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On the Theory of Reflection of Electromagnetic Waves From the Interface Between a Compressible Magnetoplasma and a Dielectric

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It is assumed that a warm plasma may be described in terms of a continuum theory of fluid dynamics in combination with Maxwell's equations. The motions of the heavy ions are neglected but collisions with electrons are accounted for by a constant collision frequency. For these conditions, a solution is given for the reflection coefficient when a plane wave is incident obliquely onto a homogeneous half space of plasma. A d–c magnetic field is superimposed on the plasma in a direction parallel to the interface between the plasma and the dielectric. It is shown that previous solutions for a cold anisotropic plasma and a warm isotropic plasma are recovered as special cases.

1. Introduction

In theoretical studies of wave propagation in ionized media, it is usually assumed that the plasma is cold. This is particularly true in the vast literature devoted to ionospheric radio propagation [e.g., Ratcliffe, 1959]. In many cases this is probably well justified because the kinetic temperature of the plasma is sufficiently low. Nevertheless, it would seem to be desirable to investigate the consequences of this cold plasma assumption in some specific problems. For example, in studying the propagation of VLF radio waves, the ionosphere may often be approximated by a sharply bounded homogeneous plasma [Wait, 1962]. In this paper an explicit expression for the reflection coefficient is derived under the condition that the electroacoustic velocity is finite. In order to simplify the discussion, the superimposed d–c magnetic field is taken to be parallel to the interface and transverse to the direction of propagation. In the limiting case of a cold plasma this problem reduces to one first solved explicitly by Barber and Crombie [1959].

2. Preliminary Considerations

The plasma medium is regarded as a one-component electron fluid. In other words, the ions are neglected in the equation of motion, yet their presence is required to neutralize the plasma. However, collisions between electrons and the neutral particles are accounted for by an energy independent collision frequency ν. It is also assumed that the amplitude of the plasma and the electromagnetic oscillations are sufficiently small that linearized equations are valid [Oster, 1960]. It is further assumed that the drift velocity of the electrons is zero, so the plasma as a whole is considered stationary. The average number density of electrons is denoted n_0, which is regarded as constant in the plasma region. The mean velocity is $\vec{v}$, while their pressure deviation is p. The electric and magnetic fields are designated by their usual symbols $\vec{E}$ and $\vec{H}$, respectively.

A uniform magnetic field B_0 is assumed to be impressed throughout the plasma. This constant magnetic field is taken to be parallel to the y axis of a suitably chosen Cartesian coordinate system (x, y, z). The field quantities $\vec{E}$, $\vec{H}$, p, and $\vec{v}$ vary with time t as exp $(i\omega t)$ where ω is the angular frequency.

For the situation described above, the linearized equation of motion for the electrons is

$$mn_0(\nu+i\omega)\vec{v}=n_0e(\vec{E}+\vec{v}\times\hat{y}B_0)-\nabla p, \tag{1}$$

where $\hat{y}$ is a unit vector in the y-direction, and e and m are the charge and mass of the electron, respectively. As a consequence of assuming a scalar pressure term, Landau damping in the plasma is neglected. The linearized equation of continuity, combined with the equation of state, leads readily to

$$u^2mn_0\nabla\cdot\vec{v}=-i\omega p, \tag{2}$$

where u is the velocity of sound in the electron gas. Maxwell's equations for the plasma in the absence of sources are given by

$$\nabla\times\vec{E}=-i\mu_0\omega\vec{H}, \tag{3}$$

and

$$\nabla\times\vec{H}=i\epsilon_0\omega\vec{E}+n_0e\vec{v}, \tag{4}$$

where μ_0 and ϵ_0 are the permeability and dielectric constant of free space, respectively.

