Reflection of VLF Radio Waves From an Inhomogeneous Ionosphere. Part I. Exponentially Varying Isotropic Model [1]

James R. Wait and Lillie C. Walters

Contribution from Central Radio Propagation Laboratory, National Bureau of Standards, Boulder, Colo.

The oblique reflection of VLF radio waves from a continuously stratified ionized medium is considered. The profile of the effective conductivity is taken to be of an exponential form. This is a fair representation for the actual D layer of the ionosphere. It is shown that the gradient of the conductivity change has a marked effect on the reflection characteristics.

1. Introduction

In the study of VLF radio wave propagation it is often assumed that the ionosphere can be regarded as a sharply bounded medium on its underside. This step may be considered reasonable in view of the long wavelength and the relatively rapid change of the electron density of the D region. Furthermore, experimental data are often in accord with calculations based on this model. The general agreement is particularly good at highly oblique incidence provided that appropriate corrections are made for earth curvature. Nevertheless, there are many occasions when the sharply bounded model appears to be inadequate. Therefore, it is worthwhile to consider a more realistic model of the lower ionosphere.

On examining much of the recent literature on the characteristics of the lower D layer, it appears that the effective dielectric constant of the medium can be well approximated by an exponential function. Then, to within this approximation, the relative permittivity may be written in the form

$$K(z) = K_0 \left(1 - i\frac{1}{L} \exp \beta z\right), \qquad (1)$$

where K_0 is a reference permittivity, and L and β are constants. The level $z=0$ may be defined as the reference level and thus

$$K(0) = K_0 \left(1 - i\frac{1}{L}\right), \qquad (2)$$

which is a familiar form.

Under the assumptions that the angular frequency ω is much less than both the collision frequency ν and the plasma frequency ω_0, it is known that [Wait, 1962]

$$K_0 = 1 \text{ and } L = \frac{\omega\nu}{\omega_0^2}, \qquad (3)$$

provided that the earth's magnetic field can be neglected. The latter assumption is strictly valid only if ν is somewhat greater than the gyrofrequency ω_H. In actual cases, however, the isotropic assumption is useful even when ν is of the order of ω_H, provided the magnetic field is not transverse to the direction of propagation [Crombie, 1961; Johler and Harper, 1962]. For the purposes of the present paper, the influence of the terrestrial magnetic field is not considered.

The constant β in the exponent of the equation for $K(z)$ is a measure of the sharpness of the gradient. For example, when $\beta = 1$ km^{-1}, it means that the ratio ω_0^2/ν or N/ν increases by a factor 2.71 for each km of vertical height. The best available information on expected values of β can be found in the recent work on pulse cross modulation [Barrington et al., 1962] and from recent rocket measurements [Kane, 1962]. From these results it appears that β is of the order of $\frac{1}{2}$ km^{-1} for quiet daytime conditions although it may differ by a factor of two or more at certain times. In the present study it appears to be desirable to allow β to vary from 0.3 to 3.0 to encompass most cases of interest.

2. Background Theory

The propagation of electromagnetic waves in a medium whose permittivity $\epsilon(z)$ varies in an exponential manner has been studied extensively. The usual case considered is for horizontal polarization where the electric vector is parallel to the stratifica-

tion. For example, if the electric vector has only a y component E_y, it is a simple matter to show that [Wait, 1962]

$$\left[\frac{d^2}{dz^2}+k^2(K(z)-S^2)\right]E_y=0, \qquad (4)$$

when the field varies in the x direction according to $\exp(-ikSx)$, where S is a dimensionless constant. Here $k=\sqrt{\epsilon_0\mu_0}\,\omega$ is the free-space wave number and ϵ_0 and μ_0 are the permittivity and permeability of free space, respectively.

At a sufficiently large negative value of z, the relative permittivity $K(z)$ becomes unity, and thus, E_y satisfies

$$\left(\frac{d^2}{dz^2}+k^2C^2\right)E_y=0, \qquad (5)$$

where $C^2=1-S^2$. The general solution of this equation has the form

$$E_y=E_0(e^{-ikCz}+R_he^{+ikCz})e^{-ikSx}, \qquad (6)$$

when E_0 and R_h are constants. Recognizing that the solution of (4) must reduce to (5), for z tending to $-\infty$, it may be shown that [Wait, 1962]

$$R_h=-\left(\frac{k}{\beta}\right)^{2\nu}\frac{1}{(-iL)^{\nu_0}}\frac{(-\nu_0)!}{(\nu_0)!}, \qquad (7)$$

where $\nu_0=2ikC/\beta$.

