JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 68B, No. 1, January–March 1964

Optimal Matchings and Degree-Constrained Subgraphs[1]

A. J. Goldman

(October 25, 1963)

The characterization of maximum-cardinality matchings in linear graphs, by the nonexistence of augmenting paths, has been extended by several authors to a similar characterization of maximum degree-constrained subgraphs. This paper contains a proof of the extended version by direct reduction to the case of matchings. Possible algorithmic implications of the reduction are suggested.

In this paper the term *graph* means a finite unoriented linear graph. The same symbol will be used for a graph and its edge-set. The *degree* $d(v, G)$ of a vertex v in a graph G is simply the number of edges of G which are incident on v, with *loops* (edges from v to v) counted twice. The term *strict graph* will be used for a graph which is loopless and contains at most one edge joining any pair of distinct vertices. A *path* in a graph is permitted to have repeated vertices, but not repeated edges.

Let G^* be a graph with vertex-set $\{v_i\}_1^n$. A *matching* in G^* is a subgraph M such that $d(v_i, M) \leq 1$ for $1 \leq i \leq n$. An *optimal* matching is one for which the number $|M|$ of edges is maximum. An *augmenting path* for a matching M is a path P in G^* with edges alternately in M and in $G^* - M$, no end edge[2] in M, and end vertices v such that $d(v, M) = 0$.

A matching M which admits an augmenting path P cannot be optimal, since the symmetric difference $M\Delta P$ is a matching with one more edge than M. The nontrivial converse is also true, a result due to Berge [1][3] and (in a different but equivalent form) to Norman and Rabin [6]. We state it in the following sharp form:

THEOREM (*M*). *Let M_1 and M_2 be matchings in a graph G^*, with $|M_1| < |M_2|$. Then $M_1\Delta M_2$ contains*[3a] *an augmenting path for M_1.*

An important consequence of the theorem is that the problem of efficiently finding an optimal matching in a graph is reduced to that of efficiently searching for an augmenting path P of a given matching M; if the search fails, so that no "improvement" of the form $M \rightarrow M\Delta P$ is possible, then no other improvement is possible and M is optimal. (For a proof by contradiction, take $M_1 = M$ and M_2 as any optimal matching in the theorem.) Theoretical and computational aspects of this topic are discussed by Edmonds [4, 5] and by Witzgall and Zahn [7].

Now let G be a graph with vertex-set $\{v_i\}_1^n$, and $\delta = \{\delta_i\}_1^n$ an n-vector of positive integers. We define a *degree-constrained subgraph* of (G, δ) to be a subgraph D of G such that $d(v_i, D) \leq \delta_i$ for $1 \leq i \leq n$, and consider the problem of finding such a subgraph which is *optimal* in the sense that $|D|$ is maximum. (When all $\delta_i = 1$ we are back to optimal matchings.) For this problem, it is no loss of generality to assume that the sets E_{ij} of edges joining particular vertex-pairs (v_i, v_j) obey the conditions

$$(1) \quad |E_{ij}| \leq \min(\delta_i, \delta_j) \qquad \text{for } i \neq j,$$

$$(2) \quad |E_{ii}| \leq \delta_i/2.$$

An *augmenting path* for a degree-constrained subgraph D is a path P in G with edges alternately in D and G–D, no end edge in D, and either distinct end vertices v_e such that $d(v_e, D) \leq \delta_e - 1$, or else coincident end vertices (i.e., P a closed path) such that $d(v_e, D) \leq \delta_e - 2$. As for matchings, it is obvious that if D admits an augmenting path then it is not optimal, and there is a nontrivial converse which we state in the following sharp form:

THEOREM (*DCSG*): *Let D_1 and D_2 be degree-constrained subgraphs of (G, δ) with $|D_1| < |D_2|$. Then $D_1\Delta D_2$ contains an augmenting path for D_1.*

Berge [2] adapted the Norman-Rabin proof of the theorem (*M*) to a proof of the theorem (*DCSG*). Zahn (unpublished manuscript) has given a direct constructive proof based on ideas in the work of Edmonds. Relevant literature includes papers [3, 5] of Edmonds.

Our purpose here is to provide a proof of the theorem (*DCSG*) by (i) reduction to the historically earliest case of matchings and (ii) application of the theorem (*M*). This is done by constructing, in a way loosely motivated by the theory of covering spaces and possibly useful in other contexts, a *strict* graph G^* such that the analysis of D_1 and D_2 can be transferred to a discussion of matchings in G^*. Thus only the restriction of theorem (*M*) to strict graphs will be used.