In the problem considered in this paper, there are two distinct regions separated by a plane interface at $z=0$. The plasma region occupies the upper half space $z>0$, while a dielectric of electrical constants μ_0 and ϵ_d occupies the lower half space $z<0$. Following the assumption of earlier workers [Cohen, 1962; Hessel et al., 1962; Fedorchenko, 1962], the boundary condition on the electron velocity $\vec{v}$ is that its normal component be zero at the interface $z=0$. The question whether an actual interface between a dielectric and plasma is totally effective in this regard is not answered here. However, the reader is referred to an illuminating discussion of this point by Cohen [1962]. Very recently the same boundary condition has been used by Yildiz [1963], who treated the oscillations of an isotropic compressible plasma sphere.

To simplify the problem, the obliquely incident wave from the lower half space propagates transverse to the magnetic field. In other words, the wave normal is contained in the (x, z) plane and makes an angle θ with the negative z-axis. Without losing further generality, it is necessary to consider two distinct cases. The first and most interesting situation is when the magnetic field of the incident wave is parallel to the interface. In other words, $\vec{H}^{(i)}$ of the incident wave has only a y component $H_y^{(i)}$. Therefore, we choose it to be of the form

$$H_y^{(i)}=H_0e^{-u_0z}e^{-i\lambda x}, \tag{5}$$

where $u_0=i(\epsilon_d\mu_0)^{1/2}\omega\cos\theta$ and $\lambda=(\epsilon_d\mu_0)^{1/2}\omega\sin\theta$. For the problem as defined, it is found that the necessary equations and the boundary conditions are satisfied if the reflected wave also has only a y component $H_y^{(r)}$. In fact, we may write

$$H_y^{(r)}=R_{||}H_0e^{+u_0z}e^{-i\lambda x}, \tag{6}$$

where, by definition, $R_{||}$ is the reflection coefficient.

The total magnetic field H_y in the region $z<0$ may be written

$$H_y=H_0[e^{-ikCz}+R_{||}e^{ikCz}]e^{-ikSx}, \tag{7}$$

where $C=\cos\theta$, $S=\sin\theta$, and $k=(\epsilon_d\mu_0)^{1/2}\omega$ is the wave number in the dielectric region. The electric field components in the dielectric are obtained from Maxwell's equations. Thus,

$$E_z=\frac{1}{i\epsilon_d\omega}\frac{\partial H_y}{\partial x}, \tag{8}$$

and

$$E_x=-\frac{1}{i\epsilon_d\omega}\frac{\partial H_y}{\partial z}, \tag{9}$$

for the region $z<0$. The normal wave impedance Z, at the interface, is defined by

$$Z=\left(\begin{matrix}\text{Limit}\\ z\to 0\end{matrix}\right)\frac{E_x}{H_y}. \tag{10}$$

On combining (7) and (9), it is seen that

$$Z=\eta C\left[\frac{1-R_{||}}{1+R_{||}}\right], \tag{11}$$

where $\eta=k/(\epsilon_d\omega)=(\mu_0/\epsilon_d)^{1/2}$ is the characteristic impedance of the dielectric. Conversely, the reflection coefficient may be expressed conveniently in terms of the normal surface impedance via the relation

$$R_{||}=\frac{\eta C-Z}{\eta C+Z}. \tag{12}$$

It is apparent from the above development that the problem boils down to finding an expression for the surface impedance of the compressible magnetoplasma at the interface $z=0$. Because of the continuity of the tangential fields E_x and H_y, this impedance is simply Z.

The second case alluded to above corresponds to the situation where the electric field of the incident wave is parallel to the interface and has only a y component. As it turns out, this is a rather trivial case from the electromagnetic viewpoint. Neither the d-c magnetic field nor the pressure play any role. Thus, the solution for the reflection coefficient may be obtained by regarding the half space as a cold isotropic plasma. This case will not be discussed any further.

