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Improvement of Bounds to Eigenvalues of Operators of the Form T^*T [1]

Norman W. Bazley [2] and David W. Fox [3]

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The authors generalize a method of Lehmann and Maehly for upper and lower bounds to eigenvalues of self-adjoint operators in Hilbert space by using a device introduced by Kato. The resulting procedure can be used to improve bounds found by the Rayleigh-Ritz method and the comparison methods of Weinstein, Aronszajn, and the authors by means of calculations involving *easily found* vectors; and it is especially suitable for application to problems of vibration of continuous elastic systems. Further a theorem of Kato is interpreted and extended by the results obtained.

1. Introduction

For the eigenvalue problems that arise in the theory of vibration of continuous elastic systems it is useful to have auxiliary methods capable of improving the rigorous upper and lower bounds that can be found by the Rayleigh-Ritz and comparison operator procedures [1, 2, 3, 4] [4]. We present here the theory of a method which should be quite useful in problems of this kind.

The procedure of Lehmann and Maehly [5, 6] is in principle applicable, since it depends on the use of an essentially arbitrary family of "trial vectors" and the knowledge of numbers that are known to separate adjacent eigenvalues. But while these numbers can be obtained by use of the upper and lower bound procedures already referred to, the trial vectors must satisfy all of the boundary conditions of the differential operators; and finding such vectors is far from an easy task. On the other hand Kato has developed [7] and applied [8] an extension of the Temple method [9] for operators of the form T^*T that uses the same numbers and avoids the difficulty of needing trial vectors that satisfy all of the boundary conditions. While the form T^*T is a common property of the operators that arise in elastic vibration theory, the procedure of Kato does not lend itself to optimization over a family of trial vectors. In the following lines we dicuss the use of a device introduced by Kato in the procedure of Lehmann and Maehly. The resulting procedure optimizes the estimate of Kato over a family of easily found trial vectors.

Section 2 sketches the needed properties of operators of the form T^*T and of the related operator H introduced by Kato. Section 3 then discusses the application of the procedure of Lehmann and Maehly to the operator H, and section 4 uses the results of section 3 to interpret and extend a theorem of Kato.

2. Operators of the Form T^*T and a Device of T. Kato

Here we are concerned with a positive self-adjoint operator A with domain $\mathfrak{D}_A$ in a Hilbert space $\mathfrak{H}^1$ for which the inner product is $(u, v)_1$. We will assume, for convenience, that the operator A has for its spectrum an infinite discrete set of eigenvalues, $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$, that diverge to infinity without finite limit points; i.e., that $(A+I)^{-1}$ is completely continuous. Further, we require A to have the form T^*T. The operator T is to be a closed operator defined on a dense domain $\mathfrak{D}_T$ in $\mathfrak{H}^1$ and having its range $\mathfrak{R}_T$ in a Hilbert space $\mathfrak{H}^2$ with inner product $(u, v)_2$; T^* is the operator adjoint to T. Thus, $\mathfrak{D}_{T^*}$ is in $\mathfrak{H}^2$, $\mathfrak{R}_{T^*}$ is in $\mathfrak{H}^1$, and together T and T^* satisfy $(Tu, v)_2 = (u, T^*v)_1$ for

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[2] Institut Battelle, Geneva, Switzerland.
[3] Applied Physics Laboratory, The Johns Hopkins University, Silver Spring, Maryland.
[4] Figures in brackets indicate the literature references at the end of this paper.

every u in $\mathfrak{D}_T$ and every v in $\mathfrak{D}_{T^*}$. The spaces $\mathfrak{H}^1$ and $\mathfrak{H}^2$ may possibly coincide. The properties of operators of the form T^*T have been widely studied by von Neumann [10], Murray [11], and others. Here we need draw on only a few of the properties of T, T^*, T^*T, and TT^*. We recall that the operators T^*T $(=A)$ and TT^* are positive self-adjoint operators which are isomorphic except for their null spaces $\mathfrak{N}_T$ and $\mathfrak{N}_{T^*}$ and that

$$\mathfrak{N}_T=\mathfrak{H}^1\ominus\overline{\mathfrak{R}}_{T^*}, \qquad \mathfrak{N}_{T^*}=\mathfrak{H}^2\ominus\overline{\mathfrak{R}}_T.$$

We define with Kato [7] the operator H in $\mathfrak{H}^1\times\mathfrak{H}^2$ with inner product (u, v) given by $(u, v)=(u^1, v^1)_1+(u^2, v^2)_2$ by

$$H[u^1, u^2]=[T^*u^2, Tu^1] \tag{1}$$

on all pairs $[u^1, u^2]$ in $\mathfrak{D}_T\times\mathfrak{D}_{T^*}$. H is a self-adjoint operator defined on $\mathfrak{D}_T\times\mathfrak{D}_{T^*}$ and has a real spectrum that is symmetric about zero. In fact, the unitary transformation U on $\mathfrak{H}^1\times\mathfrak{H}^2$, defined by

$$U[u^1, u^2]=[u^1, -u^2], \tag{2}$$

transforms H into $-H$, that is,

$$-H=U^{-1}HU, \tag{3}$$

so that H and $-H$ are isomorphic. Further the operator H^2, which has the expression and domain given by

$$H^2[u^1, u^2]=[T^*Tu^1, TT^*u^2]$$

and

$$\mathfrak{D}_{H^2}=\mathfrak{D}_{T^*T}\times\mathfrak{D}_{TT^*},$$

respectively, is reduced by $\mathfrak{H}^1$ and $\mathfrak{H}^2$ regarded as subspaces of $\mathfrak{H}^1\times\mathfrak{H}^2$. Evidently, $\mathfrak{N}_H=\mathfrak{N}_{H^2}=\mathfrak{N}_T\times\mathfrak{N}_{T^*}$. Excluding zero, H^2 has the same eigenvalues λ_ν as A but with the multiplicity doubled; since H is isomorphic with $-H$, it has just the nonzero numbers $+\lambda_\nu^{1/2}$ and $-\lambda_\nu^{1/2}$ as symmetrically arranged eigenvalues, each having the multiplicity of λ_ν in A. Thus the spectrum of H is completely described in terms of that of A with the possible addition of zero from $\mathfrak{N}_{T^*}$. On the other hand bounds for the eigenvalues of H can be converted into bounds for eigenvalues of A by squaring.