The edges of G^* will be in one-to-one correspondence with those of G, and the vertices of G^* will consist of δ_i "replicas" of the vertex v_i of G, for $1 \leq i \leq n$. Formally, the vertex-set of G^* is

$$\{(v_i, k): \ 1 \leq i \leq n, \ 1 \leq k \leq \delta_i\}.$$

[1] Supported in part by the Army Research Office (Durham), CRD–AA–L–P–3532.
[2] The possibility that P consists of a single edge accounts for this awkward wording.
[3] Figures in brackets indicate the literature references at the end of this paper.
[3a] The word "contains" can be further sharpened to "has as a connected component"; see Edmonds [4].

Let E_i be the set of edges of G incident on v_i, with loops (if any) appearing twice. Thus E_i is the disjoint union of (i) the sets E_{ij} for for $i \neq j$, and (ii) a "doubled" E_{ii}. For $i \neq j$ let ϕ_{ij} be an indexing of E_{ij} by a subset of the integers $\{1, 2, \ldots, \delta_i\}$; this is possible by condition (1). (In general $\phi_{ij} \neq \phi_{ji}$.) Also let ϕ_{ii} be an indexing of the edge-appearances in the doubled E_{ii} by a subset of the integers $\{1, 2, \ldots, \delta_i\}$, which is possible by condition (2).

The edges of G^* are constructed in one-to-one correspondence with those of G as follows. Suppose an edge e joins v_i and v_j in G. If $i \neq j$, associate to e an edge e^* in G^* which joins the vertices $(v_i, \phi_{ij}(e))$ and $(v_j, \phi_{ji}(e))$. If $i = j$ and the symbols e^+ and e^- are used for the two appearances of e in E_{ii}, then associate to e an edge e^* in G^* which joins $(v_i, \phi_{ii}(e^+))$ and $(v_i, \phi_{ii}(e^-))$. G^* has only the edges e^* thus defined.

The edge-correspondence will be denoted ϕ: $G \to G^*$. It is easily shown that G^* is indeed a strict graph, that any matching M in G^* yields a degree-constrained subgraph $D = \phi^{-1}(M)$ of (G, δ), and that an augmenting path P for M yields an augmenting path $\phi^{-1}(P)$ for D.

A degree-constrained subgraph D of (G, δ), however, yields a subgraph $\phi(D)$ of G^* which is not necessarily a matching. To overcome this difficulty we need only specialize the indexings ϕ_{ij} (which so far have been arbitrary), in a manner depending on the particular D_1 and D_2 for which the conclusion of the theorem (*DCSG*) is to be proved, so that at least $\phi(D_1)$ and $\phi(D_2)$ are matchings. (A way of doing this is given below.) Then since

$$|\phi(D_1)| = |D_1| < |D_2| = |\phi(D_2)|,$$

theorem (*M*) yields an augmenting path P for $\phi(D_1)$ contained in $\phi(D_1)\Delta\phi(D_2)$, and $\phi^{-1}(P)$ provides the desired augmenting path for D_1 contained in $D_1 \Delta D_2$.

The following choice of ϕ_{ij}'s will insure that $\phi(D_1)$ and $\phi(D_2)$ are matchings. First index the appearances in E_i of edges of $D_1 \cap D_2$ by the integers 1 through $d(v_i, D_1 \cap D_2)$ in any way. Then index the appearances in E_i of edges of $D_1 - D_2$ by the integers $d(v_i, D_1 \cap D_2) + 1$ through $d(v_i, D_1)$ in any way, and index the appearances of edges of $D_2 - D_1$ by the integers $d(v_i, D_1 \cap D_2) + 1$ through $d(v_i, D_2)$ in any way. Each appearance x of an edge of $D_1 \cup D_2$ in any E_{ij} (where $j = i$ is admitted) has now received an index, which is taken as the value of $\phi_{ij}(x)$. Note that any two edges of $E_i \cap D_1$ have received different indices, whether they were both in $D_1 \cap D_2$, both in $D_1 - D_2$, or one in $D_1 \cap D_2$ and the other in $D_1 - D_2$. This is what makes $\phi(D_1)$ a matching, and similarly for $\phi(D_2)$. The definition of the ϕ_{ij}'s can be completed by extending them in any way to the remainder of their respective domains. This completes the proof.

The same line of reasoning suggests how algorithms for finding optimal degree-constrained subgraphs might, at least in principle, be generated from algorithms for finding optimal matchings. At each stage one is trying to improve the "current" degree-constrained subgraph D of (G, δ). Form G^* as above, choosing the ϕ_{ij}'s so that $\phi(D)$ is a matching. Then use matching-algorithm techniques to improve $\phi(D)$, carry the gain back to G *via* ϕ^{-1}, and proceed to the next stage. The new degree-constrained subgraph $D' = D\Delta\phi^{-1}(P)$ is automatically such that $\phi(D')$ is a matching, so that the same G^* (i.e., the same ϕ_{ij}'s) can be used step after step so long as augmenting paths for $\phi(D)$ are found in G^*.