3. Development of Equations for the Plasma

As specified in the preceding section, the magnetic field need have only a y component H_y. In what follows, the subscript y on this quantity is dropped. Furthermore, because of the assumed direction of incidence $\partial/\partial y=0$. Consequently, (1) in component form may be written

$$(\nu+i\omega)mn_0v_x=n_0e(E_x-v_zB_0)-\partial p/\partial x, \tag{13}$$

and

$$(\nu+i\omega)mn_0v_z=n_0e(E_z+v_xB_0)-\partial p/\partial z. \tag{14}$$

By simple algebraic manipulation, (13) and (14) may be rewritten in the form

$$v_x=\frac{e}{gm\alpha}E_x+\frac{e\omega_c}{mg^2\alpha}E_z-\frac{1}{g\alpha mn_0}\frac{\partial p}{\partial x}-\frac{\omega_c}{g^2mn_0\alpha}\frac{\partial p}{\partial z}, \tag{15}$$

$$v_z=-\frac{e\omega_c}{mg^2\alpha}E_x+\frac{e}{gm\alpha}E_z+\frac{\omega_c}{g^2mn_0\alpha}\frac{\partial p}{\partial x}-\frac{1}{g\alpha mn_0}\frac{\partial p}{\partial z}, \tag{16}$$

where $\alpha=1+(\omega_c/g)^2$, where $g=\nu+i\omega$ is a convenient complex frequency parameter and $\omega_c=-eB_0/m$ is the (angular) gyrofrequency for electrons.

Maxwell's curl $\vec{H}$ equation in component form is

$$-\frac{\partial H}{\partial z}=i\epsilon_0\omega E_x+n_0ev_x, \tag{17}$$

and

$$\frac{\partial H}{\partial x}=i\epsilon_0\omega E_z+n_0 e v_z. \tag{18}$$

This pair may be used in conjunction with (15) and (16) to eliminate E_x and E_z. Thus,

$$v_x=-\frac{ie\omega_c}{mg^2\omega\epsilon_0 K\alpha}\left[\frac{\partial H}{\partial x}-\frac{i\epsilon_0\omega}{n_0 e}\frac{\partial p}{\partial z}\right]-\frac{e\left(1+\frac{\omega_0^2}{i\omega g}\right)}{mi\omega g\epsilon_0 K\alpha}\left[\frac{\partial H}{\partial z}+\frac{i\epsilon_0\omega}{n_0 e}\frac{\partial p}{\partial x}\right], \tag{19}$$

and

$$v_z=-\frac{ie\omega_c}{mg^2\omega\epsilon_0 K\alpha}\left[\frac{\partial H}{\partial z}+\frac{i\epsilon_0\omega}{n_0 e}\frac{\partial p}{\partial x}\right]+\frac{e\left(1+\frac{\omega_0^2}{i\omega g}\right)}{mi\omega g\epsilon_0 K\alpha}\left[\frac{\partial H}{\partial x}-\frac{i\epsilon_0\omega}{n_0 e}\frac{\partial p}{\partial z}\right], \tag{20}$$

where $\omega_0^2=(n_0e^2)/(\epsilon_0 m)$ is the (angular) plasma frequency,

$$\alpha=1+(\omega_c/g)^2,\ K=K_1^2-K_2^2$$

with

$$K_1=\left(1+\frac{\omega_0^2}{i\omega g\alpha}\right)\text{ and } K_2=\frac{\omega_0^2\omega_c}{g^2\omega\alpha}.$$

Equations (17) and (18), used in conjunction with (15) and (16), may be utilized to eliminate v_x and v_z. From this process, we obtain

$$E_x=-\frac{K_1}{i\epsilon_0\omega K}\frac{\partial H}{\partial z}-\frac{K_2}{\epsilon_0\omega K}\frac{\partial H}{\partial x}-\frac{K_1-K}{n_0 eK}\frac{\partial p}{\partial x}+\frac{iK_2}{n_0 eK}\frac{\partial p}{\partial z}, \tag{21}$$

and

$$E_z=\frac{K_1}{i\epsilon_0\omega K}\frac{\partial H}{\partial x}-\frac{K_2}{\epsilon_0\omega K}\frac{\partial H}{\partial z}-\frac{K_1-K}{n_0 eK}\frac{\partial p}{\partial z}-\frac{iK_2}{n_0 eK}\frac{\partial p}{\partial x}. \tag{22}$$

