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Existence of k-Edge Connected Ordinary Graphs With Prescribed Degrees[1]

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An *ordinary graph* G is a set of objects called *nodes* and a family of unordered pairs of the nodes called *edges*. The *degree* of a node in G is the number of edges in G which contain it. G is called *connected* if it is not the union of two disjoint nonempty subgraphs. A graph H is called k-edge connected if deleting any fewer than k edges from H leaves a connected graph. It is proved that there exists a k-edge connected graph H for $k>1$ with prescribed integer degrees d_i if and only if there exists an ordinary graph with these degrees and all $d_i \geqslant k$. There exists a 1-connected (i.e., connected) ordinary graph with prescribed positive integer degrees d_i if and only if there exists an ordinary graph with these degrees and $\sum_{i=1}^{n} d_i \geqslant 2(n-1)$.

An *ordinary graph* G is a finite set of objects and a family of two-member subsets of the objects. The objects are called the *nodes* of G and the pairs are called the *edges* of G. An edge and a node are said to *meet* if one contains the other. The *degree* of a node in G is the number of edges in G which it meets.

A *cut* of graph G, denoted by $(S,\bar{S})$, is a partition of the nodes of G into two nonempty subsets S and $\bar{S}$. The union of S and $\bar{S}$ is all the nodes of G. The intersection of S and $\bar{S}$ is empty. An edge e *in* cut $(S,\bar{S})$ of G is an edge of G which meets one node in S and one node in $\bar{S}$. The *value* of cut $(S,\bar{S})$ is the number of edges in it.

Graph G is called k-edge connected if it has no cut with value less than k. The usual meaning of *connected* for graphs is the same as the meaning of 1-edge connected. An equivalent definition of k-edge connected for G is that G cannot be made nonconnected by deleting fewer than k edges from G.

Hoffman, Fulkerson, and McAndrew[2] describe necessary and sufficient conditions for a set of integers to be the degrees of the nodes of some ordinary graph, and give references to related work. Here, we show that simple additional conditions are necessary and sufficient for the set of integers to be the degrees of the nodes of some k-edge connected ordinary graph (prescribed k).

THEOREM. *Integers* d_i, $i=1, \ldots, n$, $(n>1)$, *are the respective degrees of the nodes* v_i *of some* k-*edge connected ordinary graph if and only if*

(1) *they are the degrees of the nodes of some ordinary graph,*

(2) *all* $d_i \geqslant k$,

(3) *where* $k=1$, $\sum_{i=1}^{n} d_i \geqslant 2(n-1)$.

The "only if" part of the theorem is almost obvious. Certainly condition (1) is necessary. Condition (2) is necessary since a cut of a graph G which puts one node, say v, in one part and the other nodes of G in the other part has cut value equal to the degree of v.

For $k=1$, condition (2)—that the d_i's be positive—is necessary but not sufficient. Let G be a connected graph with n nodes. There must be an edge, say e_1, joining node v_1 of G to one of the other nodes of G, say v_2. Recursively, there must be an edge, say e_i, joining one of the nodes $v_1, \ldots, v_i$ $(1 \leqslant i \leqslant n-1)$ of G to one of the other nodes of G, say v_{i+1}. Clearly each of the edges $e_i (i=1, \ldots, n-1)$ is distinct. Therefore the number of edges in G is not less than $n-1$. Let d_i be the degree of node $v_i (i=1, \ldots, n)$ in G. Because each edge of G contributes twice to a degree in G, $\Sigma\, d_i$ equals twice the number of edges in G. Therefore, $\Sigma\, d_i \geqslant 2(n-1)$.

A (simple) *path joining nodes a and b* is a connected graph which has degree $=1$ at its nodes a and b and degree $=2$ at its other nodes. The edges $e_i (i=1, \ldots, n-1)$ described in the last paragraph form with their nodes a tree. A *tree* is a graph which contains exactly one path joining each pair of its nodes. A tree contains one less edge than nodes. A graph G contains a tree containing all the nodes of G if and only if G is connected. The edges and nodes of any graph partition uniquely into one or more connected *component* graphs. Two nodes in a graph G are joined by a path in G if and only if they are in the same component of G.

Suppose there exists an ordinary graph H, not necessarily connected, with positive integer degrees $d_i (i=1, \ldots, n)$ at its nodes. Suppose $\Sigma\, d_i \geqslant 2(n-1)$ and hence H has at least $n-1$ edges. If H is not connected, i.e., has more than one component, we shall see that another ordinary graph H' can be constructed from H which has the same degrees at all the nodes and which has one less component than H.

[1] Supported by the Army Research Office (Durham).
[2] D. R. Fulkerson, A. J. Hoffman, and M. H. McAndrew, Some properties of graphs with multiple edges, to appear in the Canadian Journal of Mathematics.

Thus it will follow by induction that there exists a connected ordinary graph with these degrees.

Suppose H has components $H_1, \ldots, H_p$ with $n_1, \ldots, n_p$ nodes respectively. Each $H_j (j=1, \ldots, p)$ contains a tree T_j with $n_j - 1$ edges and all n_j nodes of H_j. Since

$$\sum_{j=1}^{p} n_j = n,$$

we have

$$\sum_{j=1}^{p} (n_j - 1) < n - 1 \text{ for } p > 1.$$

Since there are at least $n-1$ edges in H, there must be an edge e of H not in any of the trees T_j. However e must have both of its nodes, say a and b, in one of these trees, say T_1.