Unfortunately the absence of augmenting paths for $\phi(D)$ in G^* does *not* imply the absence of an augmenting path for D in (G,δ). That is, D may not be an optimal degree-constrained subgraph of (G, δ) even though $\phi(D)$ is an optimal matching in G^*. This situation is readily seen to arise precisely when D has one or more augmenting paths P, but *every one* of them is "pulled apart" by ϕ so that $\phi(P)$ is not a path in G^*. From the theorem (*DCSG*) we know that this cannot happen if the ϕ_{ij}'s are so chosen that $\phi(D^0)$ is a matching for at least one optimal D^0, or even for at least one D^0 more populous than the current D. By taking D and D^0 as the D_1 and D_2 of the particular indexings used in the last proof, we see that such ϕ_{ij}'s do indeed exist; however, they are not known in advance since D^0 is not.

It is therefore necessary, when a point is reached at which $\phi(D)$ is an optimal matching in G^*, to take into account somehow the possibility of augmenting paths for D in G whose connectivity has been lost in the transformation by ϕ. For simplicity of description we distinguish two general classes of tactics; the distinction is conceptual rather than computational, and no attempt is made to consider directly the possible computational implementations, comparisons, or compromises.

First, one might attempt to *restore* possible missing connections by *reindexing*. What would be required is a systematic way of varying the ϕ_{ij}'s (i.e., G^*) until an indexing is found for which the image of D admits an augmenting path. To indicate that the optimum has been achieved, a "stop rule" for this process is also needed.

Second, one might try to *supply* the connections by *enlarging* the graph G^*, either initially, or gradually as the algorithm progresses. The enlarged graph would no longer have G as a self-intersecting copy in the sense of Edmonds [3].

The second policy can be illustrated in a rather heavy-handed way as follows. Instead of G^*, we employ a (typically enormous) graph G^{**} whose vertex-set, like that of G^*, is

$$\{(v_i, k):\ 1 \leq i \leq n,\ \ 1 \leq k \leq \delta_i\}.$$

For each edge $e \epsilon E_{ij}$ of G $(i < j)$, we construct a collection

$$\{e_{kl}:\ 1 \leq k \leq \delta_i,\ \ 1 \leq l \leq \delta_j\}$$

of $\delta_i\delta_j$ edges of G^{**}; e_{kl} joins (v_i, k) and (v_j, l). For each loop $e \epsilon E_{ii}$ of G, we construct a collection

$$\{e_{kl}:\ 1 \leq k < l \leq \delta_i\}$$

of $\delta_i(\delta_i-1)/2$ edges of G^{**}; e_{kl} joins (v_i, k) and (v_i, l). These are the only edges of G^{**}. A many-to-one onto mapping ψ: $G^{**} \to G$ is defined by the formula $\psi(e_{kl})=e$. Note that for every choice of indexings ϕ_{ij}, $G^*=\phi(G)$ is in an obvious sense a subgraph of G^{**} and $\psi\phi$ is the identity map of G.

For each matching M in G^{**}, $\psi(M)$ is a degree-constrained subgraph of (G, δ). Conversely, for each degree-constrained subgraph D of (G, δ), we can choose the ϕ_{ij}'s so that $\phi(D)$ is a matching in $G^*=\phi(G)$ and hence in G^{**}, with $\psi\phi(D)=D$. The matchings M of G^{**} obtained in this way are precisely those for which the restriction $\psi|M$ is one-to-one. Such matchings in G^{**} will be called *special.*

An augmenting path P for a special matching M will in turn be called *special* if the matching $M\Delta P$ is also special. This will be the case if and only if

$$\psi(P-M) \cap \psi(M-P)$$

is empty and in addition the matching P-M is special. A special augmenting path P for M will be called *strong* if $\psi|P$ is one-to-one.

Now consider the following process, which begins with an arbitrary special matching $M^{(1)}$ in G^{**} (e.g., the null matching). At the mth stage we have a special matching $M^{(m)}$ in G^{**}, and seek a special augmenting path (or if preferred even a strong special augmenting path) for $M^{(m)}$. If such a path P is found, we form the special matching $M^{(m+1)}=M^{(m)}\Delta P$ and go on to the next stage.

THEOREM. *This process terminates with a special matching M in G^{**} such that $D=\psi(M)$ is an optimal degree-constrained subgraph of* (G, δ).

