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Calculation of Certain Multiple Generating Functions

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This paper contains a discussion of the evaluation of generating functions of the form $F(\{x\}) = \sum_{n_1} \ldots \sum_{n_k} M_j(n_1, \ldots n_k) x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$ where $M_j(n_1, \ldots n_k)$ is the jth largest of the integers $(n_1, \ldots n_k)$. An alternate technique to one proposed by Carlitz is used in the calculation.

In a recent paper Carlitz has considered the problem of evaluating the generating functions

$$F_j(\{x\}) = \sum_{n_1=0}^{\infty} \ldots \sum_{n_k=0}^{\infty} M_j(n_1, n_2, \ldots n_k) x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} \tag{1}$$

where $M_j(n_1, n_2, \ldots n_k)$ is the jth greatest of the set of integers $\{n_j\}$, [1].[1] His method was essentially a combinatorial one. It is our purpose in the present note to reconsider this problem by evaluating, instead of F_j, the generating function

$$G_j(\{x\}; s) = \sum_{n_1=0}^{\infty} \ldots \sum_{n_k=0}^{\infty} e^{-sM_j(n_1, \ldots n_k)} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$$

from which it is possible to derive an expression for F_j by differentiation

$$F_j(\{x\}; s) = -\frac{\partial G_j}{\partial s}(\{x\}; s)\Big|_{s=0+}. \tag{3}$$

Expressions for related generating functions can also be obtained in this manner.

The principal tool in the following analysis is the identity

$$e^{-sM_1(n_1, \ldots n_k)} = s\int_{M_1(n_1, \ldots n_k)}^{\infty} e^{-st} dt = s\int_0^{\infty} e^{-st} H(t-n_1) H(t-n_2) \ldots H(t-n_k) dt \tag{4}$$

where $H(x)$ is the Heaviside step function

$$H(x) = 0,\ x < 0$$
$$= 1,\ x > 0. \tag{5}$$

We can immediately derive the expression for $G_1(\{x\}; s)$ from eq (4) by multiplying the general term of this equation by $x_1^{n_1} \ldots x_k^{n_k}$ and summing. An interchange of the orders of summation and integration is easily justified for $|x_1|, |x_2|, \ldots |x_k| < 1$ and so we have only to evaluate the sum

$$\sum_{n=0}^{\infty} H(t-n) x^n = \sum_{n=0}^{[t]} x^n = \frac{1 - x^{[t]+1}}{1-x}. \tag{6}$$

[1] L. Carlitz, The generating function for max $(n_1, \ldots n_k)$, Portugaliae, Mathematica **21**, 201 (1962).

With this result we find

$$G_1(\{x\};s)=\frac{s}{(1-x_1)(1-x_2)\ldots(1-x_k)}\int_0^\infty e^{-st}(1-x_1{}^{[t]+1})(1-x_2{}^{[t]+1})\ldots(1-x_k{}^{[t]+1})dt. \tag{7}$$

In order to evaluate this expression we need the following Laplace transform

$$s\int_0^\infty e^{-st}a^{[t]+1}dt=\frac{a(1-e^{-s})}{1-ae^{-s}}=(e^s-1)\left(\frac{1}{1-ae^{-s}}-1\right) \tag{8}$$

where a is a constant. Equations (7) and (8) together yield:

$$G_1(\{x\};s)=\frac{1}{(1-x_1)\ldots(1-x_k)}\left\{1-(e^s-1)\sum_r\left(\frac{1}{1-x_re^{-s}}-1\right)\right.$$
$$\left.+(e^s-1)\sum_{\substack{r_1\;r_2\\ r_1>r_2}}\sum\left(\frac{1}{1-x_{r_1}x_{r_2}e^{-s}}-1\right)-(e^s-1)\sum_{\substack{r_1\;r_2\;r_3\\ r_1>r_2>r_3}}\sum\sum\left(\frac{1}{1-x_{r_1}x_{r_2}x_{r_3}e^{-s}}-1\right)+\ldots\right\} \tag{9}$$

Now let us consider the evaluation of the other G_j. We require, for these generating functions, identities similar to that of eq (4). The preceding analysis suggests that we try the sum of integrals

$$s\int_0^\infty e^{-st}\{[1-H(t-n_1)]H(t-n_2)\ldots H(t-n_k)+H(t-n_1)[1-H(t-n_2)]H(t-n_3)\ldots H(t-n_k)$$
$$+\ldots+H(t-n_1)H(t-n_2)\ldots[1-H(t-n_k)]\}\,dt=U(n_1,n_2,\ldots n_k;s) \tag{10}$$

for the evaluation of G_2. By repeatedly using eq (4) we find

$$U(n_1,\ldots n_k;s)=[e^{-sM_1(n_2,\ldots n_k)}-e^{-sM_1(n_1,\ldots n_k)}]+[e^{-sM_1(n_1,n_3,\ldots n_k)}$$
$$-e^{-sM_1(n_1,\ldots n_k)}]+\ldots+[e^{-sM_1(n_1,\ldots,n_{k-1})}-e^{-sM_1(n_1,\ldots n_k)}]. \tag{11}$$

