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Generation and Composition of Functions*

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Suppose it is desired to generate some particular function, from a specified set of initial functions, using operations from a specified repertoire. Hypotheses are given which ensure that the process can be so arranged, that the intermediate functions arising at certain stages have no more arguments than does the final function sought.

Operations for producing new functions from old are studied in many branches of mathematics. (In analysis, for example, many of the standard elementary theorems concern the preservation of smoothness properties by such operations.) This topic is especially significant in both the theoretical and the concrete aspects of effective computation, e.g., in recursive-function theory on the one hand and in the programing of digital computers on the other.

Suppose it is desired to generate some particular function, from a specified set of initial functions, using operations from a specified repertoire. It is natural to ask whether the process could be so arranged, that the intermediate functions arising at certain key stages are no more complex (in some appropriate sense) than the final function sought. The present paper deals with some simple topics relating to this question, the "complexity" of a function being measured merely by the number of its arguments.

All *functions* to be considered take values in a common set S, and have finitely many variables which range independently over S. (This last restriction, though awkward in some cases, can sometimes be circumvented by adjoining to S a new element corresponding to "undefined.") A function of k variables will be called a *k-function.* If N is a subset of the natural numbers, then a function which is a k-function for some $k \epsilon N$ will be called an *N-function.*

An *operation* σ is defined to be a mapping whose domain is some subset of the collection of all finite sequences $(f_1, \ldots, f_n)$ of functions, and whose range is a subset of the collection of functions. We call σ an *N-operation* if $\sigma(f_1, \ldots, f_n)$ is defined only if—but not necessarily always if—each f_i is an N-function. For example, if the set S (in which our independent and dependent variables assume their values) happens to be well-ordered, then the operation of *inversion* given by

$$[\sigma_0(f)](x) = \min \{y : f(y) = x\}$$

is a $\{1\}$-operation; $\sigma_0(f)$ is defined if and only if f is a 1-function which assumes all members of S as values.

In any application of an operation σ to a sequence $(f_1, \ldots, f_n)$ in its domain, we call the f_i the *inputs* and $\sigma(f_1, \ldots, f_n)$ the *output.* If Φ is a family of operations, we say that a function f can be *obtained by* Φ from a class C of functions if either $f \epsilon C$ or there exists a finite sequence Σ of applications of operations in Φ, the last of which has f as output, such that the inputs to any of these applications are "available" by virtue either of lying in C or of being the output of some previous application in Σ.

Our final preliminary definition pertains to a family $\Phi_A = \{\sigma_a : a \epsilon A\}$ of operations and a transformation τ which associates to each $\sigma_a \epsilon \Phi_A$ an operation $\tau(\sigma_a)$. The pair (Φ_A, τ) will be called *N-special*, if for each set C of functions closed under all the operations of $\tau(\Phi_A)$, it is true that all N-functions obtainable by Φ_A from C already lie in C. This definition is difficult to motivate here; the reader may wish to look ahead at the definitions preceding Theorem 2, and then at the final paragraph of the paper.

THEOREM 1. *Let the collection Φ of operations consist only of* N*-operations and of the operations σ_a from some* N*-special pair (Φ_A, τ). Then all* N*-functions obtainable by Φ from a set* C *of functions are also obtainable if each $\sigma_a \epsilon \Phi_A$ is replaced in Φ by the corresponding $\tau(\sigma_a)$.*

Proof. Let C_1 consist of all functions obtainable by $(\Phi - \Phi_A) \cup \tau(\Phi_A)$ from C, C_2 consist of all functions obtainable by Φ_A from C_1 (*sic*), and C_3 consist of all functions obtainable by Φ from C. If $C_i(N)$ denotes the class of N-functions in C_i $(i=1, 2, 3)$, then the statement to be proved is $C_3(N) \subset C_1(N)$. This will be done by showing that

$$C_2(N) \subset C_1, \text{ so that } C_2(N) \subset C_1(N); \tag{1}$$

$$C_3 \subset C_2, \text{ so that } C_3(N) \subset C_2(N). \tag{2}$$

To prove (1), let $f \epsilon C_2$ be an N-function. By construction C_1 is closed under all $\tau(\sigma_a)$, and so the definition of "N-special pair" can be applied to C_1 to assert that

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all N-functions obtainable from C_1 by Φ_A, already lie in C_1. Hence $f\epsilon C_1$ as desired.

