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Magnetic Torques and Coriolis Effects on a Magnetically Suspended Rotating Sphere

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The tendency for the spin axis of a conducting ferromagnetic sphere to aline eventually with a steady uniform external magnetic field, and the effect of absolute rotation of the frame of reference are considered. If the frame of reference is stationary with respect to free space, and if the initial angle from the magnetic axis is small and equal to θ_0 at $t=0$, then $\theta=\theta_0 e^{-t/\tau}$ thereafter, where τ is a function of the imaginary part of the complex susceptibility and the square of the magnetic field (7). However, on the Earth's surface, a constant hang off angle is found, which for small θ is approximately $(\Omega_{\text{earth}}\,\tau)\sin\theta_{\text{earth}}\,(1+(\delta p/q)^2)^{-1/2}$ where p and q are the real and imaginary parts of the complex susceptibility, and δp involves both the normal and incremental permeabilities. For any one spin frequency, this results in a constant decay of the sphere's energy (13).

1. Introduction

There is increasing interest as to what actually causes otherwise freely rotating extended systems of charge or mass to dissipate energy at very slow rates and run down. Time constants for the decrease of field energy of a superconducting solenoid current on the order of 130,000 years have been measured by NMR techniques [File and Mills, 1963], while time constants on the order of 10 years have been obtained [Kenney, 1961] for the decrease in kinetic energy of spherical material rotators in a magnetic suspension system [Beams, Spitzer, and Wade, 1962].

These very small residual energy dissipations appear to border on some of the most basic phenomena of physics, such as the problem of radiation reaction and the form of coupling between spinning matter and the medium through which it rotates. There are also more standard effects, which are interesting since their small magnitude has made them previously inaccessible to measurement.

The following article treats one of the latter, the energy dissipated when external torques act on a spinning ferromagnetic ball in a steady external magnetic field (as for example upon a small rotor in a magnetic suspension system). In particular, the effect of Earth's rotation is considered, or rotation of the reference frame with respect to which the magnetic field remains fixed. For reasonable values of conductivity, permeability, and magnetic field, the kinetic energy decay time constant is found to be on the order of several years due to the effect of Earth's rotation. This can presently be measured in the laboratory.

Assume a sphere with conductivity σ and principal moments of inertia $I_1=I_2=I_3$ rotates around its spin axis $\hat{\zeta}$ with angular velocity $\dot{\psi}\hat{\zeta}$, and that this axis is inclined to the axis of an external uniform magnetic field $\vec{B}$ which is in the direction of $\vec{z}$. At time $t_0=0$ the angle between the spin axis and the magnetic field is $\theta=\theta_0$.

Assuming the magnetization is caused only by induction (that there are no permanent or fixed magnetic moments) and the sphere is magnetically isotropic, we calculate the consequent motion of the spin axis. It will be shown that if eddy currents are generated in the ball due to time varying components of the external field $\vec{B}$, the spin axis of the ball (when all other external torques are neglected) will tend to aline itself with $\vec{B}$. However, with additional external torques of a constant nature, in general there will be an equilibrium angle between the spin axis of the ball and the magnetic field resulting in continuous energy loss due to accompanying eddy currents.

Let us decompose the magnetic field along the z-axis into a component parallel to the spin axis of the sphere and another component perpendicular to the spin axis which lies in the plane of B and $\hat{\zeta}$. Thus $\vec{B}=B_\zeta\hat{\zeta}+B_\eta\hat{\eta}=B\cos\theta\hat{\zeta}+B\sin\theta\hat{\eta}$ and the magnetic moment induced in the ball may be found from the relation [Wait, 1953]

$$\vec{m}=\vec{B}/\mu_0\,(p+jq)V \tag{1}$$

$$p+jq=\frac{3}{2}\left[\frac{2\mu\,(\sinh x-x\cosh x)+\mu_0\,(\sinh x-x\cosh x+x^2\sinh x)}{\mu\,(\sinh x-x\cosh x)-\mu_0\,(\sinh x-x\cosh x+x^2\sinh x)}\right]$$

where $p+jq$ is the complex susceptibility of the sphere, μ the permeability, and $x=a(j\sigma\mu\omega)^{\frac{1}{2}}$ is the inverse ratio of the complex skin depth to the sphere radius.