Assume now that the maximum of the $\{n_j\}$ is n_1. Then since n_1 appears in each of the terms in all of the brackets except the first, these brackets must be equal to zero. Furthermore we must have

$$M_1(n_2,\ldots n_k)=M_2(n_1,\ldots n_k). \tag{12}$$

Hence we find

$$G_2(\{x\};s)-G_1(\{x\};s)=\sum_{n_1}\ldots\sum_{n_k}U_1(n_1,\ldots n_k;s)x_1{}^{n_1}\ldots x_k{}^{n_k}. \tag{13}$$

The function $G_1(\{x\};s)$ has already been calculated in eq (9) so that G_2 can be determined to be

$$G_2(\{x\};s)=\sum_{r=1}^{k}G_1(\{x\}-x_r;s)+(1-k)G_1(\{x\};s) \tag{14}$$

where $\{x\}-x_r$ is the set $\{x\}$ less the element x_r. Similar expressions can be obtained for all of the G_j by starting from a sum of integrals similar to eq (4), each of which has j factors $1-H(t-n_r)$ with the remaining factors being of the form $H(t-n_r)$. In this way we can write the recurrence relation

$$G_{j+1}(\{x\};s)=\sum_{r=1}^{k}G_j(\{x\}-x_r;s)+(1-k)G_j(\{x\};s). \tag{15}$$

The function $F_1(\{x\})$ is now obtained from eqs (3) and (9) in a straightforward manner:

$$F_1(\{x\})=\frac{1}{(1-x_1)\ldots(1-x_k)}\left\{1-\sum_r \frac{x_r}{1-x_r}+\sum_{r_1\geq r_2}\sum \frac{x_{r_1}x_{r_2}}{1-x_{r_1}x_{r_2}}-\ldots\right.$$

$$\left.+(-1)^k\frac{x_1x_2\ldots x_k}{1-x_1x_2\ldots x_k}\right\} \tag{16}$$

and others of the F_j can be derived by recurrence

$$F_{j+1}(\{x\})=\sum_{r=1}^{k} F_j(\{x\}-x_r)+(1-k)F_j(\{x\}). \tag{17}$$

By these techniques we may derive Carlitz's result

$$F_j(\{x\})=\sum_{s=j}^{k}(-1)^{s+j}\binom{s-1}{j-1}\cdot U_{ks} \tag{18}$$

where

$$U_{ks}=\frac{1}{(1-x_1)(1-x_2)\ldots(1-x_k)}J\frac{x_1x_2\ldots x_s}{1-x_1x_2\ldots x_s} \tag{19}$$

where $Jf(x_1, x_2, \ldots x_s)$ is the symmetric function determined by $f(x_1, x_2, \ldots x_s)$.

Similar techniques can also be used for the calculation of generating functions like

$$H_j(\{x\})=\sum_{n_1=0}^{\infty}\ldots\sum_{n_k=0}^{\infty}M_j(n_1^\lambda, n_2^\lambda, \ldots n_k^\lambda)x_1^{n_1}\ldots x_k^{n_k}=\sum_{n_1=0}^{\infty}\ldots\sum_{n_k=0}^{\infty}[M_j(n_1, n_2, \ldots n_k)]^\lambda x_1^{n_1}\ldots x_k^{n_k}. \tag{20}$$

When λ is an integer H_j can be expressed as a derivative of G_j. However, one can calculate H_j for any λ by the same technique as we have used for $\lambda=1$. Define a function $G_j^{(\lambda)}(\{x\}; s)$ analogous to that in eq (4) except that $M_j(n_1, \ldots n_k)$ is replaced by $M_j(n_1^\lambda, \ldots n_k^\lambda)$. Then eq (4) remains valid except that each n_j is to be replaced by n_j^λ and the succeeding steps lead, in the case $j=1$, to the expression

$$G_1^\lambda(\{x\}; s)=\frac{s}{(1-x_1)\ldots(1-x_k)}\int_0^\infty e^{-st}(1-x_1^{[t^{\frac{1}{\lambda}}]+1})\ldots(1-x_k^{[t^{\frac{1}{\lambda}}]+1})dt. \tag{21}$$

The Laplace transform of $a^{[t^{\frac{1}{\lambda}}]+1}$ is

$$s\int_0^\infty e^{-st}a^{[t^{\frac{1}{\lambda}}]+1}dt=a\left[1-(1-a)\sum_{n=0}^{\infty}a^n e^{-(n+1)^\lambda s}\right] \tag{22}$$

which leads to

$$G_1(\{x\}; s)=\frac{1}{(1-x_1)\ldots(1-x_k)}\left\{1-\sum_{j=1}^{k}x_j\left[1-(1-x_j)\sum_{n=0}^{\infty}x_j^n e^{-(n+1)^\lambda s}\right]\right.$$

$$\left.+\sum_{j>r}\sum x_jx_r[1-(1-x_jx_r)]\sum_{n=0}^{\infty}(x_jx_r)^n e^{-(n+1)^\lambda s}-\ldots\right\} \tag{23}$$

which reduces to eq (9) when $\lambda=1$.

The method suggested in this note can be generalized to deal with any functional of the form $M_j(\varphi(n_1), \varphi(n_2), \ldots \varphi(n_k))$ providing that $\varphi(n)$ is a monotone increasing function which tends to infinity with n. It can also be used to calculate Laplace transforms rather than generating functions.

(Paper 68B1–110)