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A Note on a Generalized Elliptic Integral

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An expansion of

$$\Omega_j(k)=\int_0^\pi \frac{d\theta}{(1-k^2\cos\theta)^{j+1/2}}$$

in the neighborhood of $k^2=1$ is obtained by a method based on an Abelian theorem.

In a recent paper Epstein and Hubbell have given a short table of the function

$$\Omega_j(k)=\int_0^\pi \frac{d\theta}{(1-k^2\cos\theta)^{j+1/2}},\ j=0,1,2,\ .\ .\ . \tag{1}$$

which is important in certain problems in radiation physics, [1].[1] It is simple to find a power series expansion for $\Omega_j(k)$ by expanding the denominator and integrating term by term. It is somewhat more difficult to find an approximation to $\Omega_j(k)$ which is valid for k^2 close to one, and such an approximation is not found in [1] except in the cases $j=0,1$. It is the purpose of this note to furnish such an approximation.

Let us use the identity

$$\frac{1}{z^{j+1/2}}=\frac{1}{\Gamma(j+1/2)}\int_0^\infty t^{j-1/2}e^{-zt}dt \qquad \mathrm{Re}(z)>0 \tag{2}$$

to write eq (1) in the form

$$\Omega_j(k)=\frac{1}{\Gamma(j+1/2)}\int_0^\pi d\theta\int_0^\infty t^{j-1/2}e^{-(1-k^2\cos\theta)t}dt. \tag{3}$$

It is not difficult to justify an interchange of orders of integration. The integration over θ can then be performed by making use of the Bessel function formula

$$\frac{1}{\pi}\int_0^\pi e^{k^2t\cos\theta}d\theta=I_0(k^2t)$$

where $I_0(x)$ is a modified Bessel function. (4)

Hence $\Omega_j(k)$ is

$$\Omega_j(k)=\frac{\pi}{\Gamma(j-\frac{1}{2})}\int_0^\infty t^{j-1/2}e^{-t}I_0(k^2t)dt$$

$$=\frac{\pi}{k^{2j+1}\Gamma(j-\frac{1}{2})}\int_0^\infty e^{-x\left(\frac{1-k^2}{k^2}\right)}x^{j-1/2}e^{-x}I_0(x)dx. \tag{5}$$

We have therefore exposed $\Omega_j(k)$ as a Laplace transform in which the coefficient in the first exponential, $(1-k^2)/k^2$, approaches zero as k^2 approaches 1. We can now apply an Abelian theorem for Laplace transforms, [2], to determine the behavior of $\Omega_j(k)$ in the neighborhood of $k^2=1$. To do this we note the asymptotic expansion

$$e^{-x}I_0(x)\sim\frac{1}{\sqrt{2\pi x}}\sum_{n=0}^\infty\frac{(-1)^n}{(2x)^n}\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2}-n)}\frac{1}{n!} \tag{6}$$

as x tends to infinity. Substituting eq (6) into eq (5) we find for the asymptotic expansion of $\Omega_j(k)$:

$$\Omega_j(k)\sim\sqrt{\frac{\pi}{2}}\frac{1}{k^{2j+1}\Gamma(j-\frac{1}{2})}\left\{\sum_{n=0}^{j-1}\frac{(-1)^n\Gamma(\frac{1}{2}+n)}{2^n\Gamma(\frac{1}{2}-n)}\frac{(j-n-1)!}{n!}\left(\frac{k^2}{1-k^2}\right)^{j-n}\right.$$
$$\left.+\sum_{n=j}^\infty\frac{(-1)^j}{2^n}\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2}-n)}\frac{1}{(n-j)!}\left(\frac{1-k^2}{k^2}\right)^{n-j}ln\left(\frac{1-k^2}{k^2}\right)\right\}. \tag{7}$$

Another useful representation for $\Omega_j(k)$ can be obtained by noting that the Legendre function $P_{n-1/2}$ $(\cosh\eta)$ can be written, [3],

$$P_{n-1/2}(\cosh\eta)=\frac{1}{\pi}\int_0^\pi\frac{d\theta}{(\cosh\eta+\sinh\eta\cos\theta)^{n+1/2}} \tag{8}$$

[1] Figures in brackets indicate the literature references at the end of this paper.

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from which it follows that

$$\Omega_j(k)=\frac{\pi}{(1-k^2)^{\frac{2j+1}{4}}}P_{j-1/2}\left(\frac{1}{(1-k^4)^{1/2}}\right)$$

$$=\frac{\pi}{(1+k^2)^{\frac{2j+1}{2}}}{}_2F_1\left(\tfrac{1}{2},j+\tfrac{1}{2};1;\frac{2k^2}{1+k^2}\right). \tag{9}$$

However, a derivation of asymptotic behavior starting with eq (9) is not as direct as the proof we have given.

References

[1] L. F. Epstein and J. H. Hubbell, Evaluation of a generalized elliptic-type integral, J. Res. **67B** (Math. and Math. Phys.) No. 1, 1 (1963).

[2] G. Doetsch, Theorie und Anwendung der Laplace Transformation (Dover Reprint, 1943).

[3] I. M. Ryshik and I. S. Gradstein, Tables of series, products and integrals (VEB Deutscher Verlag der Wissenschaften, Berlin, 1957).

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