Using (1) and decomposing the perpendicular component $B\sin\theta$ onto two perpendicular axes rotating with the ball, we find that if the external $\vec{B}$ field is written as $\vec{B}=B\cos\theta\hat{\zeta}+B\sin\theta\hat{\eta}+0\hat{\xi}$, that the magnetic moments induced in the spinning ball in these same directions are [Landau and Lifshitz, 1960]:

$$m_\eta=p^*VB/\mu_0\sin\theta$$

$$m_\xi=-q^*VB/\mu_0\sin\theta$$

$$m_\zeta=p_0VB/\mu_0\cos\theta \qquad p_0=p_{DC} \tag{2}$$

$$q_0=0$$

$$p^*=p_{AC}$$

$$q^*=q_{AC}$$

and the torques $(\vec{m}\times\vec{B})$ in the same direction are

$$T_\eta=-m_\xi B_\zeta=+qVB^2/\mu_0\sin\theta\cos\theta=k''\sin\theta\cos\theta$$

$$T_\xi=m_\eta B_\zeta-m_\zeta B_\eta=-qVB^2/\mu_0\sin\theta\cos\theta\left(\frac{p_0-p^*}{q}\right)$$

$$T_\zeta=m_\xi B_\eta=-qVB^2/\mu_0\sin^2\theta=-k''\sin^2\theta$$

$$k''=\frac{qVB^2}{\mu_0}. \tag{3}$$

Using these torques, we may set up Lagrange's equations for the motion of the spinning sphere in Euler's angles. Referring to the figure 1, $T_\xi=-k''\dfrac{\delta p}{q}\sin\theta\cos\theta$. $T_\phi=T_\eta\sin\theta+T_\zeta\cos\theta=0$ and $T_\zeta=-k''\sin^2\theta$, so that Lagrange's equations in terms of Euler's angles become

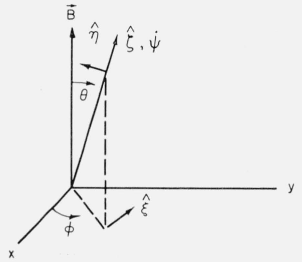

FIGURE 1. *Euler coordinates for sphere with spin axis along ζ.*

[Slater and Frank, 1947]

$$I\ddot{\theta}=-I\dot{\phi}\dot{\psi}\sin\theta+T_\xi \qquad \frac{dp_\phi}{dt}=\frac{d}{dt}I(\dot{\phi}+\dot{\psi}\cos\theta)=T_\phi$$

$$\frac{dp_\psi}{dt}=\frac{d}{dt}I(\dot{\psi}+\dot{\phi}\cos\theta)=T_\zeta=-k''\sin^2\theta. \tag{4}$$

These equations are not easily integrated without additional assumptions. The first of (4) shows that $\dot{\phi}$, the angular velocity of precession of the spin axis ζ, is not generally zero even when T_ξ vanishes. Let us, however, assume that $\dot{\phi}$ is small in comparison with $\dot{\psi}$, and in addition is nearly constant in time. The last two of (4) then yield with good approximation:

$$\ddot{\psi}\cong-\frac{k''}{I}\sin^2\theta$$

$$\ddot{\psi}\cos\theta\cong\dot{\theta}\sin\theta \qquad \therefore\int_\theta^{\theta_0}d\theta\,\frac{\sin\theta}{\cos\theta}=\int_{\dot{\psi}}^{\dot{\psi}_0}\frac{d\dot{\psi}}{\dot{\psi}}$$

$$\dot{\psi}=\dot{\psi}_0\frac{\cos\theta_0}{\cos\theta} \qquad \ddot{\psi}=\frac{\dot{\psi}_0\cos\theta_0}{\cos^2\theta}\dot{\theta}\sin\theta. \tag{5}$$

