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Oblique Propagation of Groundwaves Across a Coastline—Part I[1]

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The amplitude and phase of the groundwave are calculated for oblique propagation across a flat lying coastline. The land and sea are assumed to be smooth and homogeneous. Attention is focused on the effects which take place near the coastline. It is shown that the reflected wave depends critically on the angle of incidence, θ_0, while the transmitted wave has only a weak dependence on θ_0.

1. Introduction

In certain radio navigational systems it is important to estimate the influence of inhomogeneities of the earth's surface. A striking example corresponds to the situation when the transmission is over mixed land/sea paths. Here, not only the conductivity contrast, but also the change of elevation of the various portions of the path may influence the amplitude and the phase of the received signal.

The problem of calculating fields over mixed paths has been investigated by Feinberg [1946], Millington [1949], Clemmow [1953], Bremmer [1954], Furutsu [1956], Wait [1956], Wait and Householder [1957], Godzinski [1958], Kalinin and Feinberg [1958], and others. Surveys of this and related work can be found in a recent review article [Wait, 1963].

It is the purpose of the present paper to consider propagation across a straight coastline for oblique incidence. Special attention is given to the behavior of the fields near the coastline itself since most previous theories are not valid there. Also, the reflection from the coastline and the change of wavefront direction are evaluated. In part II, the influence of a gradual elevation change between land and sea is treated using an approximate condition due originally to Feinberg. Finally, in part III, the effect of allowing for the nonabruptness of the effective conductivity change between land and sea is considered.

2. Formulation

To simplify the problem at the outset, a flat coastline is considered and the influence of earth curvature is neglected. The situation is illustrated in figure 1. With respect to a Cartesian coordinate system, the xy plane is taken to be the plane surface of the earth and the coastline is the y axis. For purposes of discussion, the medium to the left (i.e., $x<0$) is described as land with surface impedance Z while the medium to the right (i.e., $x>0$) is described as the sea with surface impedance Z'. The transmitter at A with coordinates $(-x_0, y_0)$ is regarded as a vertical electric dipole, of effective height h_a, on the surface of the land. The receiving antenna, of effective height h_b, is located at B with coordinates $(d_1, 0)$ where d_1 may be positive or negative. For convenience, the receiving antenna at B is also located on the earth's surface and it is assumed that it responds only to the vertical electric field component at $z=0$.

In formulating mixed path problems of this kind, it is usually assumed that the surface impedance Z for the land, and Z' for the sea, hold right up to the boundary. Actually, even if the boundary were abrupt, there would be a violation of the basic premises in the use of the surface impedance concept. The conditions for the applicability of the impedance relations

$$E_x=-ZH_y \text{ and } E_y=ZH_x$$

are that Z should change slowly in a distance equal to the effective wavelength λ_e in the lower medium. Because λ_e is much less than λ_0, the free space wavelength, it means that the surface impedance relations may be valid at distances from the boundary small

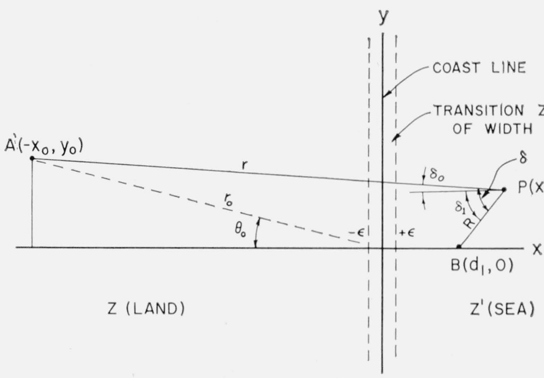

FIGURE 1. *Plane view of the mixed path showing the location of the dipoles* A *and* B.

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compared with λ_0. To extend the usefulness of the surface impedance concept, it is desirable to insert a transition zone such that Z changes gradually to Z'. The width of the transition zone is equal to 2ϵ as indicated in figure 1. It may be defined such that Z and Z' are essentially constant when $x<-\epsilon$ and $x>+\epsilon$, respectively. Apart from mathematical convenience, the existence of such a transition zone has a practical significance in studying the influence of a variable water depth in the vicinity of the shoreline.

If the problem involved only a homogeneous land of surface impedance Z, the mutual impedance z_m between dipoles A and B would be readily calculated. Thus, it is convenient to regard the medium to the right of the coastline as a perturbation and the resulting change of the mutual impedance is denoted Δz_m. From previous work [Wait, 1956], it is known that Δz_m can be expressed in terms of a surface integral over the plane $z=0$, of the form

$$\Delta z_m=\frac{1}{I^2}\iint_S (Z'-Z)\vec{H}_{at}\cdot\vec{H}'_{bt}dS, \tag{1}$$

where $\vec{H}_{at}$ is the tangential magnetic field of dipole A over the reference earth of constant surface impedance Z and where $\vec{H}'_{bt}$ is the tangential magnetic field of dipole B over the actual inhomogeneous earth. In formulating this integral, the current at the terminals of dipoles A and B is I.

