission lines which is associated with mode coupling in the waveguide.

Upon examining the coupling terms appearing in these last equations, it is clear that much has been gained by choosing to employ a modal representation which is charasteristic of the waveguide region. At any cross section of the waveguide region, the electromagnetic field has been represented as a superposition of the characteristic modes. The result is that there is no coupling between modes when the cross section becomes z-independent. This is in contrast to the work of Schelkunoff [1955] who reduced this same problem to a system of generalized telegraphist's equations using modal functions characteristic of the perfectly conducting case (the familiar sinusoidal functions). The resulting equations are continuously coupled, even in the cylindrical case. In the case of the metallic waveguides, Schelkunoff's equations are to be preferred because the surface impedance is so small as to be negligible, reducing the coupling to a usable form. In addition, the modal functions used are obtained by inspection. In contrast, the modal functions used in this paper are not easily found and constitute a major hurdle in the final solution to any problem. Nevertheless, this is a mathematical problem which can be solved in any given instance and our results are predicated upon a knowledge of the dependence of Γ, Z, and Y on the structure of the cylindrical waveguide. On that basis the equations are minimally coupled and in fact reduce to uncoupled equations in the case of the cylindrical waveguide. It would therefore seem that these equations are to be preferred in the general impedance case (impedance not necessarily small).

5. Forward and Backward Traveling Waves

Considerable physical significance can be injected into the generalized telegraphist's equations by introducing the familiar forward and backward traveling waves.[3] This concept has been used successfully by Morgan [1957] and by Solymar [1959], among others, and can lead to useful approximate solutions for the amplitudes of spurious modes in a nonuniform waveguide. Because of the complexity of the equations encountered here, it seems appropriate to make the change of variable which formulates this concept.

We introduce as new variables the amplitudes of the forward (A_i^+) and backward (A_i^-) traveling waves:

$$V_i=\sqrt{Z_i}\,(A_i^+ + A_i^-), \tag{48a}$$

$$I_i=\frac{1}{\sqrt{Z_i}}\,(A_i^+ - A_i^-). \tag{48b}$$

The radical is introduced for convenience. It is implicitly assumed that Z_i is finite and nonzero. The equations to be satisfied by the new variables can be obtained by inserting (48) into (38)–(41). The result is

$$\frac{-dA_i^+}{dz}=j\Gamma_i A_i^+ +\frac{1}{2}A_i^- \frac{d}{dz}(\ln Z_i)+\sum_k [M_{ik1}^+ A_k^+ + M_{ik1}^- A_k^-], \tag{49a}$$

$$\frac{-dA_i^-}{dz}=-j\Gamma_i A_i^- +\frac{1}{2}A_i^+ \frac{d}{dz}(\ln Z_i)+\sum_k [M_{ik2}^+ A_k^- + M_{ik2}^- A_k^+]. \tag{49b}$$

The primes are omitted since the equations have the same form in each case. Here M_{ik}^+ and M_{ik}^- are the forward and backward coupling coefficients, respectively. The subscript 1 or 2 is attached to designate the ith mode as being forward or backward traveling, respectively.

The coupling coefficients take the form

$$M_{ik1}'^{\pm}=\frac{1}{2}\left[\pm R_{ki}'\frac{1}{\sqrt{Z_k'Z_i'}}-P_{ki}'\sqrt{\frac{Z_k'}{Z_i'}}\mp S_{ki}'\sqrt{\frac{Z_i'}{Z_k'}}\right], \tag{50}$$

$$M_{ik2}'^{\pm}=\frac{1}{2}\left[\mp\frac{R_{ki}'}{\sqrt{Z_k'Z_i'}}-P_{ki}'\sqrt{\frac{Z_k'}{Z_i'}}\mp S_{ki}'\sqrt{\frac{Z_i'}{Z_k'}}\right], \tag{51}$$

$$M_{ik1}''^{\pm}=\frac{1}{2}\left[G_{ki}''\sqrt{Z_k''Z_i''}-P_{ki}''\sqrt{\frac{Z_k''}{Z_i''}}\mp S_{ki}''\sqrt{\frac{Z_i''}{Z_k''}}\right], \tag{52}$$

$$M_{ik2}''^{\pm}=\frac{1}{2}\left[-G_{ki}''\sqrt{Z_k''Z_i''}-P_{ki}''\sqrt{\frac{Z_k''}{Z_i''}}\mp S_{ki}''\sqrt{\frac{Z_i''}{Z_k''}}\right]. \tag{53}$$

Equations (49a) and (49b) are in a form well suited to approximate solution based on the assumption that the mode conversion comes predominately from the main mode.