3. Application of the Lehmann-Maehly Procedure to the Operator H

In this section we apply the Lehmann-Maehly procedure to the operator H of section 2. We designate the *separation constant* by ρ or $-\rho$, where ρ is a non-negative real number. Our discussion considers first ρ in the resolvent set of H and then turns to ρ in the spectrum of H; finally, we consider what happens as ρ passes from the resolvent set to the spectrum.

The procedure of Lehmann and Maehly applied to a self-adjoint operator B makes use of a real constant τ that separates adjacent eigenvalues of B and amounts to the calculation of Rayleigh-Ritz bounds for the eigenvalues of the bounded operator $(B-\tau)^{-1}$ based on trial vectors φ of the form $\varphi=(B-\tau)v$, where the v's are linearly independent vectors from $\mathfrak{D}_B$. The resulting bounds are a consequence of the fact that for bounded operators the Rayleigh-Ritz method gives "inner bounds" for "outer eigenvalues".

3.1. ρ in the Resolvent Set of H

Let us suppose, at first, that ρ is a positive real number not in the spectrum of H. Then $-\rho$ is not in the spectrum of H, and $-\rho$ and ρ satisfy

$$-\lambda_{n+1}^{1/2}<-\rho<-\lambda_n^{1/2} \text{ and } \lambda_n^{1/2}<\rho<\lambda_{n+1}^{1/2},$$

respectively, for some [5] n. The operator $(H-\rho)^{-1}$ is bounded. The positive spectrum of $(H-\rho)^{-1}$

[5] We introduce the "conventional" eigenvalue λ_0 equal to zero whenever zero is not in the spectrum of A in order to unify the presentation.

consists of an infinite decreasing sequence of eigenvalues,

$$(\lambda_{n+1}^{1/2}-\rho)^{-1} \geqslant (\lambda_{n+2}^{1/2}-\rho)^{-1} \geqslant \ . \ . \ . > 0.$$

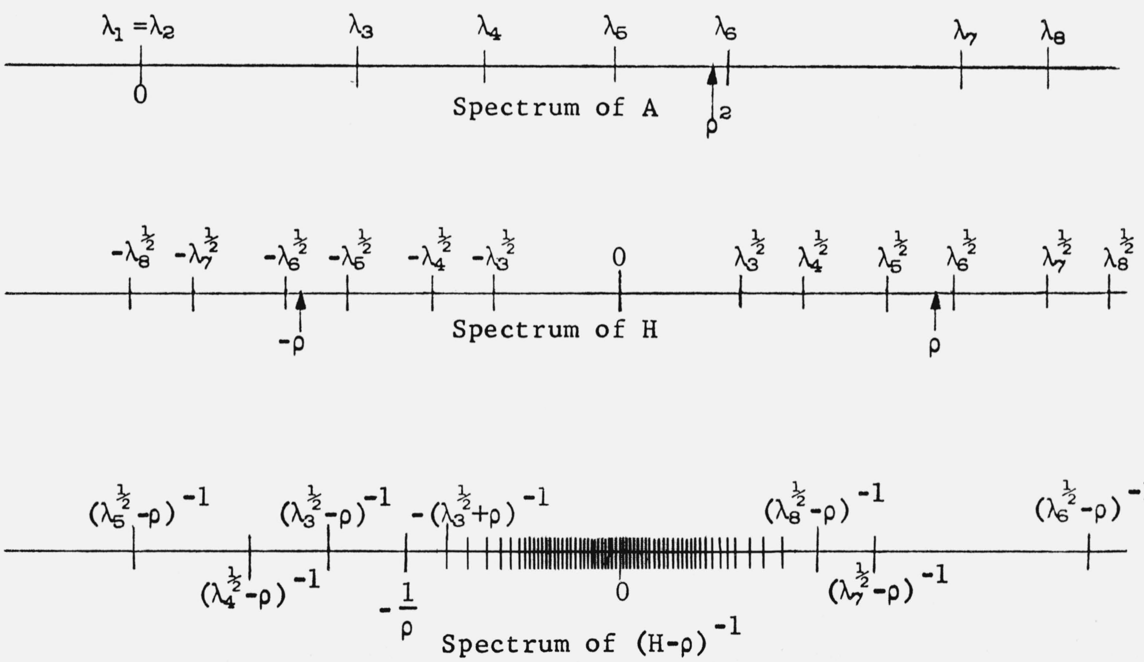
The negative spectrum begins with the increasing finite set (which is empty if $\lambda_n = \lambda_0 = 0$),

$$-(\rho-\lambda_n^{1/2})^{-1} \leqslant -(\rho-\lambda_{n-1}^{1/2})^{-1} \leqslant \ . \ . \ . \leqslant -(\rho-\lambda_p^{1/2})^{-1},$$

where λ_p designates the first nonzero eigenvalue of A; the point $-1/\rho$ has an associated characteristic subspace equal to $\mathfrak{N}_H$; and the interval $(-1/\rho, 0)$ contains the infinite set of negative eigenvalues,

$$-(\rho+\lambda_p^{1/2})^{-1} \leqslant -(\rho+\lambda_{p+1}^{1/2})^{-1} \leqslant \ . \ . \ . < 0.$$

The following spectral diagrams may be helpful. For the sake of illustration we have assumed $\rho = 3$ so that $\lambda_1 = \lambda_2 = 0$, $\lambda_3 > 0$ and $\lambda_5 < \rho^2 < \lambda_6$.