To prove (2), observe that C_2 includes C_1 and thus the initial class C, and also is closed under all $\sigma_a\epsilon\Phi_A$; we need only show in addition that it is closed under each $\sigma\epsilon\Phi-\Phi_A$. But if σ (which by hypothesis is an N-operation) is to be applicable to $(f_1, \ldots, f_n)$ where each $f_i\epsilon C_2$, then each f_i must be an N-function and hence $f_i\epsilon C_1$ by (1), implying

$$\sigma(f_1, \ldots, f_n)\epsilon C_1\subset C_2$$

as desired. This completes the proof of the theorem.

To see how theorem 1 is relevant to the question raised in the second paragraph of the paper, suppose τ can be so chosen relative to Φ_A that the output of any application of any $\tau(\sigma_a)$ is an N-function. If f is any N-function obtainable by Φ from C, then by the theorem f is also obtainable by $(\Phi-\Phi_A)\cup\tau(\Phi_A)$ from C, and in the latter process the intermediate functions resulting from the use of $\tau(\Phi_A)$ are all N-functions; if for example $N=\{1, 2, \ldots, m\}$ where f is an m-function, then these intermediate products are at most m-functions. If in addition the operations in $\Phi-\Phi_A$ produce only N-functions, then *all* the intermediate products are at most m-functions.

Justification of the previous material requires presentation of at least one significant instance to which theorem 1 and the comments of the last paragraph apply. For this purpose we consider the operation σ_c of *composition*, given by

$$[\sigma_c(f_1, \ldots, f_n)](x_1, \ldots, x_k) = f_1(f_2(x_1, \ldots, x_k), \ldots, f_n(x_1, \ldots, x_k))$$

where $n>1$; here f_1 will be called the *outer* input and the other f_i will be called the *inner* inputs. For any subset N of the natural numbers, we let $\tau_N(\sigma_c)$ be the restriction of σ_c to those sequences $(f_1, \ldots, f_n)$ for which all the inner inputs are N-functions.

[For the following proof, we shall need the observation that the output of an application of composition is a k-function if and only if every inner input is a k-function. The "only if" may seem unduly restrictive; e.g., if

$$f_1(x_1, x_2)=x_1+x_2, f_2(x_1, x_2)=x_1x_2, f_3(x_1)=x_1,$$

one would expect to be able to obtain the function $f_4(x_1, x_2)=x_1x_2+x_1$ by composition. This is not in general possible with our definition of composition, essentially because there is no mechanism provided for "inflating" f_3 (by adjunction of a dummy variable) to $f_3^*(x_1, x_2)=x_1$. Such inflations *would* be possible if, as is usually assumed in recursive-function theory, the stock of "available" functions includes the generalized identity functions U_i^k defined by

$$U_i^k(x_1, \ldots, x_k)=x_i \qquad (i=1, 2, \ldots, k).$$

Then we would use two applications of composition,

$$f_4=\sigma_c(f_1, f_2, f_3^*)=\sigma_c(f_1, f_2, \sigma_c(f_3, U_1^2)),$$

to obtain f_4 from f_1, f_2, and f_3. The functions U_i^k are also necessary if, for example, we wish to obtain $f_5(x_1, x_2, x_3)=x_1x_2+x_2x_3$ from f_1 and f_2:

$$f_5=\sigma_c(f_1, \sigma_c(f_2, U_1^3, U_2^3), \sigma_c(f_2, U_2^3, U_3^3)).$$

In summary, the composition operation defined above appears "weaker" than usual simply because we do not supplement it by explicitly postulating the availability of the generalized identity functions.]