We assume that the mth mode (the main mode) is traveling in the positive z-direction and encounters at $z=0$ a changing cross section. It is of interest to know the amplitude of the spurious modes at $z=z_0(z_0>0)$ and the amplitude of the mode reflected at $z=0$. The constant z_0 may be taken to be the end of the nonuniform region. We assume that all mode conversion comes from the main mode, neglecting the interaction between the pth and the nth modes when $p\neq m$, $n\neq m$. We also neglect the interaction between the spurious modes and the main mode

[3] The term forward traveling refers to propagation in the z-direction; backward traveling refers to the negative z-direction.

(the reflection of the main mode is here considered a spurious mode). The amplitude of the incident main mode is taken to be A_0. We thus seek the solution to the following system of equations:

$$\frac{dA_m^+}{dz}+j\Gamma_m A_m^+=0, \tag{54a}$$

$$A_m^+(0)=A_0, \tag{54b}$$

$$\frac{dA_m^-}{dz}-j\Gamma_m A_m^- +\frac{1}{2}A_m^+\frac{d}{dz}(\ln Z_m)+M_{mm2}^- A_m^+=0, \tag{55a}$$

$$A_m^-(z_0)=0, \tag{55b}$$

$$\frac{dA_i^+}{dz}+j\Gamma_i A_i^+ +M_{im1}^+ A_m^+=0, \tag{56a}$$

$$A_i^+(0)=0, \tag{56b}$$

$$\frac{dA_i^-}{dz}-j\Gamma_i A_i^- +M_{im2}^- A_m^+=0, \tag{57a}$$

$$A_i^-(z_0)=0. \tag{57b}$$

The boundary condition on the backward traveling waves insures that all reflection takes place between $z=0$ and $z=z_0$.

The first equation may be easily solved after a rearrangement of terms. The remaining equations can be solved by the method of Green's function. Concentrating on the solution for the magnitude of the spurious modes, it is found that

$$A_i^+(z)=-A_0 \exp\left(-j\int_0^z \Gamma_i(t)dt\right)\int_0^z M_{im1}^+(y) \exp\left(j\int_0^y [\Gamma_i(t)-\Gamma_m(t)]dt\right)dy, \tag{58}$$

$$A_i^-(z)=A_0 \exp\left(j\int_{z_0}^z \Gamma_i(t)dt\right)\int_z^{z_0} M_{im2}^-(y) \exp\left(-j\int_{z_0}^y \Gamma_i(t)dt\right)\exp\left(-j\int_0^y \Gamma_m(t)dt\right)dy. \tag{59}$$

6. A Numerical Example

In this section the results of the preceding sections are used in a numerical example. It would of course be convenient if the example to be considered had some significance; i.e., if it could be related to a physical problem. Such a relation does in fact exist if we consider the parallel-plate waveguide as representing the earth-ionosphere waveguide and focus attention on the problem of VLF mode conversion due to a localized ionosphere height perturbation. Some work has already been done on this general problem by J. R. Wait [1961b, 1962a, b]. The approach used here differs from that of Wait, who used three essentially independent methods of attack on this problem.