The operator $(H+\rho)^{-1}$ is also bounded and has the same norm as $(H-\rho)^{-1}$. In fact, since H and $-H$ are isomorphic, it follows that $(H+\rho)^{-1}$ is isomorphic with $-(H-\rho)^{-1}$, so that the spectrum of $(H+\rho)^{-1}$ is the same as that of $(H-\rho)^{-1}$ after a reflection about the origin.

The Lehmann-Maehly procedure can be applied to H using ρ or $-\rho$ as the separating constant. If we use ρ and base the procedure on an m^+-dimensional manifold $\mathfrak{M}^+$ of $\mathfrak{D}_H$, then the resulting eigenvalues, designated by μ^+, are the stationary values over vectors v in $\mathfrak{M}^+$ of the ratio

$$\frac{((H-\rho)v, v)}{((H-\rho)v, (H-\rho)v)},$$

and may be determined from a matrix eigenvalue equation of the form

$$(A-\mu^+ B)\alpha = 0.$$

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The matrices A and B are given in terms of a linearly independent basis $\{v_i\}$ for $\mathfrak{M}^+$ by

$$A=\{((H-\rho)v_i,\ v_j)\}\ \text{and}\ B=\{((H-\rho)v_i,\ (H-\rho)v_j)\}\,.$$

Let us agree to enumerate the nonzero eigenvalues by the useful but unconventional scheme,

$$\mu_n^+\leqslant\mu_{n-1}^+\leqslant\ .\ .\ .\ \leqslant\mu_{n+1-j^+}^+<-\frac{1}{\rho}\leqslant\mu_{p-s}^+\leqslant\mu_{p-s+1}^+\leqslant\ .\ .\ .\ \leqslant\mu_{p+k^+-s-1}^+<0,$$

and

$$0<\mu_{n+l^+}^+\leqslant\ .\ .\ .\ \leqslant\mu_{n+2}^+\leqslant\mu_{n+1}^+,$$

where s is the dimension (possibly infinite) of $\mathfrak{N}_H$, and the quantities j^+, k^+, and l^+ are nonnegative integers that satisfy $j^+\leqslant n-p+1$ and $j^++k^++l^+\leqslant m^+$. Since the negative eigenvalues give upper bounds to the negative eigenvalues of $(H-\rho)^{-1}$ and the positive give lower bounds to the positive, we have [6]

$$\mu_r^+\geqslant\ \ (\lambda_r^{1/2}-\rho)^{-1},\qquad r=n,\ n-1,\ .\ .\ .,\ n+1-j^+,\tag{4}$$

$$\mu_r^+\geqslant-(\lambda_r^{1/2}+\rho)^{-1},\qquad r=p,\ p+1,\ .\ .\ .,\ p+k^+-s-1,\tag{5}$$

and

$$\mu_r^+\leqslant\ \ (\lambda_r^{1/2}-\rho)^{-1},\qquad r=n+1,\ n+2,\ .\ .\ .,\ n+l^+,\tag{6}$$

or equivalently,

$$\lambda_r^{1/2}\geqslant\rho+\frac{1}{\mu_r^+},\qquad r=n,\ n-1,\ .\ .\ .,\ n+1-j^+,\tag{7}$$

$$-\lambda_r^{1/2}\geqslant\rho+\frac{1}{\mu_r^+},\qquad r=p,\ p+1,\ .\ .\ .,\ p+k^+-s-1,\tag{8}$$

and

$$\lambda_r^{1/2}\leqslant\rho+\frac{1}{\mu_r^+},\qquad r=n+1,\ n+2,\ .\ .\ .,\ n+l^+.\tag{9}$$

When they are squared, the inequalities (7), (8), and (9) give lower bounds for λ_n, λ_{n-1}, . . ., λ_{n+1-j^+} and upper bounds for λ_p, λ_{p+1}, . . ., λ_{p+k^+-s-1}, and for λ_{n+1}, λ_{n+2}, . . ., λ_{n+l^+}.

If $-\rho$ is used in place of ρ in the Lehmann-Maehly procedure on H, we designate the eigenvalues based on an m^--dimensional subspace $\mathfrak{M}^-$ in $\mathfrak{D}_H$ by μ^- and enumerate those that are nonzero according to

$$\mu_{n+1}^-\leqslant\mu_{n+2}^-\leqslant\ .\ .\ .\ \leqslant\mu_{n+l^-}^-<0,$$

and

$$0<\mu_{p+k^--s-1}^-\leqslant\ .\ .\ .\ \leqslant\mu_{p-s+1}^-\leqslant\mu_{p-s}^-\leqslant\frac{1}{\rho}<\mu_{n+1-j^-}^-\leqslant\ .\ .\ .\ \leqslant\mu_{n-1}^-\leqslant\mu_n^-,$$

[6] We omit the valid but useless bounds $\mu_n^+\geqslant-\frac{1}{\rho}$, $n=p-s,\ p-s+1,\ .\ .\ .,\ p-1$ and note that (5) gives useful information only when s is finite and $k^+>s$.

where j^-, k^-, and l^- are nonnegative integers that satisfy $j^- \leqslant n-p+1$ and $j^-+k^-+l^- \leqslant m^-$. By arguments parallel with those for ρ we find the bounds

$$\lambda_r^{1/2} \geqslant \rho - \frac{1}{\mu_r^-}, \qquad r=n, n-1, \ldots, n+1-j^-, \tag{10}$$

$$-\lambda_r^{1/2} \geqslant \rho - \frac{1}{\mu_r^-}, \qquad r=p, p+1, \ldots, p+k^- - s-1, \tag{11}$$

and

$$\lambda_r^{1/2} \leqslant \rho - \frac{1}{\mu_r^-}, \qquad r=n+1, n+2, \ldots, n+l^-. \tag{12}$$

When squared, these inequalities give lower bounds for $\lambda_n, \lambda_{n-1}, \ldots, \lambda_{n+1-j^-}$, and upper bounds for $\lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+k^- -s-1}$ and for $\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{n+l^-}$.