Since component H_2 contains a node and all d_i's are positive, H_2 contains an edge, say e_2, which meets nodes, say a_2 and b_2, in H_2. Form a new graph H' from H by deleting from H edges e and e_2 and then adjoining a new edge e_a joining a and a_2 and a new edge e_b joining b and b_2. Because a_2 and b_2 are in a different component of H than a and b, there are no edges in H already joining a_2 to a and b_2 to b. Hence H' is a validly constructed ordinary graph with the same degrees at its nodes as in H. It is easy to verify that H_1 and H_2 are hereby replaced by a single component in H'. Therefore H' has one less component than H. This completes the proof of the theorem for the case $k=1$.

The proof for the case $k>1$ is more tricky. Here we don't have an analog of trees or a straightforward decomposition into k-edge connected components to help us.

For prescribed integer $k>1$, assume that all $d_i \geq k (i=1, \ldots, n)$ and assume there exists an ordinary graph whose nodes v_i have degrees d_i. It follows that there is a connected ordinary graph H whose nodes v_i have degrees d_i, since $d_i \geq 2$ implies

$$\sum d_i \geq 2n > 2(n-1).$$

Assuming that H is only h-edge connected where $1 \leq h < k$, we shall show how to construct a new ordinary graph H' whose nodes v_i have degrees d_i, and which is at least h-edge connected, and which has fewer cuts of value h than H does. From this it will follow by induction that there is a k-edge connected ordinary graph G whose nodes v_i have degrees d_i.

Clearly H has h-valued cuts $(A, \overline{A})$ and $(B, \overline{B})$ such that sets A and B are disjoint and such that neither A nor B contain a proper subset S for which $(S, \overline{S})$ is only an h-valued cut. It may or may not be true that $A=\overline{B}$ and $\overline{A}=B$. In any case, because H is connected, there must be a path P joining a node, a_1, in A to a node, b_1, in B where P has no other nodes in A or B.

Because node a_1 has degree at least $k>h$, and because there are only h edges with one node in A and one node not in A, node a_1 must meet an edge e_a whose other node a_2 is in A and meets no edge with a node not in A. Similarly, node b_1 must meet an edge e_b whose other node b_2 is in B and meets no edge with a node not in B.

Construct a new ordinary graph H' by deleting edges e_a and e_b and then adjoining a new edge e'_a joining a_1 to b_2 and a new edge e'_b joining b_1 to a_2. Graph H' has the same degrees at the various nodes as H. The value of the cuts $(A, \overline{A})$ and $(B, \overline{B})$ is increased to $h+2$.

It remains to show that replacing H by H' introduces no new cuts of value $\leq h$. Any cut, say $(D, \overline{D})$, whose value decreases in going from H to H' must have a_1 and b_2 together in one of the parts, say D, and a_2 and b_1 together in the other part, $\overline{D}$. Suppose cut $(D, \overline{D})$ in H' has value h or less.

Consider the mutually disjoint sets of nodes $A \cap D$, $A \cap \overline{D}$, $B \cap D$, $B \cap \overline{D}$. Path P, defined with respect to H, is also entirely in H' and it joins node a_1 in $A \cap D$ to node b_1 in $B \cap \overline{D}$. Hence some edge e_0 of P is an edge of the cut $(D, \overline{D})$ of H'. Since P contains no nodes of $A \cap \overline{D}$ or $B \cap D$, edge e_0 neither meets one node in $A \cap \overline{D}$ and one node in $A \cap D$ nor meets one node in $B \cap D$ and one node in $B \cap \overline{D}$.

Partition the edges of the cut $(D, \overline{D})$ in H' into three classes E_A, E_B, E_C: E_A consisting of all edges connecting $A \cap D$ with $A \cap \overline{D}$, E_B consisting of all edges connecting $B \cap D$ with $B \cap \overline{D}$, E_C being formed by the remaining edges. As has been seen above, E_C contains e_0; hence the cardinality of E_C is at least one. Since the cardinalities of E_A, E_B, E_C add up to at most h, either the cardinality of E_A or the cardinality of E_B must be bounded by[3] $[(h-1)/2]$. By symmetry, suppose the former is true. Then we have, counting e_a, at most $[(h-1)/2]+1$ edges in H with a node in $A \cap D$ and a node in $A \cap \overline{D}$.

Since the value of cut $(A, \overline{A})$ in H is h, either $A \cap D$ or $A \cap \overline{D}$ contains nodes of no more than $[h/2]$ edges of cut $(A, \overline{A})$. Every edge of H with precisely one node in $A \cap D$ (in $A \cap \overline{D}$) has its other node in $A \cap \overline{D}$ (in $A \cap D$) or in $\overline{A}$. Therefore either cut $(A \cap D, \overline{A \cap D})$ or cut $(A \cap \overline{D}, \overline{A \cap \overline{D}})$ of H has value at most

$$[h/2] + [(h-1)/2] + 1 = h.$$

However this contradicts the fact that A contains no proper subset S such that cut $(S, \overline{S})$ in H has value at most h. Therefore, our assumption that cut $(D, \overline{D})$ in H' has value at most h is incorrect. Therefore replacing H by H' introduces no new cuts of value at most h. The theorem is proved.

Mr. Alvin Owen (in conversation) has posed an interesting related problem. What are necessary and sufficient conditions for a set of integers

$$\{d_i\} \; (i=1, \ldots, n)$$

so that *all* ordinary graphs whose nodes have degrees precisely $\{d_i\}$ are connected? Or, looking for the negative of these conditions, what are necessary and sufficient conditions for $\{d_i\}$ to be the degrees of the nodes of some nonconnected ordinary graph?

Note added in proof: D. R. Fulkerson has called to my attention a related paper by himself and L. S. Shapley, Minimal k-arc connected graphs, RAND Report P–2371 (1961). Their main result states that the fewest edges required in a k-edge connected ordinary graph with n nodes is $[kn-1)/2]$.

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[3] Here $[x]$ means "the greatest integer not greater than x."