For the proof, first observe that the finiteness of G^{**} insures termination with some special matching M. Since M is special, $\psi|M$ is a one-to-one mapping of M onto the degree-constrained subgraph $D=\psi(M)$ of (G, δ). If D were *not* optimal, there would be a degree-constrained subgraph D^0 of (G, δ) such that $|D|<|D^0|$. Exploiting our freedom in selecting indexings, we choose ϕ_{ij}'s such that $\phi(D)=M$ and $\phi(D^0)$ is a matching. (A way of doing this is given below.) Then since

$$|\phi(D)|=|D|<|D^0|=|\phi(D^0)|,$$

theorem (M) yields [4] an augmenting path P for $\phi(D)$, contained in $\phi(D)\Delta\phi(D^0)$ and thus in $\phi(G)$. This containment guarantees that P is a strong special augmenting path for $\phi(D)=M$, contradicting the assumption that the process terminated with M.

The following choice of ϕ_{ij}'s will insure that $\phi(D)=M$ and that $\phi(D^0)$ is a matching. Consider any $e\epsilon D \cap E_{ij}$, where $i\leqslant j$. There is a unique $e_{kl}\epsilon M$, for which $\psi(e_{kl})=e$. If $i<j$, set $\phi_{ij}(e)=k$ and $\phi_{ji}(e)=l$. If $i=j$, which implies $k<l$, then set $\phi_{ii}(e^-)=k$ and $\phi_{ii}(e^+)=l$. This yields $\phi(D)=M$. Now fix the value of i. For each j (with $j\leqslant i$ admitted), the ϕ_{ij}-values assigned to appearances in E_i of edges in $D \cap D^0$ are distinct (since $M=\phi(D)$ is a matching) and form a subset S_{ij} of $\{1,2,\ldots,\delta_i\}$. For the same reason, the union $\bigcup_j S_{ij}$ is a disjoint union and has cardinality $d(v_i, D \cap D^0)$. Because

$$d(v_i, D^0-D)=d(v_i, D^0)-d(v_i, D\cap D^0)\leqslant \delta_i-|\bigcup_j S_{ij}|,$$

the appearances x of edges of D^0-D in E_i can be indexed by a subset of

$$\{1,2,\ldots,\delta_i\}-\bigcup_j S_{ij},$$

and we can take $\phi_{ij}(x)$ to be the index of x. This makes $\phi(D^0)$ a matching; the definitions of the ϕ_{ij}'s can be completed by extending them in any way to the remainder of their respective domains. This completes the proof, and shows that finding optimal degree-constrained subgraphs in (G, δ) can in principle be "reduced" to finding strong special augmenting paths for special matchings in the very large graph G^{**}.

This "reduction" is not quite of the form desired, since G^{**} will not be strict unless G is such that $|E_{ij}|\leqslant 1$ for $1\leqslant i, j\leqslant n$. For many situations of interest that assumption is satisfied, and in any case the work on matching optimization [4] does not require strictness of the underlying graph.

It is hoped that further exploration of the ideas presented above will lead to a computationally useful reduction of the degree-constrained subgraph optimization problem to the matching optimization problem or its associated techniques. To avoid possible misunderstanding, it should be emphasized that one efficient algorithm for the former problem is already known [5].

I am indebted to NBS colleagues J. Edmonds, C. Witzgall, and C. T. Zahn, Jr., for many discussions of their recent work in this area of combinatorial extremization. Zahn's direct proof of the theorem (*DCSG*) was the specific stimulus for this paper, and his scrutiny of an early draft uncovered the need for several modifications.

References

[1] C. Berge, Two theorems in graph theory, Proc. Nat. Acad. Sci. USA **43**, 842–844 (1957).

[2] C. Berge, The Theory of Graphs and its Applications (London, Methuen 1962).

[3] J. Edmonds, Covers and packings in a family of sets, Bull. Amer. Math. Soc. **68**, 494–499 (1962).

[4] J. Edmonds, Paths, trees and flowers, to appear in Can. J. Math.

[5] J. Edmonds, Maximum matching and a polyhedron with (0, 1) vertices, to be submitted for publication.

[6] R. Z. Norman and M. O. Rabin, An algorithm for a minimum cover of a graph, Proc. Amer. Math. Soc. **10**, 315–319 (1959).

[7] C. Witzgall and C. T. Zahn, Jr., A modification of Edmonds' algorithm for maximum matching, to be submitted for publication.

(Paper 68B1–112)

---

[4] Theorem (M) specialized to strict graphs can be used even though G^{**} is not necessarily strict, since we are working in the strict graph $\phi(G)$.