Although, in general, the bounds given by the procedure using ρ will differ from those obtained using $-\rho$, even when the manifolds $\mathfrak{M}^+$ and $\mathfrak{M}^-$ are the same, there is an important case in which they will be identical. This happens when

$$U\mathfrak{M}^+ = \mathfrak{M}^-, \tag{13}$$

where U is the unitary transformation given by (2), as is clear from the equality,

$$\frac{((H-\rho)v, v)}{((H-\rho)v, (H-\rho)v)} = -\frac{((H+\rho)Uv, Uv)}{((H+\rho)Uv, (H+\rho)Uv)},$$

that follows from (3). In fact, when (13) holds we have $\mu_r^+ = -\mu_r^-$, and $j^+=j^-$, $k^+=k^-$, and $l^+=l^-$. When $\mathfrak{M}^+$ and $\mathfrak{M}^-$ are equal and (13) is satisfied, then it is always possible to choose a basis $\{v_i\}$ such that each vector has the form $[v^1, 0]$ or $[0, v^2]$, as is evident from the definition (2) of U.

No matter whether ρ or $-\rho$ is used, the bounds given on the right hand sides of (7), (8), (9), (10), (11), and (12) are increasing with ρ. In fact, if we designate a right hand side of an inequality (7), (8) or (9) by ω^+, then it follows from the stationary property of μ^+ that ω^+ is a stationary value of the quotient

$$\frac{((H-\rho)v, Hv)}{((H-\rho)v, v)} \tag{14}$$

over the vectors of $\mathfrak{M}^+$, and that a vector v^+ that makes the quotient stationary gives

$$\omega^+ = \frac{((H-\rho)v^+, Hv^+)}{((H-\rho)v^+, v^+)}.$$

When the eigenvalues ω^+ and corresponding eigenvectors v^+ are considered as functions of ρ they are analytic.[7] Considering the variation of a ω^+ with respect to ρ we find that

$$\frac{d\omega^+}{d\rho} = \frac{(Hv^+, Hv^+)(v^+, v^+) - (Hv^+, v^+)^2}{((H-\rho)v^+, v^+)^2} \geqslant 0.$$

In computing this derivative the contribution from the variation of v^+ with ρ is zero since the ratio (14) is stationary at v^+. Similarly, for ω^-, a right hand side in (10), (11), or (12), we find

$$\omega^- = -\frac{((H+\rho)v^-, Hv^-)}{((H+\rho)v^-, v^-)},$$

[7] Strictly speaking, it may be necessary in order to preserve their analyticity in ρ to rename the eigenvalues and eigenvectors in neighborhoods of ρ where they are multiple. This in no way affects the validity of the argument since all the ω^+'s are nondecreasing in ρ.

and

$$\frac{d\omega^-}{d\rho}=\frac{(Hv^-, Hv^-)(v^-, v^-)-(Hv^-, v^-)^2}{((H+\rho)v^-, v^-)^2}\geqslant 0.$$

Since all of the bounds are increasing with ρ, the best lower bounds will be obtained when ρ^2 is the largest known lower bound to λ_{n+1}; and, conversely, the best upper bounds will be taken when ρ^2 is the smallest known upper bound to λ_n.

3.2. ρ in the Spectrum of H

Let us now suppose that ρ coincides with an eigenvalue of H. Our considerations can be conveniently divided into the two cases: $\rho=0$ and $\rho>0$. (The results for $-\rho$ follow analogously to those for ρ.)

If ρ is zero then the Lehmann-Maehly procedure can be applied to the restriction of H to the orthogonal complement of the null space $\mathfrak{N}_H$. As we shall see, this produces upper bounds for the *strictly positive* eigenvalues of A, but these upper bounds may not be as good as those obtained from the usual Rayleigh-Ritz procedure based on essentially the same family of trial vectors. Let us designate the restriction of H to the orthogonal complement of $\mathfrak{N}_H$ by H^0 and the orthogonal projection on this complement by P^0. We apply the procedure using an m^0-dimensional manifold $\mathfrak{M}^0$ of $\mathfrak{D}_{H^0}$. Let us designate the resulting nonzero eigenvalues according to

$$\mu_p^0\leqslant\mu_{p+1}^0\leqslant\ .\ .\ .\ \leqslant\mu_{p+t_1-1}^0<0<\tilde{\mu}_{p+t_2-1}^0\leqslant\ .\ .\ .\ \leqslant\tilde{\mu}_{p+1}^0\leqslant\tilde{\mu}_p^0,$$

where t_1 and t_2 are nonnegative integers that satisfy $t_1+t_2\leqslant m^0$. Since the negative eigenvalues are upper bounds to the negative eigenvalues of $(H^0)^{-1}$ and the positive to the positive, we have

$$\lambda_r^{1/2}\leqslant-\frac{1}{\mu_r^0},\qquad r=p, p+1, \ldots, p+t_1-1, \tag{15}$$

and

$$\lambda_r^{1/2}\leqslant\frac{1}{\tilde{\mu}_r^0},\qquad r=p, p+1, \ldots, p+t_2-1. \tag{16}$$

Further, if $U\mathfrak{M}^0=\mathfrak{M}^0$ we have $t_1=t_2$ and $\tilde{\mu}_r^0=-\mu_r^0$, as follows from (3).