Following earlier work [Wait, 1956], it is convenient to express the tangential field vectors at the variable point $P(x, y)$ in the following forms

$$\vec{H}_{at}=\frac{ikh_aI}{2\pi r}\left(1+\frac{1}{ikr}\right)e^{-ikr}F(r, Z)(\vec{i}_r\times\vec{i}_z), \tag{2}$$

and

$$\vec{H}'_{bt}=\frac{ikh_bI}{2\pi R}\left(1+\frac{1}{ikR}\right)e^{-ikR}F'(R, Z, Z')(\vec{i}_R\times\vec{i}_z), \tag{3}$$

where

$$r=[(x+x_0)^2+(y-y_0)^2]^{\frac{1}{2}},$$

$$R=[(x-d_1)^2+y^2]^{\frac{1}{2}},$$

and F and F' are slowly varying quantities. $\vec{i}_r$, $\vec{i}_R$, and $\vec{i}_z$ are unit vectors in the directions of increasing r, R, and z, respectively.

The attenuation function F is a known quantity from the theory of groundwave propagation over a flat homogeneous earth of surface impedance Z. The explicit form [Wait, 1963] is given by

$$F(r, Z)\cong 1-i(\pi p)^{\frac{1}{2}}e^{-p}\ \mathrm{erfc}\ (ip^{\frac{1}{2}}), \tag{4}$$

where $p=-\frac{ikr}{2}\left(\frac{Z}{\eta_0}\right)^2$. When $|p|<<1$, $F(r, Z)\cong +1$ while, if $|p|>>1$, $F(r, Z)\cong -1/(2p)$. The function $F'(R, Z, Z')$ is not known except to say that it is a function of R, Z, and Z' and that it approaches unity if R is sufficiently small.

Using the above representations for $\vec{H}_{at}$ and $\vec{H}_{bt}$, (1) for Δz_m may be written

$$\Delta z_m=-\frac{k^2h_ah_b}{4\pi^2}\int_{x=-\epsilon}^{+\infty}\int_{y=-\infty}^{+\infty}\frac{e^{-ik(r+R)}}{rR}F(r, Z)F'(R, Z, Z')$$
$$\times(Z'-Z)\left(1+\frac{1}{ikR}\right)\left(1+\frac{1}{ikr}\right)\cos\delta dxdy, \tag{5}$$

where δ is the angle subtended by the units vectors $\vec{i}_r$ and $\vec{i}_R$. Here the element of area dS has been replaced by $dxdy$. Also, it should be noted that the surface S is the region $x>-\epsilon$ since $Z'-Z\cong 0$ for $x<-\epsilon$.

3. Approximate Solution

A number of approximations are now made in order to facilitate the integrations. First, it is assumed that dipole A is in the far field such that the incident wave fronts are approximately plane in the neighborhood of the boundary under consideration. Then the slowly varying attenuation function $F(r, Z)$ may be replaced by $F(r_0, Z)$ which is the value appropriate to the origin. Also, it is assumed that the attenuation function $F'(R, Z, Z')$ may be replaced by unity if quantities of first order only are to be retained.

Within the approximations stated above it is evident that

$$\cos\delta=\cos(\delta_0+\delta_1)=\cos\delta_0\cos\delta_1-\sin\delta_0\sin\delta_1,$$

where

$$\sin\delta_1=S_1=y_0/r_0,$$

$$\cos\delta_1=C_1=(1-S_1^2)^{\frac{1}{2}}=x_0/r_0,$$

$$\sin\delta_0=y/R \text{ and } \cos\delta_0=\frac{x-d_1}{R}.$$

Thus

$$\cos\delta\cong\frac{x-d_1}{R}C_1-\frac{y}{R}S_1.$$

In the phase factor $\exp(-ik(r+R))$ it is desirable to simplify r by retaining only first-order variations in x and y. For example,

$$r^2=(x+x_0)^2+(y-y_0)^2$$
$$\cong x_0^2+y_0^2+2xx_0-2yy_0$$

and, therefore,

$$r\cong r_0+xC_1-yS_1.$$

Furthermore, in the integrand of (5), $1/r$ may be replaced by $1/r_0$, and the factor $1+1/ikr$ by unity.

Using the simplifications indicated, it readily follows that

$$\Delta z_m \cong -\frac{k^2 h_a h_b}{4\pi^2} \frac{e^{-ikr_0}}{r_0} F(r_0, Z)$$

$$\times \int_{-\epsilon}^{\infty} (Z'-Z) e^{-ikC_1x} \int_{-\infty}^{+\infty} \frac{e^{-ikR}}{R} \left(1+\frac{1}{ikR}\right)$$

$$\times \left[\left(\frac{x-d_1}{R}\right) C_1 - \frac{y}{R} S_1\right] e^{ikS_1y} dy dx. \quad (6)$$