Eliminating $\ddot{\psi}$ from the equations in (5), one finds approximately how θ varies in time from its initial value θ_0:

$$\ddot{\psi}=\dot{\psi}\frac{\sin\theta}{\cos\theta}\dot{\theta}=-\frac{k''}{I}\sin^2\theta$$

$$\int\frac{d\theta}{\sin\theta\cos\theta}=-\int\frac{k''}{I\dot{\psi}}dt \qquad \dot{\psi}=\text{constant}$$

$$\tan\frac{\theta}{2}\cong\tan\frac{\theta_0}{2}\,e^{-\frac{k''}{I\dot{\psi}}t}. \tag{6}$$

For small θ one easily finds from (6) the exponential dependence of θ on time when no other external torques act except those generated by eddy currents in the spinning sphere. Using this result we can solve the first of (4) for $\dot{\phi}$ in the limit of small θ:

$$\theta\cong\theta_0 e^{-\frac{k''}{I\dot{\psi}}t} \qquad \ddot{\theta}\cong\frac{1}{\tau^2}\theta \qquad \text{(for small }\theta)$$

$$=\theta_0 e^{-t/\tau}$$

$$\dot{\phi}=\frac{T_\xi-I\ddot{\theta}}{I\dot{\psi}\sin\theta} \qquad \cong-\frac{1}{\tau}\left(\frac{\delta p}{q}+\frac{1}{\tau\dot{\psi}}\right) \qquad \text{(for small }\theta)$$

$$\text{for }|\dot{\psi}|>>\frac{1}{\tau}\left(\frac{\delta p}{q}+\frac{1}{\tau\dot{\psi}}\right). \tag{7}$$

Here we have used the following notations:

$$\frac{qB^2V}{\mu_0}=k'' \qquad \tau=\frac{1}{\dfrac{qB^2V}{\mu_0 I\dot{\psi}}}$$

$$\left(\frac{k''}{I\dot{\psi}}\right)_{\text{low }\dot{\psi}}=-\frac{1}{4}\frac{\sigma}{\rho}B^2(1+2\beta_0)^2$$

$$\text{for }\dot{\psi}<\frac{1}{\sigma\mu a^2} \qquad \beta_0=\frac{1-\dfrac{\mu_0}{\mu}}{1+2\dfrac{\mu_0}{\mu}}.$$

Finally, substituting (7) in the last of (4) or in the approximation giving the first of (6), the decay ratio turns out to be:

$$\frac{\ddot{\psi}}{\dot{\psi}}=\frac{\dot{\omega}}{\omega}=-\frac{k''}{I}\sin^2\theta\cong+(\dot{f}/f)_0\sin^2\theta_0 e^{-2t/\tau}$$

$$(\dot{f}/f)_0=-\frac{1}{4}\frac{\sigma}{\rho}B^2(1+2\beta_0)^2F(\omega)\qquad F(\omega)=1 \text{ at low } \omega$$

$$\rightarrow 0 \text{ at high } \omega. \tag{8}$$

Now let us consider what happens to a conducting rotor spinning in a magnetic field B which is along a vertical axis z. Since the axis of the rotor tends to remain fixed in space, one should look at the Earth's rotation as causing a relative rotation of the magnetic field away from the spin axis. With a constant rotation of the magnetic field, the rotor never can quite catch up with the magnetic field axis because the eddy currents generating the alining moment drop to zero from (8) as θ goes to zero. Thus, there will always be a constant lag angle of the spin axis behind the magnetic axis, and this will cause constant eddy current dissipation in the spinning ball.