It is important to note that to apply the Lehmann-Maehly procedure with ρ equal to zero, it is no way necessary to know explicitly the operators H^0 or P^0 or to find vectors orthogonal to $\mathfrak{N}_H$. In fact, if $\tilde{\mathfrak{M}}^0$ is any m^0-dimensional manifold in $\mathfrak{D}_H$ for which rank $\{(Hv_i, Hv_j)\}=m^0$, where $\{v_i\}$ is a basis for $\tilde{\mathfrak{M}}^0$, then the manifold $\mathfrak{M}^0$ given by $P^0\tilde{\mathfrak{M}}^0$ is of dimension m^0 as well and lies in the orthogonal complement of $\mathfrak{N}_H$. Further, the inner products used in the procedure are just (Hv_i, v_j) and (Hv_i, Hv_j), as follows from the relation $P^0H^0P^0=H^0=H$ on $\mathfrak{D}_H$.

If A has no zero eigenvalues, then the upper bounds given by (15) and (16) are always weaker than those given by the usual Rayleigh-Ritz procedure for A based on the projection of $\tilde{\mathfrak{M}}^0$ on $\mathfrak{H}^1$. In fact, we may assume that $\tilde{\mathfrak{M}}^0$ satisfies $U\tilde{\mathfrak{M}}^0=\tilde{\mathfrak{M}}^0$, for if it does not, we can replace it by the even larger space obtained by forming $\tilde{\mathfrak{M}}^0+U\tilde{\mathfrak{M}}^0$ and then removing the submanifold that lies in $\mathfrak{N}_H$. The bounds obtained from the enlarged space will be better than those given by (10) and (11), and they will be symmetrically arranged about the origin. As we shall see, the Rayleigh-Ritz bounds will be even better. Let us suppose, then, that $\tilde{\mathfrak{M}}^0$ satisfies $U\tilde{\mathfrak{M}}^0=\tilde{\mathfrak{M}}^0$ and that a basis is taken using vectors of the form $[v^1, 0]$ and $[0, v^2]$. Those of the first set form a basis for the projection of $\tilde{\mathfrak{M}}^0$ on $\mathfrak{H}^1$, and those of the second for the projection of $\tilde{\mathfrak{M}}^0$ on $\mathfrak{H}^2$ after removal of the part in $\mathfrak{N}_H$. The matrix eigenvalue problem takes the form

$$\left\{\left(\begin{array}{c|c} 0 & C \\ \hline C^* & 0 \end{array}\right)-\mu\left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right)\right\}\left(\begin{array}{c} \alpha^1 \\ \hline \alpha^2 \end{array}\right)=0, \tag{17}$$

where B_1 and B_2 are the positive definite matrices given by

$$B_1 = \{(Tv_i^1, Tv_j^1)_2\}, \qquad \text{and } B_2 = \{(T^*v_i^2, T^*v_j^2)_1\},$$

and C and C^* are the matrices given by

$$C = \{(Tv_i^1, v_j^2)_2\}, \qquad \text{and } C^* = \{(T^*v_i^2, v_j^1)_1\}.$$

But the matrix equation (17) has just the same nonzero eigenvalues as the equation

$$(CB_2^{-1}C^* - \mu^2 B_1)\alpha^1 = 0. \tag{18}$$

However, since the matrix $CB_2^{-1}C^*$ is the Gram matrix of the orthogonal projection in $\mathfrak{H}^1$ of the vectors $\{v_i^1\}$ on the subspace spanned by the vectors $\{T^*v_i^2\}$, we have the matrix inequality,

$$CB_2^{-1}C^* \leq \Gamma,$$

in which Γ is the Gram matrix of the vectors $\{v_i^1\}$. Thus, according to the minimum-maximum principle, the eigenvalues, $\hat{\mu}_1^2 \geq \hat{\mu}_2^2 \geq \ldots$, of the matrix equation

$$(\Gamma - \hat{\mu}^2 B_1)\alpha^1 = 0 \tag{19}$$

are larger than those of (18). That is,

$$(\mu_i^0)^2 \leq \hat{\mu}_i^2, \qquad i = 1, 2, \ldots, t,$$

and hence,

$$\lambda_i \leq \frac{1}{\hat{\mu}_i^2} \leq \frac{1}{(\mu_i^0)^2}, \qquad i = 1, 2, \ldots, t.$$

But the matrix equation (19) is just that which arises from the Rayleigh-Ritz procedure for A using the projection of $\tilde{\mathfrak{M}}^0$ on $\mathfrak{H}^1$.

If, however, A has zero in its spectrum, our conclusion is no longer valid, for the Rayleigh-Ritz procedure gives upper bounds starting with the lowest, which is zero, while the Lehmann-Maehly procedure starts with the first *strictly positive* eigenvalue.

When ρ is equal to a nonzero eigenvalue of H, that is, $0 < \rho = \lambda_\nu^{1/2}$, we proceed in much the same way as for ρ equal to zero. Suppose λ_ν is an eigenvalue of A of multiplicity t, i.e.,

$$\lambda_{\nu-1} < \lambda_\nu = \lambda_{\nu+1} = \ldots = \lambda_{\nu+t-1} < \lambda_{\nu+t}, \tag{20}$$

and that $\mathfrak{N}_\nu^+$ is the characteristic subspace associated with $\lambda_\nu^{1/2}$ as an eigenvalue of H. We designate by H_ν^+ the restriction of H to the orthogonal complement of $\mathfrak{N}_\nu^+$ and apply to it the procedure using ρ and an m_ν^+-dimensional manifold $\mathfrak{M}_\nu^+$ of vectors in $\mathfrak{D}_{H_\nu^+}$. The negative eigenvalues less than $-1/\rho$ of the procedure give rise to lower bounds for eigenvalues of H_ν^+ below ρ. Since the spectrum of H_ν^+ coincides with that of H after the omission of $\lambda_\nu^{1/2}$, we find bounds to eigenvalues of A. Designating the nonzero eigenvalues according to