By making use of the identity

$$\frac{\partial}{\partial R} \frac{e^{-ikR}}{R} = -ik\left(1+\frac{1}{ikR}\right)\frac{e^{-ikR}}{R},$$

it readily follows that

$$\Delta z_m = \frac{-ikh_a h_b}{4\pi^2} \frac{e^{-ikr_0}}{r_0} F(r_0, Z) \int_{-\epsilon}^{\infty} (Z'-Z) \exp[-ikC_1x]$$

$$\times \left\{ C_1 \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} \frac{e^{-ikR}}{R} e^{ikS_1y} dy \right.$$

$$\left. -S_1 \int_{-\infty}^{+\infty} e^{ikS_1y} \frac{\partial}{\partial y} \frac{e^{-ikR}}{R} dy \right\} dx. \quad (7)$$

The infinite integrals with respect to y may be expressed in terms of the Hankel function of order zero [Campbell and Foster, 1949]. Explicitly

$$\int_{-\infty}^{+\infty} e^{ikS_1y} \frac{e^{-ikR}}{R} dy = -i\pi H_0^{(2)}[kC_1|x-d_1|], \quad (8)$$

which is valid because k can be regarded to have a vanishingly small but finite negative imaginary part.

To simplify the final expression and to cast it into a dimensionless form, the results are normalized by dividing by the mutual impedance z_m, defined by

$$z_m = \frac{ik\eta_0 h_a h_b}{2\pi_0 \hat{r}} \left(1+\frac{1}{ik\hat{r}_0} - \frac{1}{k^2\hat{r}_0^2}\right) e^{-ik\hat{r}_0} F(\hat{r}_0, Z), \quad (9)$$

where $\hat{r}_0 = r_0 + d_1C_1$. This expression corresponds to the mutual impedance between dipoles A and B, at separation distance $\hat{r}_0$, if they were located on a flat homogeneous earth of surface impedance Z. Within the limitations already imposed,

$$z_m \cong \frac{ik\eta_0 h_a h_b}{2\pi r_0} e^{-ikr_0} e^{-ikC_1d} F(r_0, Z). \quad (10)$$

Making use of (7) and (8), it readily follows that

$$\frac{\Delta z_m}{z_m} \cong \frac{e^{ikC_1d_1}}{2\pi} \int_{-\epsilon}^{\infty} \frac{Z'-Z}{\eta_0} e^{-ikC_1x}$$

$$\times \left\{ -i\pi C_1 \frac{\partial}{\partial x} H_0^{(2)}[kC_1|x-d_1|] \right.$$

$$\left. +\pi k S_1^2 H_0^{(2)}[kC_1|x-d_1|] \right\} dx. \quad (11)$$

Following an integration by parts,

$$\frac{\Delta z_m}{z_m} \simeq -\frac{i}{2} e^{ikC_1d_1} \int_{-\epsilon}^{\infty} \left[C_1 \frac{df(x)}{dx} - ikf(x)\right]$$

$$\times e^{-ikC_1x} H_0^{(2)}[kC_1|x-d_1|] dx, \quad (12)$$

where $f(x) = (Z'-Z)/\eta_0$. Introducing the dimensionless quantities

$$\alpha = kC_1x,\ \alpha_1 = kC_1d_1, \text{and } \alpha_0 = kC_1\epsilon,$$

it is evident that

$$\frac{\Delta z_m}{z_m} = -\frac{i}{2} e^{i\alpha_1} \int_{-\alpha_0}^{\infty} H_0^{(2)}[|\alpha-\alpha_1|] e^{-i\alpha}$$

$$\times \left[C_1 \frac{df(\alpha)}{d\alpha} - \frac{if(\alpha)}{C_1}\right] d\alpha, \quad (13)$$

where $f(\alpha) = (Z'-Z)/\eta_0$ is regarded as a function of α.

When the vertical stratification in the earth may be neglected, the surface impedance function can be expressed in terms of the conductivities σ_g and σ' and dielectric constants ϵ_g and ϵ'. Thus

$$f(\alpha) \cong (i\epsilon_0\omega)^{\frac{1}{2}} \left[\frac{1}{(\sigma'+i\epsilon'\omega)^{\frac{1}{2}}} - \frac{1}{(\sigma_g+i\epsilon_g\omega)^{\frac{1}{2}}}\right],$$

where

$$\epsilon_0 = 8.85 \times 10^{-12}.$$

4. Special Case of Abrupt Boundary

To effect the integration with respect to α requires that the function $f(\alpha)$ is specified. The simplest case, of course, is the coastline which can be regarded as abrupt. For example, one may consider

$$f(\alpha) = -\Delta_0 e^{i\pi/4} u(\alpha)$$

where

$$u(\alpha) = 1 \text{ for } \alpha > 0,$$

$$= 0 \text{ for } \alpha < 0.$$

Here, Δ_0 is to be regarded as a constant and defined such that

$$\Delta_0 = \frac{Z-Z'}{\eta_0} e^{-i\pi/4}.$$

If both media (i.e., land and sea) are sufficiently well conducting that displacement currents can be neglected, Δ_0 is a real quantity. In general, however, Δ_0 is complex and is a measure of the contrast in surface impedances between the two homogeneous regions.