It is evident from the above that all quantities of interest are derivable from the two scalars p and H. By inserting v_x and v_z, as given by (19) and (20), into (2), we obtain

$$\frac{\left(\frac{i\omega_0^2}{g\omega}\right)B_0}{\left(1+\frac{\omega_0^2}{i\omega g}\right)}\nabla^2 H-[\nabla^2-\Gamma_p^2]p=0, \tag{23}$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2},\text{ and } \Gamma_p^2=\frac{i\omega g\alpha K}{u^2\left(1+\frac{\omega_0^2}{i\omega g}\right)}.$$

In a similar manner, by working with (3), (21), and (22), we arrive at

$$(\nabla^2-\Gamma_e^2)H+\frac{K_2\epsilon_0\omega}{K_1 n_0 e}\nabla^2 p=0, \tag{24}$$

where

$$\Gamma_e^2=-\frac{\mu_0\epsilon_0\omega^2 K}{K_1}.$$

Equations (23) and (24) are coupled differential equations for H and p. An important special case is when $B_0=0$. Then, because $K_2=0$, these two equations decouple. Thus,

$$(\nabla^2-\gamma_p^2)p=0, \tag{25}$$

where

$$\gamma_p^2=\frac{i\omega g}{u^2}\left(1+\frac{\omega_0^2}{i\omega g}\right),$$

and

$$(\nabla^2-\gamma_e^2)H=0, \tag{26}$$

where

$$\gamma_e^2=-\mu_0\epsilon_0\omega^2\left(1+\frac{\omega_0^2}{i\omega g}\right).$$

These are the governing equations for propagation in an isotropic compressible plasma when collisions are accounted for [Wait, 1964].

Another significant special case is when the temperature of the plasma is set equal to zero. This corresponds to $\Gamma_p\to\infty$ whence, from (23), the pressure p vanishes. Equation (24) then simplifies to

$$(\nabla^2-\Gamma_e^2)H=0, \tag{27}$$

where Γ_e^2 is defined exactly as in (24). This is the wave equation for transverse propagation in a cold anisotropic plasma [Wait, 1962].

The general coupled differential equations given by (23) and (24), in the case of zero collisions, are the same as the ones derived by Seshadri [1963] who treated the radiation from a magnetic line source embedded in a homogeneous compressible plasma.

4. Solution of the Coupled Equations

To obtain the solution of the coupled equations, it is assumed that the waves in the plasma vary according to the factor exp $(-uz)$ exp $(-i\lambda x)$ where $\lambda=k\sin\theta$, and u is to be determined. Because ∇^2 may be replaced by $u^2-\lambda^2$ it follows, from (23) and (24), that

$$(u^2-\lambda^2-\Gamma_e^2)(u^2-\lambda^2-\Gamma_p^2)-\delta(u^2-\lambda^2)^2=0, \tag{28}$$

where

$$\delta=\frac{i\omega_c K_2}{gK_1}\left(1+\frac{\omega_0^2}{i\omega g}\right)^{-1}.$$

When the gyrofrequency approaches zero, the coupling factor δ vanishes, and the two solutions are $u^2=\lambda^2+\Gamma_e^2$ and $\lambda^2+\Gamma_p^2$. In accordance with earlier work, these may be described as electromagnetic and acoustic waves, respectively.

Equation (28) is now conveniently written in the form

$$(u^2-\lambda^2-\hat{\Gamma}_e^2)(u^2-\lambda^2-\hat{\Gamma}_p^2)=0, \tag{29}$$

where

$$\hat{\Gamma}_e^2=v+(v^2-w)^{1/2},$$

and

$$\hat{\Gamma}_p^2=v-(v^2-w)^{1/2},$$

with

$$v=\frac{\Gamma_p^2+\Gamma_e^2}{2(1-\delta)} \text{ and } w=\frac{\Gamma_p^2\Gamma_e^2}{1-\delta}.$$