Here we shall assume the conductivity of the ball is high enough so that along with a sufficiently large magnetic field the lag angle is very small and the spin axis is very nearly parallel to the z-axis. For the purpose of solving the problem in terms of Lagrange's equations (4) for the spinning ball, one may consider the B-axis as fixed in the rest frame of the observer standing on Earth's surface, in which case certain apparent or fictitious torques arise on the rotor because of the Earth's rotation. Since these are of order $-\vec{\Omega}_e\times I\dot{\psi}\hat{\zeta}$ and the rotor spin axis is nearly vertical, to good approximation this torque is always in the West-East direction and of constant magnitude $\Omega I\dot{\psi}\sin\theta_e$ where θ_e is $\pi/2$ minus the latitude angle, i.e., it is the polar angle. We note the apparent Coriolis type torque $-\vec{\Omega}\times\vec{J}$ which the rotor encounters in various positions ϕ about the vertical axis in figure 2. Qualitatively, T_ξ causes the rotor spin axis to precess about z. However, the apparent torque due to the Earth's rotation $-\vec{\Omega}\times\vec{J}$ is always in one direction, so that any torques causing the rotor to precess must eventually fight and overcome T_e in order for actual precession to occur. But if $-\vec{\Omega}\times\vec{J}=T_\xi$, there will be a stable equilibrium condition in which the torques exactly balance out one another.

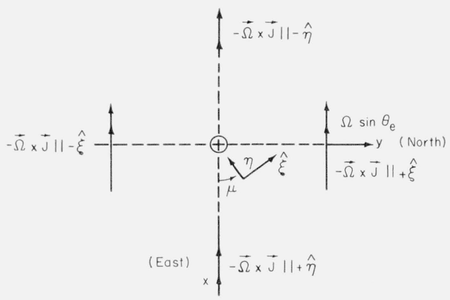

FIGURE 2. *Coriolis torques on vertically constrained sphere.*

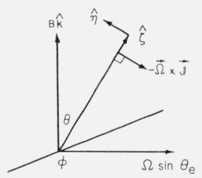

FIGURE 3. *Decomposition of* $-\Omega\times J$ *along coordinate axes.*

Neglecting small oscillations about this equilibrium position (θ_0, ϕ_0), $-\vec{\Omega}\times\vec{J}$ must be added into (3), and now (see fig. 3) we find that:

$$T_\xi=-\frac{\delta p}{q}k''\sin\theta\cos\theta+(\sin\phi\cos\theta\sin\theta_e-\sin\theta\cos\theta_e)I\dot{\psi}\Omega$$

$$T_\eta = k'' \sin\theta\cos\theta + I\dot{\psi}\Omega\sin\theta_e\cos\phi$$

$$T_\xi = -k''\sin^2\theta$$

$$T_\phi = +I\dot{\psi}\Omega\sin\theta_e\sin\theta\cos\phi. \qquad (9)$$

Using these torques in Lagrange's equations (4) with the assumptions θ_0=constant, ϕ_0=constant, and θ close to zero,

$$I\ddot{\theta}=0 \qquad \dot{\phi}=0 \qquad T_\xi=0$$

$$\frac{d}{dt}I\dot{\psi}\cos\theta = T_\phi = -I\dot{\psi}\Omega\sin\theta_e\sin\theta\cos\phi$$

$$\frac{d}{dt}I\dot{\psi} = -k''\sin^2\theta \qquad (10)$$

and therefore,

$$I\dot{\psi}\Omega\sin\theta_e\sin\phi \simeq +\frac{\delta p}{q}k''\sin\theta$$

$$I\dot{\psi}\Omega\sin\theta_e\cos\phi = -k''\sin\theta\cos\theta. \qquad (11)$$

The equilibrium angle is found by solving (11) simultaneously:

$$\cos\theta\approx 1$$

$$\frac{\sin\phi_0}{\cos\phi_0}\simeq -\frac{\delta p}{q} \qquad \sin\theta\cos\theta = -\frac{I\dot{\psi}\Omega\sin\theta_e\cos\phi}{k''}$$

$$\text{small } \theta \qquad \theta_0 \simeq \frac{\Omega_e\sin\theta_e\tau}{\sqrt{1+\left(\frac{\delta p}{q}\right)^2}}. \qquad (12)$$