For the assumed sharp boundary, it should be noted that

$$\frac{df(\alpha)}{d\alpha} = -\Delta_0 e^{i\pi/4} \delta(\alpha),$$

where $\delta(\alpha)$ is the Dirac impulse function at $\alpha=0$.

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Therefore

$$\frac{\Delta z_m}{z_m}=\frac{\Delta_0}{2}e^{i3\pi/4}e^{i\alpha_1}\left[C_1 H_0^{(2)}(|\alpha_1|)\right.$$
$$\left.-\frac{i}{C_1}\int_0^\infty H_0^{(2)}(|\alpha-\alpha_1|)e^{-i\alpha}d\alpha\right]. \quad (14)$$

It is immediately evident that this expression for the impedance or the field possesses a singularity as α_1 tends to zero. For example, when $|\alpha_1|<<1$,

$$H_0^{(2)}(|\alpha_1|)\simeq 1-i\frac{2}{\pi}(\log|\alpha_1|-\log 2+0.5773), \quad (15)$$

which would indicate that the field has a logarithmic singularity at the boundary. As will be indicated below, the singularity is spurious since for any finite transition between Z and Z' the field no longer behaves in this manner very near the coastline. However, provided $|\alpha_1|$ is not too small compared with unity, the assumption of an abrupt or sharp boundary is satisfactory.

Using the identity

$$\frac{\partial}{\partial\alpha}[\alpha e^{\pm i\alpha}(H_0^{(2)}(\alpha)\mp iH_1^{(2)}(\alpha))]=e^{\pm i\alpha}H_0^{(2)}(\alpha), \quad (16)$$

it is a straightforward task to show that

$$\frac{\Delta z_m}{z_m}=\frac{\Delta_0}{2C_1}e^{i3\pi/4}[(C_1^2-i\alpha_1)H_0^{(2)}(-\alpha_1)$$
$$+\alpha_1 H_1^{(2)}(-\alpha_1)]e^{i\alpha_1}, \quad (17)$$

for $\alpha_1<0$, while

$$\frac{\Delta z_m}{z_m}=\frac{\Delta_0}{2C_1}e^{i3\pi/4}[(C_1^2-i\alpha_1)H_0^{(2)}(\alpha_1)-\alpha_1 H_1^{(2)}(\alpha_1)]e^{i\alpha_1}, \quad (18)$$

for $\alpha_1>0$.

For purposes of computation, it is convenient to write

$$\frac{\Delta z_m}{z_m}=C_1\Delta_0 g(\alpha_1)=\Delta_0 C_1\left[g_1(\alpha_1)+\frac{1}{C_1^2}g_2(\alpha_1)\right], \quad (19)$$

where g_1 and g_2 do not depend on C_1 or Δ_0. Noting that $H_m^{(2)}(\chi)=J_m(\chi)-iY_m(\chi)$ where J_m and Y_m are Bessel functions of order m, it is a simple matter to obtain the following explicit formulas in terms of real quantities.

For $\chi=\alpha_1>0$,

$$g_1(\chi)=\frac{1}{2}\left[\cos\left(\frac{3\pi}{4}+\chi\right)J_0(\chi)+\sin\left(\frac{3\pi}{4}+\chi\right)Y_0(\chi)\right]$$
$$+\frac{i}{2}\left[\sin\left(\frac{3\pi}{4}+\chi\right)J_0(\chi)-\cos\left(\frac{3\pi}{4}+\chi\right)Y_0(\chi)\right], \quad (20)$$

and

$$g_2(\chi)=\frac{\chi}{2}\left[\cos\left(\frac{\pi}{4}+\chi\right)(J_0-Y_1)+\sin\left(\frac{\pi}{4}+\chi\right)(J_1+Y_0)\right]$$
$$+i\frac{\chi}{2}\left[\sin\left(\frac{\pi}{4}+\chi\right)(J_0-Y_1)-\cos\left(\frac{\pi}{4}+\chi\right)(J_1+Y_0)\right], \quad (21)$$

where the arguments of the Bessel functions are $\chi(=\alpha_1)$ throughout.

For $\chi=\alpha_1<0$,

$$g_1(\chi)=\frac{1}{2}\left[\cos\left(\frac{3\pi}{4}+\chi\right)J_0(-\chi)+\sin\left(\frac{3\pi}{4}+\chi\right)Y_0(-\chi)\right]$$
$$+\frac{i}{2}\left[\sin\left(\frac{3\pi}{4}+\chi\right)J_0(-\chi)-\cos\left(\frac{3\pi}{4}+\chi\right)Y_0(-\chi)\right], \quad (22)$$

and

$$g_2(\chi)=\frac{\chi}{2}\left[\cos\left(\frac{\pi}{4}+\chi\right)(J_0+Y_1)-\sin\left(\frac{\pi}{4}+\chi\right)(J_1-Y_0)\right]$$
$$+i\frac{\chi}{2}\left[\sin\left(\frac{\pi}{4}+\chi\right)(J_0+Y_1)+\cos\left(\frac{\pi}{4}+\chi\right)(J_1-Y_0)\right], \quad (23)$$

where the arguments of the Bessel functions are $-\chi(=-\alpha_1)$ throughout.