The term $E_0e^{-ikCz}e^{-ikSx}$ may be regarded as an incident wave whose direction of propagation makes angle θ with the z axis where $C=\cos\theta$ or $S=\sin\theta$. Then $E_0Re^{+ikCz}e^{-ikSx}$ can be interpreted as a reflected wave and R can be defined as the reflection coefficient of the exponential layer. It should be stressed that this reflection coefficient, while referred to the level $z=0$, is only valid in the free space region at large negative values of z.

The reflection coefficient R_h, given by (7), may be conveniently written in the form

$$R_h=\exp\left(-\frac{2\pi^2}{\lambda_0\beta}C\right)\exp i\Phi, \qquad (8)$$

where

$$\Phi=\frac{8\pi}{\lambda_0\beta}C\log\left(\frac{2\pi}{\lambda_0\beta L^{1/2}}\right)\pm\pi+2\arg\left[\left(-i\frac{4\pi C}{\lambda_0\beta}\right)!\right],$$

where $\lambda_0=2\pi/k$ is the free-space wavelength. Thus, the amplitude of the reflection coefficient is given by the very simple form

$$|R_h|=\exp\left(-\frac{2\pi^2}{\lambda_0\beta}C\right), \qquad (9)$$

which is independent of L. On the other hand, the phase factor Φ is relatively complicated since it involves the factorial function of imaginary argument. Nevertheless, (8), for the complex value of R_h can

be used to obtain quantitative results for exponential-type layers.

3. Extension to Vertical Polarization

In the VLF radio problem one is mainly interested in vertical polarization such that the magnetic vector is parallel to the stratification. For example, if the magnetic field has only a y component, H_y, and assuming again that the fields vary in the x direction according to $\exp(-ikSx)$, it follows that [Wait, 1962]

$$\left[\frac{d^2}{dz^2}-\frac{1}{K(z)}\frac{dK(z)}{dz}\frac{d}{dz}+k^2(K-S^2)\right]H_y=0. \qquad (10)$$

Intrinsically, this is a more complicated form than (4) for horizontal polarization.

In the region of large negative z the (10) reduces to the elementary form

$$\left(\frac{d^2}{dz^2}+k^2C\right)H_y=0, \qquad (11)$$

and thus

$$H_y=H_0(e^{-ikCz}+R_ve^{+ikCz})e^{-ikSx}, \qquad (12)$$

where H_0 is a constant and R_v is, by definition, the reflection coefficient for vertical polarization.

Unfortunately, for the exponentially varying permittivity of the form defined by (1), it does not appear to be possible to obtain a closed form expression for R_v. For this reason, the required quantitative results were obtained by a numerical method. Essentially, the procedure is to replace the continuous $K(z)$ profile by a finite number of steps. The situation is illustrated in figure 1 where the function

$$\frac{1}{L(z)}=\frac{1}{L}\exp\beta z, \qquad (13)$$

is shown plotted versus z along with its step approximation. Thus, between the limits $z=-z_0$ and $z=+T$, the medium is divided into M homogeneous layers of width $h_1, h_2 \ldots h_m, \ldots h_{M-1}, h_M$. The value of $L(z)$ in each of these slabs is then replaced by a
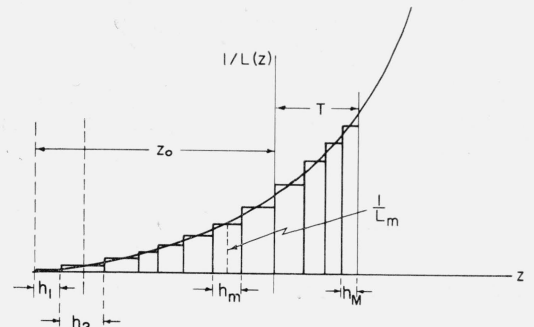

FIGURE 1. *The step approximation to an exponential profile.*

constant value L_m. The method is really equivalent to the usual method [Budden, 1961] of numerically integrating the basic differential equations.

The problem now boils down to finding the wave impedance Z_1 at $z=-z_0$ in terms of the properties of all the individual slabs. Here $Z_1=\dfrac{E_x}{H_y}$ which is to be evaluated at $z=-z_0$. The reflection coefficient R_v, which is referred to the level $z=0$, is then found from

$$R_v=\frac{C-Z_1/\eta_0}{C+Z_1/\eta_0}\exp\left[-i\frac{4\pi}{\lambda_0}Cz\right], \quad (14)$$

for $z=-z_0$, and where $\eta_0=\sqrt{\mu_0/\epsilon_0}=120\pi$. Here z_0 is chosen sufficiently large that $1/L(z)$ may be regarded as zero. (Strictly speaking, $z_0\to\infty$.)