Although the methods developed in the preceding sections are applicable to surface impedances which are different on the two bounding surfaces, much simplification can be realized when they are identical. The simplification comes about through the integrals representing the mode conversion. When the surface impedance terms are identical, several of the integrals vanish. Further, the case of identical surface impedance on the two bounding walls can be related to a useful first approximation to the earth-ionosphere problem. For if the earth is taken to be perfectly conducting, the method of images can be used to remove the (perfectly conducting) earth and to place another ionosphere below the earth-plane.[4] The finite con-

[4] The application of the method of images to the earth-ionosphere problem was originally suggested by J. R. Wait in connection with a model study of the problem under consideration in this section.

ductivity of the earth plays a significant role in VLF propagation, but Wait [1957] has shown that the lower boundary may be taken to be perfectly conducting for an over-sea path. In the problem to be considered here, then, we consider a parallel-plate waveguide with identical values of surface impedance on the two walls. In so doing, the earth (or sea) corresponds to the center of the waveguide. The ionosphere, then, is assumed to be homogeneous, isotropic, and sharply bounded. For the TM modes (which are the only modes of interest and the only modes to be considered) the surface impedance of the ionosphere is taken to be [Wait, 1960]

$$Z_z = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{3+j1}{5}. \tag{60}$$

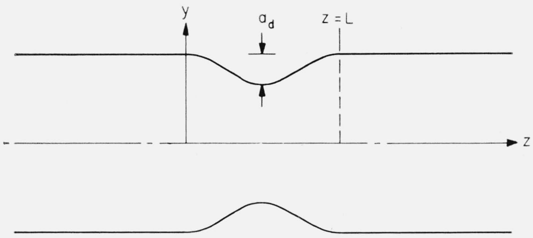

FIGURE 2. *Configuration of ionosphere height anomaly.*

The nature of the ionosphere height anomaly to be used in this example will be sinusoidal. The configuration is shown in figure 2. The analytic expression for $a(z)$ is

$$a(z) = a_0 + \frac{a_d}{2}\left[-1+\cos\frac{2\pi}{L} z\right], \tag{61}$$

where a_d is the depression magnitude, a_0 is the unperturbed ionosphere height, and L is the length of the disturbance.

Using a frequency of 16.6 kc/s and letting a_0 be 70 km, the integral for mode conversion (58) was evaluated to determine the magnitude of the forward traveling second, third, and fourth modes generated in a height anomaly one wavelength long and one wavelength deep. The solution, as obtained on the IBM 709 computer, leads to results given in table 1. The results are tabulated for half of the depression ($A_i^+(L/2)$, $i=2, 3, 4$) as well as for the entire depression ($A_i^+(L)$, $i=2, 3, 4$).

TABLE 1. *Mode conversion from main mode*

	Half depression length	Entire depression length
2d Mode:		
Amplitude	0.204	0.017
Phase radians	3.04	.57
3d Mode:		
Amplitude	0.095	0.0254
Phase radians	2.86	.54
4th Mode:		
Amplitude	0.0507	0.0297
Phase radians	2.85	2.05

In terms of the earth-ionosphere waveguide, the results in table 1 suggest that the mode conversion for propagation across a night-day transition is not negligible, the magnitude of the generated second mode being 0.2 of the magnitude of the incident mode. We assume, of course, the proper ionosphere-height function in the transition so the results of this study apply. When considering propagation across a depression such as might be produced by a nuclear explosion in the atmosphere, the mode conversion is small. Again we must assume the proper form for the manmade ionosphere depression in order that the results given here apply.

The conclusion to be drawn from these results is that, in a perturbation which is symmetrical with respect to length, the energy coupled from the main mode to a higher order mode over half the length may be partially cancelled by the energy coupled over the last half of the length. In that regard it must be emphasized that coupling between only two modes is considered in each case. The two values of amplitude of the second mode (for example) given in table 1 reflect only the interaction between the forward traveling first and second modes. The amplitude differences cannot be due to coupling from or to the third mode (say). Likewise, the values do not include the coupling from or to the reflected main mode, which must be considered as yet another spurious mode.

7. Experimental Results

Some work has been done [Gallawa, 1964] using the method outlined here to evaluate the accuracy of the technique. Work was restricted to the perfectly conducting waveguide to simplify the experimental procedure. The configuration consisted of a rectangular waveguide eight wavelengths wide and of sufficiently small height to exclude any variation of the field quantities in that direction. The width of the waveguide was perturbed in the manner shown in figure 2. The perturbation depth and length were taken to be one wavelength and ten wavelengths, respectively.