Employing the preceding formulas, the real and imaginary parts of $g(\alpha_1)$ are calculated for a range of α_1 from -20 to $+20$. These results are shown in figures 2 to 6 for selected values of θ_0. Additional curves of this type may be readily calculated from the numerical values of g_1 and g_2 given in the appendix.

When the dipole at A is transmitting, the fractional change of the field at B, resulting from the presence of the medium of surface impedance Z', is $(\Delta z_m/z_m)$ or $C_1\Delta_0 g(\alpha_1)$. Thus, the curves in figures 2 and 3 show the nature of the phenomenon near the coastline. As expected, the field exhibits a singular behavior near the boundary. For positive values

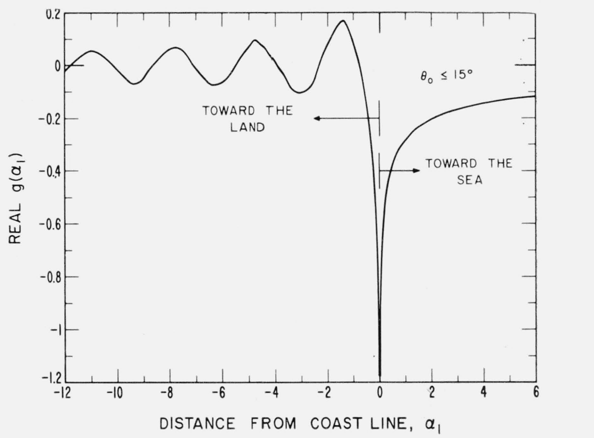

FIGURE 2. *Real part of the function* $g(\alpha_1)$ *for near normal incidence.*

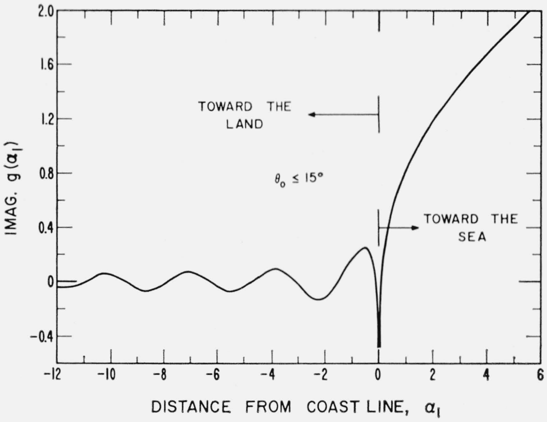

FIGURE. 3. *Imaginary part of the function* $g(\alpha_1)$ *for near normal incidence.*

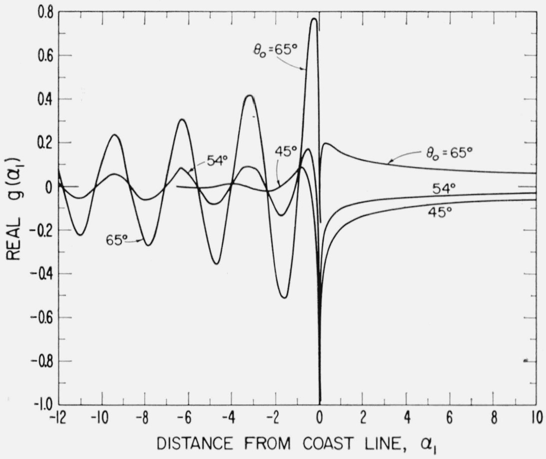

FIGURE 4. *Real part of the function* $g(\alpha_1)$ *for various oblique angles of incidence.*

of α (i.e., B located over the sea), the field is smoothly varying and for α_1 greater than 2 or 3, the imaginary part of $g(\alpha_1)$ is predominant and it increases to relatively large values as α_1 is increased further. For negative values of α_1, (i.e., B located over the land) there is strong evidence that reflection takes place at the boundary. Presumably, the standing wave pattern results from the interference between the incident and the reflected waves. It is interesting to note that at an angle of $\theta=45°$ the reflection from the coastline appears to be very weak. This fact is confirmed by noting in the asymptotic approximation for $(-\alpha_1)>>1$, that

$$g(\alpha_1)\cong-\frac{1}{\sqrt{2\pi(-\alpha_1)}}\,e^{i2\alpha_1}\left[\left(1+\frac{i}{8(-\alpha_1)}+\ldots\right)\right.$$
$$\left.-\frac{1}{2C_1^2}\left(1+\frac{9i}{32(-\alpha_1)}+\ldots\right)\right]. \quad (24)$$

It is evident that, if $C_1^2=\cos\,\theta_0=\frac{1}{2}$, the leading terms in the expansion cancel. In this case, $g(\alpha_1)$ is varying as $(-\alpha_1)^{-3/2}$ and thus the reflected wave would be attenuated quite rapidly. This same asymptotic expansion shows that the reflection is strong at highly oblique incidence (where C_1 is small).