$$\mu_{\nu-1}^{+\nu} \leq \mu_{\nu-2}^{+\nu} \leq \ldots \leq \mu_{\nu-j_\nu^+}^{+\nu} < -\frac{1}{\rho} \leq \mu_{p-s}^{+\nu} \leq \mu_{p-s+1}^{+\nu} \leq \ldots \leq \mu_{p+k_\nu^+-s-1}^{+\nu} < 0$$

and

$$0 < \mu_{\nu+t+l_\nu^+-1}^{+\nu} \leq \ldots \leq \mu_{\nu+t+1}^{+\nu} \leq \mu_{\nu+t}^{+\nu},$$

where j_ν^+, k_ν^+, and l_ν^+ are nonnegative integers that satisfy $j_\nu^+ \leqslant \nu - p$, $j_\nu^+ + k_\nu^+ + l_\nu^+ \leqslant m_\nu^+$, we obtain the bounds

$$\lambda_r^{1/2} \geqslant \rho + \frac{1}{\mu_r^{+\nu}}, \qquad r = \nu - 1, \nu - 2, \ldots, \nu - j_\nu^+, \tag{21}$$

$$-\lambda_r^{1/2} \geqslant \rho + \frac{1}{\mu_r^{+\nu}}, \qquad r = p, p+1, \ldots, p + k_\nu^+ + s - 1, \tag{22}$$

and

$$\lambda_r^{1/2} \geqslant \rho + \frac{1}{\mu_r^{+\nu}}, \qquad r = \nu + t, \nu + t + 1, \ldots, \nu + t + l_\nu^+ - 1. \tag{23}$$

Parallelly, using $-\rho$ we find bounds in terms of the eigenvalues,

$$\mu_{\nu+t}^{-\nu} \leqslant \mu_{\nu+t+1}^{-\nu} \leqslant \ldots \leqslant \mu_{\nu+t+l_\nu^- -1}^{-\nu} < 0,$$

and

$$0 < \mu_{p+k_\nu^- - s - 1}^{-\nu} \leqslant \ldots \leqslant \mu_{p-s+1}^{-\nu} \leqslant \mu_{p-s}^{-\nu} \leqslant \frac{1}{\rho} < \mu_{\nu - j_\nu^-}^{-\nu} \leqslant \ldots \leqslant \mu_{\nu-2}^{-\nu} \leqslant \mu_{\nu-1}^{-\nu},$$

according to

$$\lambda_r^{1/2} \geqslant \rho - \frac{1}{\mu_r^{-\nu}}, \qquad r = \nu - 1, \nu - 2, \ldots, \nu - j_\nu^-, \tag{24}$$

$$-\lambda_r^{1/2} \geqslant \rho - \frac{1}{\mu_r^{-\nu}}, \qquad r = p, p+1, \ldots, p + k_\nu^- - s - 1, \tag{25}$$

and

$$\lambda_r^{1/2} \leqslant \rho - \frac{1}{\mu_r^{-\nu}}, \qquad r = \nu + t, \nu + t + 1, \ldots, \nu + t + l_\nu^- - 1. \tag{26}$$

As in the previous cases, we find that $\mu_r^{+\nu} = \mu_r^{-\nu}$ provided that $U\mathfrak{M}_\nu^+ = \mathfrak{M}_\nu^-$. The characteristic subspace $\mathfrak{N}_\nu^+$ does not need to be known, nor do vectors orthogonal to it have to be found explicitly. In fact, if $\tilde{\mathfrak{M}}_\nu^+$ is an m_ν^+-dimensional manifold in $\mathfrak{D}_H$ that satisfies rank $\{((H-\rho)v_i, (H-\rho)v_j)\} = m_\nu^+$ for a basis $\{v_i\}$ for $\tilde{\mathfrak{M}}_\nu^+$, then $\tilde{\mathfrak{M}}_\nu^+$ can be used in place of $\mathfrak{M}_\nu^+$, and the inner products needed are just $((H-\rho)v_i, (H-\rho)v_j)$ and $((H-\rho)v_i, v_j)$.

The manifolds $\mathfrak{M}$ that we have introduced when ρ is in the spectrum of H have the advantage that in each case the eigenvalue problem has the matrix form, $(A - \mu B)\alpha = 0$, in which B is positive definite. This restriction is not really necessary. The eigenvalues may be defined to be stationary values of the quotient.

$$\frac{((H-\rho)v, v)}{((H-\rho)v, (H-\rho)v)}$$

over vectors v in an arbitrary m-dimensional manifold $\mathfrak{M}$ of $\mathfrak{D}_H$ under the subsidiary condition that $((H-\rho)v, (H-\rho)v) > 0$. In fact, the eigenvalues may be defined[8] as recursive maxima or minima or by minimum-maxima or maximum-minima of $((H-\rho)v, v)$ under the condition $((H-\rho)v, (H-\rho)v) = 1$. Evidently the eigenvalues μ are uniquely defined and are equivalent to those found using only that part of $\mathfrak{M}$ in the orthogonal complement of $\mathfrak{N}$, while the eigenvectors in $\mathfrak{M}$ are undetermined to the extent of an arbitrary vector v in $\mathfrak{M} \cap \mathfrak{N}$.

[8] The orthogonality statements in the variational principles are to be understood in terms of the form $((H-\rho)v, (H-\rho)v)$.

3.3. ρ Passes From the Resolvent Set to the Spectrum

We now examine what happens when ρ passes from the resolvent set of H to a point σ of the spectrum. We shall see that the bounds pass smoothly, indeed analytically, into those given at the spectral point. In order to preserve the proper enumeration, and thus obtain the strongest bounds, it is necessary to consider the limits from both sides of the spectral point.