The sign of the radicals is chosen so that $\hat{\Gamma}_e^2$ approaches Γ_e^2 and $\hat{\Gamma}_p^2$ approaches Γ_p^2 as δ tends uniformly to zero. The solutions of (29) then lead to two kinds of waves which are described by the factors

$$\exp(-u_e z)\exp(-i\lambda x) \text{ and } \exp(-u_p z)\exp(-i\lambda x),$$

where

$$u_e=(\lambda^2+\hat{\Gamma}_e^2)^{1/2} \text{ and } u_p=(\lambda^2+\hat{\Gamma}_p^2)^{1/2}.$$

It is appropriate to describe these as quasi-electromagnetic and quasi-acoustic waves, respectively. The radicals in the expression for u_e and u_p must be chosen to satisfy radiation conditions as $z\to\infty$. In the case of a finite collision frequency ν, this is equivalent to assuring that the real parts of u_e and u_p are positive.

5. Derivation of the Surface Impedance

General expressions for the magnetic field H and the pressure p within the plasma may be expressed in the form

$$H=[f_e\exp(-u_e z)+f_p\exp(-u_p z)]\exp(-i\lambda x), \tag{30}$$

and

$$p=[g_e\exp(-u_e z)+g_p\exp(-u_p z)]\exp(-i\lambda x), \tag{31}$$

where f_e, g_e, f_p, and g_p are coefficients which do not depend on the coordinates. By using (24), which is valid for all $z>0$, it follows that

$$g_e=\Lambda_e\frac{n_0 e}{\epsilon_0\omega}f_e, \tag{32}$$

and

$$g_p=\Lambda_p\frac{n_0 e}{\epsilon_0\omega}f_p, \tag{33}$$

where

$$\Lambda_e=\frac{\Gamma_e^2-\hat{\Gamma}_e^2}{\hat{\Gamma}_e^2}\frac{K_1}{K_2} \text{ and } \Lambda_p=\frac{\Gamma_e^2-\hat{\Gamma}_p^2}{\hat{\Gamma}_p^2}\frac{K_1}{K_2}.$$

Thus, the pressure is given in terms of f_e and f_p by

$$p=\frac{n_0 e}{\epsilon_0\omega}[f_e\Lambda_e\exp(-u_e z)+f_p\Lambda_p\exp(-u_p z)]\exp(-i\lambda x). \tag{34}$$

Using (20), (30), and (34) and the boundary condition $v_z=0$, we obtain

$$\frac{f_p}{f_e}=-\frac{(i\omega_c/g)[u_e-\Lambda_e\lambda]-\left(1+\frac{\omega_0^2}{i\omega g}\right)(\lambda-\Lambda_e u_e)}{(i\omega_c/g)[u_p-\Lambda_p\lambda]-\left(1+\frac{\omega_0^2}{i\omega g}\right)(\lambda-\Lambda_p u_p)}. \tag{35}$$

This means that the rigidity condition at the interface fixes the ratio of the quasi-acoustic wave to the quasi-electromagnetic wave. In the special case of a cold plasma (i.e., $u=0$), $\Lambda_p\to\infty$ and, therefore, the ratio f_p/f_e vanishes. Another limiting case is when the plasma becomes isotropic (i.e., $\omega_c=0$). Then, again, $\Lambda_p\to\infty$ and f_p/f_e become vanishingly small. However, in this case, a more meaningful ratio is g_p/f_e, which approaches the finite value $n_0 e\lambda/(\epsilon_0\omega u_p)$ in accordance with the required behavior for an isotropic compressible plasma [Wait, 1964].