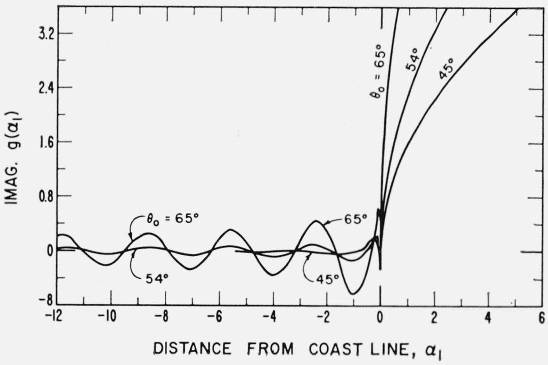

FIGURE 5. *Imaginary part of the function* $g(\alpha_1)$ *for various oblique angles of incidence.*

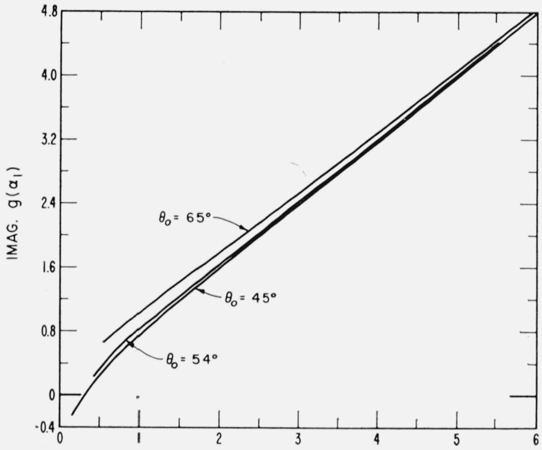

FIGURE 6. *Imaginary part of* $g(\alpha_1)$ *for various values of* θ_0 *when plotted as a function of* $(\alpha_1)^{\frac{1}{2}}/\cos^2\theta_0$.

The asymptotic behavior of the field for large positive values of α_1 is also very interesting. For example, if $\alpha_1>>1$,

$$g(\alpha_1)\sim\frac{e^{i3\pi/4}}{2C_1^2}\left[(C_1^2-i\alpha_1)\sqrt{\frac{2}{\pi\alpha_1}}\,e^{i\pi/4}\left(1+\frac{i}{8\alpha_1}+\ldots\right)\right.$$
$$\left.-\alpha_1\sqrt{\frac{2}{\pi\alpha_1}}\,e^{i3\pi/4}\left(1-\frac{3i}{8\alpha_1}+\ldots\right)\right]. \quad (25)$$

On retaining terms of first order only,

$$g(\alpha_1)\sim\frac{i}{C_1^2}\left(\frac{2\alpha_1}{\pi}\right)^{\frac{1}{2}},$$

and if r_1 denotes the distance, measured in the direction of propagation, from the coastline to the observer, geometry indicates that

$$\alpha_1=kd_1C_1=kr_1C_1^2.$$

Therefore, within a first order,

$$\frac{\Delta z_m}{z_m}\cong\Delta_0C_1g(\alpha_1)\sim i\Delta_0\left(\frac{2r_1k}{\pi}\right)^{\frac{1}{2}}, \quad (26)$$

which is independent of the angle C_1. To demonstrate how closely this law is followed in the general case, Im $g(\alpha_1)$ for positive α_1 is plotted as a function of $(\alpha_1)^{\frac{1}{2}}/C_1^2$ using the complete expressions derived above. It is quite evident that for larger values of α_1, the curves for all angles up to 70° approach a straight line with a slope of $(2/\pi)^{\frac{1}{2}}$.

5. Coastal Refraction

An extremely interesting aspect of oblique propagation across a coastline is the resulting change of wave front direction. To simplify the discussion, it is assumed that the field, incident on the boundary at $x=0$, has the form

$$E_o=e^{-ikC_1x}e^{ikS_1y}, \tag{27}$$

which is appropriate if the transmitter at A is sufficiently removed to the left in figure 1. Then the resultant field has the form

$$E=E_o(1+\Delta_0 gC_1)\simeq E_o \exp(\Delta_0 gC_1), \tag{28}$$

where Δ_0 g, defined by (19) et seq., is a function of x only.

By definition the phase velocities in the x and y directions are given by

$$v_x/c=-(1/k)(\partial/\partial x) \text{ phase } E, \tag{29}$$

and

$$v_y/c=(1/k)(\partial/\partial y) \text{ phase } E, \tag{30}$$

where c is the velocity of light. Then it readily follows that

$$v_x/c\cong C_1-(1/k)(\partial/\partial x) \text{ Im } (C_1\Delta_0 g), \tag{31}$$

and

$$v_y/c\cong S_1. \tag{32}$$

The effective direction θ_{eff} of phase propagation is then obtained from

$$\tan\theta_{\text{eff}}\cong(v_y/v_x), \tag{33}$$

or

$$\tan\theta_{\text{eff}}\cong\tan\theta_0\left[1+\frac{1}{k}\frac{\partial}{\partial x}\text{ Im }(\Delta_0 g)\right]. \tag{34}$$