When θ is small and equals the equilibrium hang off angle θ_0 in (12), $\ddot{\theta}$, $\dot{\theta}$, $\dot{\phi}$, T_ξ, T_η are all nearly zero, and the sphere's spin axis stays fixed in the frame of the observer, e.g., the observer on Earth at polar angle θ_e. The slight equilibrium misalinement of spin and magnetic axes is forced by absolute rotation of the reference frame. Substituting θ_0 from (12) into the last of (10) gives finally the corresponding rate of decay of the sphere's angular frequency:

$$\frac{d\dot{\psi}/dt}{\dot{\psi}} = -\frac{k''}{I\dot{\psi}}\sin^2\theta_0 \simeq -\frac{\Omega_e^2\sin^2\theta_e}{\frac{B^2V}{\mu_0 I}\frac{q}{\dot{\psi}}\left[1+\left(\frac{\delta p}{q}\right)^2\right]} \qquad (13)$$

p and q can be found from (1). For low frequencies (the skin depth greater than the sphere radius) (13) reduces to:

$$\frac{d\dot{\psi}/dt}{\dot{\psi}} = -\frac{\Omega_e^2\sin^2\theta_e}{\frac{9}{4}\frac{\sigma}{\rho}\frac{B^2}{\left(1+2\frac{\mu_0}{\mu}\right)^2}\left(1+c'\sigma^2\mu^2\dot{\psi}^2a^4+\frac{d'}{\sigma^2\mu^2\dot{\psi}^2a^4}\right)} \qquad \text{for } \dot{\psi}<\frac{2}{\sigma\mu a^2}. \qquad (14)$$

Since the fluctuating magnetic fields induced in the sphere are very small in comparison with the support field, μ/μ_0 in (14) (and in p and q in (13)) can be taken as the reversible permeability (with the fluctuating field perpendicular to the biasing field). However δp also involves the normal permeability B/H (see (2) and (3)) associated with the main $DC\hat{\xi}$ component of B and

H induced in the sphere. Both permeabilities therefore enter into the constants c' and d' in (14) and $(\delta p/q)^2$ in (13).

Here the principal moments of inertia have been taken equal and the body is assumed to be shaped like a perfect sphere. (However, see appendix A.) In addition the sphere material is assumed to be isotropic, homogeneous with uniform permeability μ/μ_0, μ_r/μ_0, and high conductivity σ, with $\Omega\tau << 1$.

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3. Appendix A

Among the more important factors influencing the frequency decay in (13) and (14) which have not been discussed here are the following [Bozorth 1951]:

1. Domain wall movement, internal strain and associated energy losses.
2. Rotational hysteresis and other magnetic lags due to crystal anisotropy, impurities, etc.
3. Additional T_ξ magnetic and mechanical torques due to nonspherical shape of the rotor and unequal moments of inertia.

The first and second factors can add additional constant T_ξ and T_ζ torques in (9) and also influence the reversible and normal permeabilities.

The third factor introduces additional T_ξ torques of the form [Slater and Frank, 1947]:

$$\begin{aligned} T'_\xi &= \left(a_1 \frac{\delta I}{I} I\omega^2 + a_2 \frac{B^2V}{\mu_0} \frac{\delta c}{c} + a_3 \frac{B^2V}{\mu_0} \sigma^2\mu^2a^4\omega^2\right) \times \sin\theta\cos\theta \\ &= \pm X' \sin\theta\cos\theta. \end{aligned} \tag{A.1}$$

When added into the equilibrium equations (10) and (12) we find that the hang off angle is modified:

$$\frac{\sin\phi}{\cos\phi} = \left(-\frac{\delta p}{q} \pm \frac{X'}{k''}\right) \qquad \theta_0 \simeq \frac{\Omega_e \sin\theta_e \tau}{\sqrt{1+\left(\frac{\delta p}{q} \pm \frac{X'}{k''}\right)^2}}. \tag{A.2}$$

Thus the frequency decay (13) may be either increased or decreased depending on the magnitude and sign of X'/k''. If either $\delta p/q$ or X'/k'' predominates the magnitude of the effect (13) may be reduced considerably; if they cancel each other, the effect may be noticeably increased.

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