In figures 3 through 6 are plotted the electric field pattern across the waveguide width, taken to be the y-axis, for various values of distance from the end of transition. The two graphs in each case represent the theoretical pattern, based on this study, and the experimental pattern taken at the University of Colorado. The experimental curve given in figure 5 is the averaged curve of figure 6; i.e., the asymmetrical nature of the experimental curve of figure 6 was eliminated by averaging the values of the ordinate for conjugate values of the abscissa. This was done in order to maintain a reasonable balance between the intended philosophy of the experimental work (the waveguide was intended to be symmetrical about the axis) and the theoretical work.

The graphs in figures 3 through 5 indicate that the theory given here is in substantial agreement with expermental work. Yet there is a detectable discrepancy which suggests that the actual mode conversion is more extensive than that predicted by the theory. One would expect the experimental mode conversion to be greater than the theoretical, but the extent of the discrepancy is difficult to predict. The imperfections which are surely present in the waveguide and which lead to further mode conversion must also be considered. These imperfections are excluded from theoretical consideration.

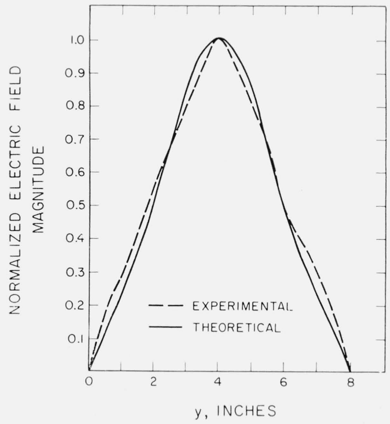

FIGURE 3. *Normalized transverse field pattern at a plane 21.906 in. from the end of a sinusoidal 10 wavelength perturbation.*

The perturbation changes the waveguide width from 8 wavelengths to 6 wavelengths.

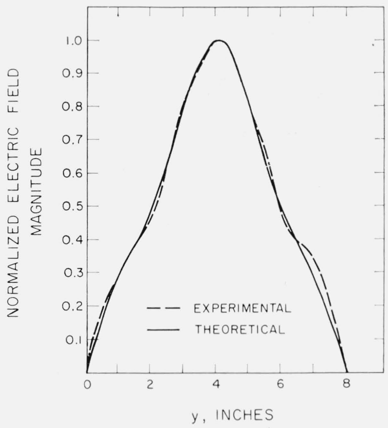

FIGURE 4. *Normalized transverse field pattern at a plane 24.094 in. from the end of a sinusoidal 10 wavelength perturbation.*

The perturbation changes the waveguide widthfrom 8 wavelengths to 6 wavelengths.

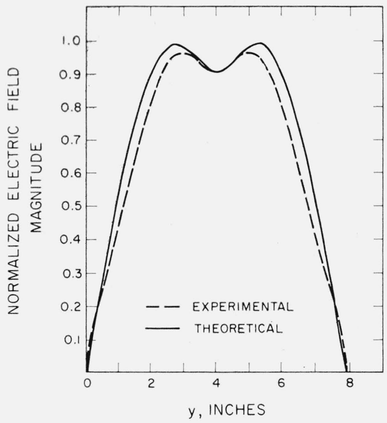

FIGURE 5. *Normalized transverse field pattern at a plane 45.000 in. from the end of a sinusoidal 10 wavelength perturbation.*

The perturbation changes the waveguide width from 8 wavelengths to 6 wavelengths.

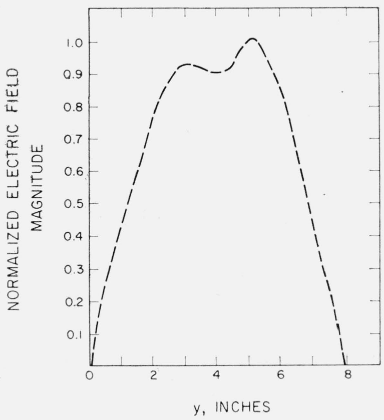

FIGURE 6. *Normalized experimental transverse field pattern at a plane 45.000 in. from the end of a sinusoidal 10 wavelength perturbation.*

The perturbation changes the waveguide width from 8 wavelengths to 6 wavelengths.

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