The refraction error $\delta\theta$ is given by $\delta\theta=\theta_{\text{eff}}-\theta_0$. Thus, to within a first order of smallness in $\delta\theta$,

$$\tan(\theta_0+\delta\theta)=\tan\theta_0+\frac{1}{\cos^2\theta_0}\delta\theta, \tag{35}$$

which leads to the simple but important result that

$$\delta\theta\cong\frac{\sin\theta_0}{k}\frac{\partial}{\partial x}\text{ Im }(\Delta_0 gC_1). \tag{36}$$

In terms of the variable $\alpha_1=kC_1d_1$, and for an observer at $x=d_1$, the refraction error may be expressed in the form

$$\delta\theta\cong S_1C_1^2\frac{\partial}{\partial\alpha_1}[\text{Im }\Delta_0 g(\alpha_1)]. \tag{37}$$

Using the general expression for $g(\alpha_1)$ given by (19), it is found that, for $\alpha_1>0$,

$$\delta\theta=\frac{S_1}{2}\text{ Im }\left\{\Delta_0 e^{i\left(\alpha_1+\frac{3\pi}{4}\right)}[i(C_1^2-1)H_0^{(2)}(\alpha_1)-C_1^2H_1^{(2)}(\alpha_1)]\right\}, \tag{38}$$

If Δ_0 is regarded as real,

$$\delta(\theta)\simeq\frac{S_1}{2}\Delta_0\left\{\sin\left(\alpha_1+\frac{\pi}{4}\right)[J_0-C_1^2(J_0+Y_1)]-\cos\left(\alpha_1+\frac{\pi}{4}\right)[Y_0+C_1^2(J_1-Y_0)]\right\}, \tag{39}$$

where, as usual, the arguments of the Bessel functions are α_1.

At reasonably large distances from the boundary, where $\alpha_1>>1$, (38) simplifies to

$$\delta\theta\cong S_1\left(\frac{1}{2\pi\alpha_1}\right)^{\frac{1}{2}}\text{Re }\Delta_0$$

$$\simeq\sin\theta_0(2\pi kd_1\cos\theta_0)^{-\frac{1}{2}}\text{ Re }\Delta_0, \tag{40}$$

which is in agreement with a formula given by Feinberg [1946]. It is evident that the effect is very small when d_1 is greater than a few wavelengths. (When propagating from land toward the sea Re Δ_0 may be replaced by $(\epsilon_0\omega/\sigma_g)^{\frac{1}{2}}$ where $\epsilon_0=8.85\times10^{-12}$ and σ_g is the conductivity of the land.)

6. Magnetic Field Near the Coastline

In the analysis in this paper, the mutual impedance ratio $\Delta z_m/z_m$ is valid when the antennas at A and B are vertical electric dipoles. Actually, if neither A or B is near the boundary, the results are still applicable for other antennas which transmit or receive vertically polarized waves. However, if one of the antennas, say B is close to the boundary, the mutual impedance ratio will depend on whether the antenna is a vertical electric dipole (e.g., a whip) or a horizontal magnetic dipole (e.g., a vertical loop). For the general case of a loop antenna at B and a vertical dipole at A, it is possible to use the same method outlined in this paper. However, a complexity arises since the tangential magnetic field vector $\vec{H}_{bt}$ has additional terms proportional to $1/R^3$ and these tend to complicate the integrations. However, if the direction of propagation is perpendicular to the coastline, the final results may be obtained in a relatively straightforward manner.

If the field incident on the boundary, at $x=0$, has the form

$$E_0=e^{-ikx},$$

then the resultant vertical electric field may be written

$$E_z(x)=e^{-ikx}[1+F(x)], \text{ at } z=0, \tag{41}$$

where

$$F(x)=\frac{\Delta z_m}{z_m},$$

is the mutual impedance ratio given by (13) with $C_1=1$ and $d_1=x$.

The magnetic field which has only a y component is obtained from the Maxwellian equation, written in integral form,

$$H_y(x)-H_y(\hat{x}_0)=i\epsilon_0\omega\int_{\hat{x}_0}^{x}E_z(x')dx', \tag{42}$$

where $\hat{x}_0$ is some reference distance. Now, if $k\hat{x}_0>>1$, it is safe to assume that

$$\eta_0 H_y(\hat{x}_0)\cong -E_z(\hat{x}_0)=-e^{-ik\hat{x}_0}[1+F(\hat{x}_0)]. \tag{43}$$

Then, without difficulty one finds

$$-\eta_0 H_y=e^{-ikx}[1+G(x)], \tag{44}$$

where

$$G(x)=F(\hat{x}_0)e^{-ik(\hat{x}_0-x)}+ike^{ikx}\int_x^{\hat{x}_0}F(x')e^{-ikx'}dx'. \tag{45}$$

Since $F(x)$ is known, the function $G(x)$ can be found from an integration which can be best done numerically. It can be seen that, when $kx>>1$, $G(x)\sim F(x)$, as it must.