Let σ be a nonnegative point of the spectrum of H and let $\mathfrak{N}^{\sigma}$ be the associated characteristic subspace. We take $\mathfrak{M}$ to be an m-dimensional subspace of $\mathfrak{D}_H$ and suppose that $\{v_i\}$ is an orthonormal basis for $\mathfrak{M}$ such that $v_i \epsilon \mathfrak{N}^{\sigma}$, $i=m'+1, m'+2, \ldots, m$, and that the other vectors in the basis have no nontrivial linear combinations in $\mathfrak{N}^{\sigma}$. The manifold spanned by the first m' of the v's will be denoted by $\tilde{\mathfrak{M}}_{\sigma}$. We restrict our attention to ρ's that are distinct from σ and lie in an interval about σ that contains no other spectral points of H. The Lehmann-Maehly eigenvalue equation for H based on ρ and $\mathfrak{M}$ takes the form,

$$\left\{\left(\begin{array}{c|c} A_1(\rho) & 0 \\ \hline 0 & (\sigma-\rho)I \end{array}\right) - \mu \left(\begin{array}{c|c} B_1(\rho) & 0 \\ \hline 0 & (\sigma-\rho)^2 I \end{array}\right)\right\}\left(\begin{array}{c} \alpha^1 \\ \hline \alpha^2 \end{array}\right)=0,$$

in which A_1 and B_1 are given by $((H-\rho)v_i, v_j)$ and $((H-\rho)v_i, (H-\rho)v_j)$, respectively, $i,j=1, 2, \ldots, m'$. Equivalently, we have

$$(A_1-\mu B_1)\alpha^1=0, \tag{27}$$

and

$$\{I-\mu(\sigma-\rho)\}\alpha^2=0. \tag{28}$$

As ρ approaches σ the values μ determined by (28) become infinite while the numbers $\rho+1/\mu$ have the constant value σ. The μ's determined by (27) depend analytically on ρ and pass to the values found with ρ equal to σ.

Let us assume that σ is $\lambda_{\nu}^{1/2}$, where λ_{ν} is the same as in (20). We identify $\mathfrak{M}$ with $\mathfrak{M}^+$, $\tilde{\mathfrak{M}}_{\sigma}$ with $\tilde{\mathfrak{M}}_{\nu}^{+}$, m with m^+ and m' with m_{ν}^{+}. As ρ approaches $\lambda_{\nu}^{1/2}$ from above the bounds given by (8) decrease to those of (22) and the bounds given by (9) with n equal to $\nu+t-1$ decrease to those of (23).[9] The numbers obtained as the limiting values of the right side of (7) contain $\lambda_{\nu}^{1/2}$ repeated $m-m'$ times and the others are the bounds found on the right hand side of (21), but the enumeration obtained by the limiting process will not be as strong as that in (21) unless $m-m'$ equals t. On the other hand as ρ increases to $\lambda_{\nu}^{1/2}$ the bounds given by (7) with n equal to $\nu-1$ and by (8) increase with ρ to those given by (21) and (22) respectively. The numbers obtained as the limits of the right hand side of (9) will contain $\lambda_{\nu}^{1/2}$ repeated $m-m'$ times and the others are the bounds found on the right hand side of (23). Again, if $m-m'$ is less than t, the enumeration of the bounds found from (9) by the limiting process will not be as strong as that of (23). Evidently, a parallel discussion can be made for the bounds found with $-\rho$ to relate the limiting values of the bounds of (10), (11), and (12) to those of (24), (25), and (26).

If we assume that σ is zero, we identify $\mathfrak{M}$ with $\mathfrak{M}^+$, $\tilde{\mathfrak{M}}_{\sigma}$ with $\tilde{\mathfrak{M}}^0$, m with m^+ and m' with m^0. As ρ approaches zero from above the bounds given by (9) with n equal to $p-1$ decrease to those of (16). The set of bounds given by (7) is empty, and the limiting values of the right-hand side of (8) contain zero repeated $m-m'$ times and the others are the numbers that appear on the right in (15), but the enumeration obtained in the limit will not be as strong as that of (15) unless $m-m'$ equals s. To obtain the bounds (15) directly as limits we use $-\rho$. For this we identify $\mathfrak{M}$ with $\mathfrak{M}^-$ and m with m^-. As ρ approaches zero from above, the bounds of (12) with n equal to $p-1$ decrease to those of (15). The set of bounds given by (11) is empty, and the limiting values of the right hand side of (10) contain zero $m-m'$ times and those of the right hand side of (16). Again precise enumeration is lost unless $m-m'$ equals s.

[9] The numbers j^+, k^+, and l^+ change with ρ as values of μ pass through zero or $-1/\rho$.

4. An Interpretation of a Theorem of T. Kato

T. Kato [7], has proved the following theorem:

Let H *be the operator described in sec. 2, and suppose that the interval* (α,β), $0 \leq \alpha < \beta$, *contains at most a nondegenerate eigenvalue of* H. *If* v, *a nonzero vector in* $\mathfrak{D}_H$, *satisfies*

$$\epsilon^2 < (\eta-\alpha)(\beta-\eta), \tag{29}$$

where

$$\eta = \frac{(\mathrm{Hv}, \mathrm{v})}{(\mathrm{v},\mathrm{v})} \text{ and } \epsilon^2 = \frac{(\mathrm{Hv}, \mathrm{Hv})}{(\mathrm{v},\mathrm{v})} - \eta^2,$$

then there is an eigenvalue $\lambda^{1/2}$ *of* H *in* (α,β) *and the following bounds hold:*

$$\alpha < \eta - \frac{\epsilon^2}{\beta-\eta} \leq \lambda^{1/2} \leq \eta + \frac{\epsilon^2}{\eta-\alpha} < \beta. \tag{30}$$