THEOREM 2. *The pair* $(\{\sigma_c\}, \tau_N)$ *is* N*-special.*

PROOF. Let C be a set of functions closed under $\tau_N(\sigma_c)$, and let f be an N-function obtainable by $\{\sigma_c\}$ from C. We are to prove that f is obtainable by $\{\tau_N(\sigma_c)\}$ from C. It will be convenient to refer to an application of σ_c as an N-composition or an $\overline{N}$-composition, according as all the inner inputs (and hence the output) are N-functions or not.

If $f\epsilon C$, or if there is a sequence of N-compositions leading from C to f, then by the hypothesis on C we have $f\epsilon C$ as desired. (A formal proof would involve induction on the length of the shortest such sequence.) If however every sequence of applications of composition leading from C to f contains at least one $\overline{N}$-composition, then consider such a sequence Σ for which the number of $\overline{N}$-compositions is *minimum*, and let

$$G(y_1, \ldots, y_m)=H(h_1(y_1, \ldots, y_m), \ldots, h_p(y_1, \ldots, y_m)) \quad (3)$$

be the *last* $\overline{N}$-composition in Σ. Thus m is not in N.

By the minimality of Σ, G must be used as an outer or inner input in at least one composition of Σ appearing after (3). It cannot be used as an inner input, since the outputs and hence the inner inputs of all compositions following (3) in Σ are N-functions. Let its *first* use be in

$$F(x_1, \ldots, x_k)=G(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k)). \quad (4)$$

Then $k\epsilon N$, and the availability of the g_j for use just before (4) would not be affected if (3) were deleted from Σ. In addition, the functions H and h_i were available just before (3), and hence would be available just before (4) even if (3) were deleted from Σ.

We show in the next paragraph how to replace (4) in Σ by a sequence of N-compositions, the last of which has F as output. It will follow from the previous remarks that the functions used are in fact available, and would remain so even if (3) were deleted. None of these N-compositions will involve G, so that in the resulting modification Σ' of Σ (which still leads from C to f) there is one less use of G and the same number of $\bar{N}$-compositions. Continuing the process, we arrive at a sequence Σ'' leading from C to f, with the same number of $\bar{N}$-compositions as in Σ, and such that (3) appears in Σ'' but is *never* used subsequently. Thus deletion of (3) from Σ'' yields a sequence leading from C to f which contradicts the minimality of Σ.

The sequence of N-compositions which can replace (4) in Σ is given by

$$f_i(x_1, \ldots, x_k) = h_i(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k))$$

for $i=1, 2, \ldots, p$, followed by

$$F(x_1, \ldots, x_k) = H(f_1(x_1, \ldots, x_k), \ldots, f_p(x_1, \ldots, x_k)).$$

This completes the proof.

Theorems 1 and 2 together immediately imply the following result.

COROLLARY. *If Φ consists of σ_c and* N*-operations, then all* N*-functions obtainable by Φ from a set* C *of functions, are also obtainable if $\tau_N(\sigma_c)$ replaces σ_c in Φ.*

The special case of this corollary in which S consists of the natural numbers, $N=\{1\}$, and Φ consists of σ_c and the inversion operation σ_0 defined earlier, appears in a paper of J. Robinson.[1] The present paper was motivated by a desire to abstract the essentials of this special case. Our proof of Theorem 1 was patterned after the proof given by R. M. Robinson[2] of a precursor of J. Robinson's theorem. Additional interesting applications seem likely to exist, but are apparently difficult to recognize. The subject arose in connection with Davis' characterization[3] of universal Turing machines.

[1] J. Robinson, General recursive functions, Proc. Amer. Math. Soc. **1** (1950).
[2] R. M. Robinson, Primitive recursive functions, Bull. Amer. Math. Soc. **53** (1947).
[3] M. Davis, A note on universal Turing machines, in "Automata Studies," Princeton Annals of Math. Study No. 34.

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