It is interesting to note that, for an abrupt boundary, $G(x)$ does not have the singular behavior characteristic of $F(x)$, at the boundary. While $F(x)$ behaves as $\log x$, the function $G(x)$ behaves as $x(\log x)-x$ when $|kx|<<1$.

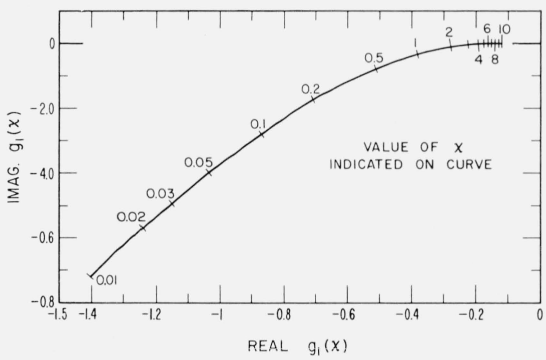

FIGURE 7. *Argand plot of* $g_1(\chi)$ *for* $\chi>0$.

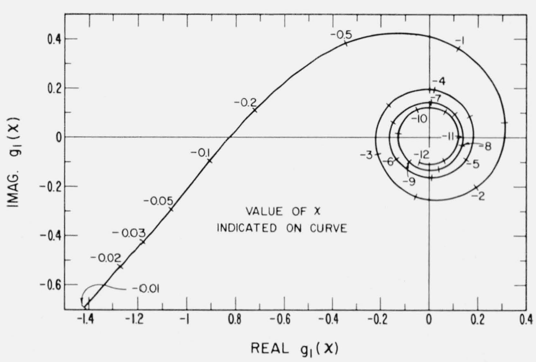

FIGURE 8. *Argand plot of* $g_1(\chi)$ *for* $\chi<0$.

7. Appendix

For convenience in auxiliary calculations, a graphical plot of the functions $g_1(\chi)$ and $g_2(\chi)$ is presented in figures 7 to 10. In complex form these quantities are defined by

$$g_1(\chi)=(1/2)\exp[i\chi+i(3\pi/4)]H_0^{(2)}(|\chi|)\text{ for }\chi\gtrless 0, \tag{46}$$

$$g_2(\chi)=(\chi/2)\exp[i\chi+i(\pi/4)][H_0^{(2)}(\chi)-iH_1^{(2)}(\chi)], \tag{47}$$

for $\chi>0$, and

$$g_2(\chi)=(\chi/2)\exp[i\chi+i(\pi/4)][H_0^{(2)}(-\chi)+iH_1^{(2)}(-\chi)], \tag{48}$$

for $\chi<0$.

The mutual impedance increment Δz_m may then be calculated for any value of θ_0 by noting that

$$\frac{\Delta z_m}{z_m}=\Delta_0\left[C_1 g_1(\alpha_1)+\frac{1}{C_1}g_2(\alpha_1)\right], \tag{49}$$

where $C_1=\cos\theta_0$.

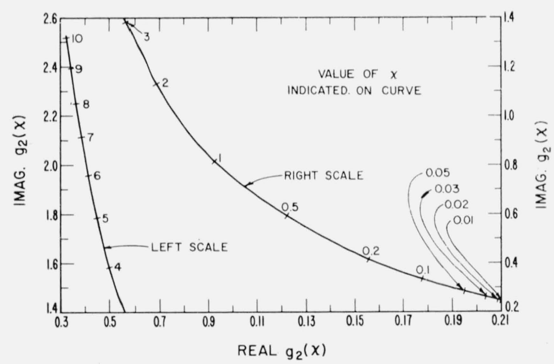

FIGURE 9. *Argand plot of* $g_2(\chi)$ *for* $\chi>0$.

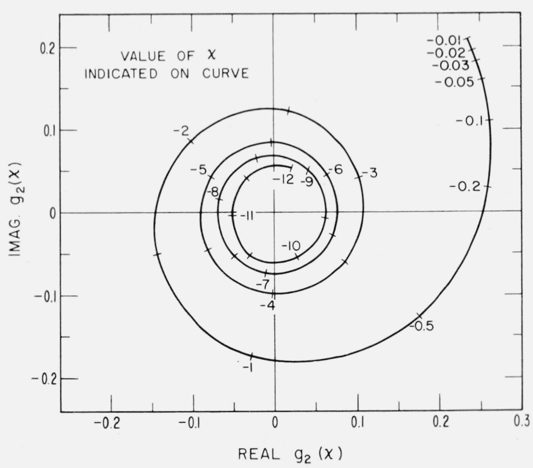

FIGURE 10. *Argand plot of* $g_2(\chi)$ *for* $\chi<0$.

I thank Carolen M. Jackson for carrying out the calculations of $g_1(\chi)$ and $g_2(\chi)$ on a desk computer, using available tables for the Bessel functions. I am also indebted to Mrs. Eileen Brackett who typed the manuscript. I would like to thank S. W. Maley for his constructive criticisms.

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