We first note that the inequality (29) is equivalent to each of the inequalities

$$\frac{((H-\beta)v, v)}{((H-\beta)v, (H-\beta)v)} < \frac{-1}{\beta-\alpha}, \tag{31}$$

and

$$\frac{((H-\alpha)v, v)}{((H-\alpha)v, (H-\alpha)v)} < \frac{1}{\beta-\alpha}, \tag{32}$$

and that the bounds (30) can be written

$$\alpha < \beta + \frac{((H-\beta)v, (H-\beta)v)}{((H-\beta)v, v)} \leq \lambda^{1/2} \leq \alpha + \frac{((H-\alpha)v, (H-\alpha)v)}{((H-\alpha)v, v)} < \beta. \tag{33}$$

From the form of the bounds in (33) it is clear that the theorem of Kato can be interpreted as the application of the Lehmann-Maehly procedure based on the one-dimensional manifold spanned by v and using in turn α and β as ρ. The inequality (29) suffices to insure that the bounds give improvements over α and β. Further, the procedure applied to H using a fixed finite-dimensional manifold $\mathfrak{M}$ and the constants α and β gives the best possible bounds [10] of the Kato type (33) that can be obtained using v's from $\mathfrak{M}$. In fact, the smallest eigenvalue $\mu(\beta)$ of the procedure based on $\mathfrak{M}$ and β is the minimum of the quotient $\frac{((H-\beta)v, v)}{((H-\beta)v, (H-\beta)v)}$ over $\mathfrak{M}$, and if there is any v in $\mathfrak{M}$ that satisfies (29) then it satisfies (31), which guarantees that

$$\mu(\beta) < \frac{-1}{\beta-\alpha}.$$

In addition, we have by the minimum property of $\mu(\beta)$ that

$$\mu(\beta) \leq \frac{((H-\beta)v, v)}{((H-\beta)v, (H-\beta)v)}$$

for any v in $\mathfrak{M}$ for which $(H-\beta)v \neq 0$. Consequently, any lower bound of the form $\beta + \frac{((H-\beta)v, (H-\beta)v)}{((H-\beta)v, v)}$ can be no better than that given by $\beta + \frac{1}{\mu(\beta)}$, which shows that the Lehmann-Maehly procedure is optimum for the lower bound. Similarly, the largest eigenvalue $\mu(\alpha)$ of the procedure based on α and $\mathfrak{M}$ is the maximum value of the quotient $\frac{((H-\alpha)v, v)}{((H-\alpha)v, (H-\alpha)v)}$, and by (32) it satisfies

$$\mu(\alpha) > \frac{1}{\beta-\alpha}.$$

[10] Kato suggested [7] and used [8] a process that tends to minimize ϵ^2 instead.

The maximum characterization yields

$$\mu(\alpha) \geqslant \frac{((H-\alpha)v,\, v)}{((H-\alpha)v,\, (H-\alpha)v)}$$

for any v in $\mathfrak{M}$ for which $(H-\alpha)v \neq 0$, and thus the procedure yields the best upper bound of the form $\alpha + \frac{((H-\alpha)v,\, (H-\alpha)v)}{((H-\alpha)v,\, v)}$. According to the equivalence of the inequalities (31) and (32), a vector v_β that yields $\mu(\beta)$ less than $-(\beta-\alpha)^{-1}$ satisfies (31) and hence (32), and conversely a vector v_α that yields $\mu(\alpha)$ greater than $(\beta-\alpha)^{-1}$ satisfies (32) and thus (31). Nevertheless, the vectors v_α and v_β will be different in general, even when neither α nor β is in the spectrum of H; that is to say that the upper and lower bounds are optimized separately by the procedure of Lehmann and Maehly.

Let us note that it is not necessary to assume that the interval (α, β) contains at most one eigenvalue of H. In fact, if the procedure based on $\mathfrak{M}$ and β yields j^+ eigenvalues $\mu_r(\beta)$ that are less than $-(\beta-\alpha)^{-1}$ then from these μ's there result j^+ positive lower bounds of the form $\beta + \frac{1}{\mu_r(\beta)}$ for the eigenvalues of H that lie just below β, and these bounds satisfy

$$\alpha < \beta + \frac{1}{\mu_r(\beta)} < \beta.$$

Consequently, there are at least j^+ eigenvalues of H in the interval $[\beta + (\min \mu_r(\beta))^{-1}, \beta)$. However, since the inequality (31) is equivalent to (32), the procedure for H based on $\mathfrak{M}$ and α has also exactly j^+ eigenvalues $\mu_r(\alpha)$ greater than $(\beta-\alpha)^{-1}$. These give j^+ upper bounds of the form $\alpha + \frac{1}{\mu_r(\alpha)}$ for the eigenvalues of H that lie just above α, and they satisfy

$$\alpha < \alpha + \frac{1}{\mu_r(\alpha)} < \beta.$$

Hence there are also at least j^+ eigenvalues of H in the interval $(\alpha, \alpha + (\max \mu_r(\alpha))^{-1}]$. If it is known that there are at most j^+ eigenvalues of H in (α, β), then the results from the use of α and β combine to assert that there are exactly j^+ eigenvalues of H in the smaller interval $[\beta + (\min \mu_r(\beta))^{-1}$, $\alpha + (\max \mu_r(\alpha))^{-1}]$, and that the numbers $\alpha + \frac{1}{\mu_r(\alpha)}$ and $\beta + \frac{1}{\mu_r(\beta)}$ provide j^+ pairs of upper and lower bounds for those eigenvalues.

From the results of section 3, it is clear that the interval $(-\beta, -\alpha)$ could be used in place of (α, β) and results parallel to those of this section would be obtained.

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