Using (21), (30), and (34), the following expression is obtained for the tangential electric field at the interface:

$$E_x]_{z=0}=\frac{1}{\epsilon_0\omega}\left[\frac{K_1 u_e}{iK}+\frac{K_2 i\lambda}{K}+\frac{K_1-K}{K}\Lambda_e i\lambda-\frac{K_2}{K}\Lambda_e i u_e\right]f_e$$
$$+\frac{1}{\epsilon_0\omega}\left[\frac{K_1 u_p}{iK}+\frac{K_2 i\lambda}{K}+\frac{K_1-K}{K}\Lambda_p i\lambda-\frac{K_2}{K}\Lambda_p i u_p\right]f_p. \tag{36}$$

According to (10), it is noted that

$$E_x]_{z=0}=ZH_y]_{z=0}=Z(f_e+f_p). \tag{37}$$

Therefore, on equating (36) and (37), we arrive at the rather cumbersome expression for the surface impedance.

$$Z=\left(\frac{\lambda}{\epsilon_0\omega}\right)\frac{i}{\left(1+\frac{f_p}{f_e}\right)K}\left\{\left[K_2-\frac{K_1 u_e}{\lambda}+(K_1-K)\Lambda_e-\frac{K_2\Lambda_e u_e}{\lambda}\right]+\frac{f_p}{f_e}\left[K_2-\frac{K_1 u_p}{\lambda}+(K_1-K)\Lambda_p-\frac{K_2\Lambda_p u_p}{\lambda}\right]\right\}, \tag{38}$$

where f_p/f_e is given explicitly by (35).

As a partial check on the correctness of (38), we consider the case of a cold anisotropic plasma (i.e., $u=0$). Then, since $f_p/f_e\to 0$ and $\Lambda_e\to 0$, it immediately follows that $Z\to Z_c$ where

$$Z_c=\frac{i}{\epsilon_0\omega K}(K_2\lambda-K_1 u_e) \text{ with } u_e=(\lambda^2+\Gamma_e^2)^{\frac{1}{2}},$$

which is in agreement with a result derived originally by Barber and Crombie [1959]. Another special case is a warm isotropic plasma (i.e., $\omega_c=0$). Then, $\delta=0$, $\hat{\Gamma}_e \rightarrow \Gamma_e$, $\hat{\Gamma}_p \rightarrow \Gamma_p$, $K_2 \rightarrow 0$, and $K/K_1 \rightarrow [1+\omega_0^2/(i\omega g)]=\epsilon/\epsilon_0$ where ϵ is the dielectric constant of the isotropic plasma. In this limiting case, $(f_p/f_e) \rightarrow 0$ and $\Lambda_p \rightarrow \infty$ but in such a way that $(\Lambda_p f_p/f_e) \rightarrow \lambda/u_p$. As a result of these considerations, $Z \rightarrow Z_i$ where

$$Z_i=\frac{u_e}{i\epsilon\omega}\left[1-\frac{\lambda^2}{u_p u_e}\frac{i\omega_0^2}{g\omega}\right], \tag{39}$$

where

$$u_e=(\lambda^2+\gamma_e^2)^{\frac{1}{2}} \text{ and } u_p=(\lambda^2+\gamma_p^2)^{\frac{1}{2}},$$

with

$$\gamma_p^2=\frac{i\omega g}{u^2}\left(1+\frac{\omega_0^2}{i\omega g}\right)=-\frac{\omega^2}{u^2}\left(1+\frac{\nu}{i\omega}\right)\frac{\epsilon}{\epsilon_0},$$

and

$$\gamma_e^2=-\epsilon_0\mu_0\omega^2\left(1+\frac{\omega_0^2}{i\omega g}\right)=-\epsilon\mu_0\omega^2.$$

The above result for Z_i is in agreement with an independent derivation for an isotropic warm plasma [Wait, 1964]. It is immediately evident that the bracketed term in (39) may be replaced by unity if $u/c << 1$, which is the condition for the validity of the cold plasma assumption.

6. Concluding Remarks

An explicit expression for the reflection coefficient for a compressible magnetoplasma has been given in this paper. It is shown that it reduces to the well-known result for a cold magnetoplasma when the ratio of the acoustic velocity u is small compared with the velocity c of electromagnetic waves in free space. The general expression also reduces to a previously derived result for reflection from an isotropic compressible half space of plasma.

7. References

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