4. Iterative Process

From the theory of wave propagation in stratified media [e.g., Wait, 1962], it is known that Z_1 can be obtained by a series of iterative processes. Thus

$$Z_1=K_1Q_1, \quad (15)$$

where

$$Q_1=\frac{K_{2,1}Q_2+\tanh D_1}{1+K_{2,1}Q_2\tanh D_1}, \quad (16)$$

$$Q_2=\frac{K_{3,2}Q_3+\tanh D_2}{1+K_{3,2}Q_3\tanh D_2}, \quad (17)$$

$$\cdots$$

$$Q_m=\frac{K_{m+1,m}Q_{m+1}+\tanh D_m}{1+K_{m+1,m}Q_{m+1}\tanh D_m}, \quad (18)$$

and so on. The various factors are defined below. The parameter D_m which is a measure of the thickness of an individual slab is defined by

$$D_m=\frac{i2\pi h_m}{\lambda_0}N_m\left(1-\frac{S^2}{N_m^2}\right)^{1/2}, \quad (19)$$

where N_m is the refractive index of the mth slab given by [2]

$$N_m=\left(1-\frac{i}{L_m}\right)^{1/2}.$$

The quantity $K_{m+1,m}$ is the ratio of the wave impedances of layer $m+1$ and layer m. Thus

$$K_{m+1,m}=\frac{K_{m+1}}{K_m}, \quad (20)$$

where

$$K_m=\frac{\eta_0}{N_m}\left(1-\frac{S^2}{N_m^2}\right)^{1/2},$$

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[2] The square roots are *defined* such that $D_m=\left(\dfrac{i2\pi h_m}{\lambda_0}\right)N_m$ if $N_m\to\infty$, while $D_m=\left(\dfrac{i2\pi h_m}{\lambda_0}\right)C$ if $N_m\to 1$. Furthermore, $N_m\cong(1/L_m)^{1/2}e^{-i\pi/4}$ if $|N_m|>>1$ and $N_m=1$ if $L_m\to\infty$.

is the wave impedance in the mth layer. Alternately,

$$K_{m+1,m}=\frac{\left(1-\dfrac{i}{L_m}\right)}{\left(1-\dfrac{i}{L_{m+1}}\right)}\left[\frac{C^2-\dfrac{i}{L_{m+1}}}{C^2-\dfrac{i}{L_m}}\right]^{1/2}. \quad (21)$$

Finally, because the quantities Q_m are given in terms of Q_{m+1} it is necessary to know the initial value Q_{M+1} at $z=T$. For example,

$$Q_M=\frac{K_{M+1,M}Q_{M+1}+\tanh D_M}{1+K_{M+1,M}[\tanh D_M]Q_{M+1}}, \quad (22)$$

and similarly, Q_{M-1} is expressed in terms of Q_M. The process is continued until Q_1 is obtained.

In some problems of this kind it is not necessary to know precisely the value of Q_{M+1}. However, the economy of the calculations is greatly improved if a good starting value of Q_{M+1} is known. Recognizing that the wave impedance at $z=T$ is given by

$$Z_{M+1}=\frac{K_{M+1}}{\eta_0}Q_{M+1}, \quad (23)$$

and utilizing the condition $|N_{M+1}|>>1$, it follows from previous work [Wait, 1962] that

$$Q_{M+1}\cong\frac{K_0\left[\sqrt{i}\dfrac{2\pi}{\lambda_0}N_{M+1}\dfrac{2}{\beta}\right]}{K_1\left[\sqrt{i}\dfrac{2\pi}{\lambda_0}N_{M+1}\dfrac{2}{\beta}\right]}, \quad (24)$$

where K_0 and K_1 are modified Bessel functions.

In a practical sequence of calculations it is necessary to choose T large enough that the final result for R_v is insensitive to further changes in T. For example, if $\beta=1$ km^{-1} and $\lambda_0=15$ km, it was found that $T=4$ km was sufficiently large to obtain four-figure accuracy in R_v. Furthermore, for some values of β and λ_0, it was found that R_v approaches a constant as z_0 was increased to 20 km. In general, for smaller values of β it was necessary to increase T and z_0 to larger values in order to achieve stability of the final results. The width of the steps, h_m, must also be chosen sufficiently small to achieve an adequate simulation of the smooth profile. Generally, it was found that $|D_m|<10^{-2}$ was a satisfactory criterion.

5. Discussion of Reflection Coefficient Calculations

Using the basic definition of the reflection coefficient R_v, given by (14), the amplitude and phase of R_v were calculated for a wide range of conditions. While, strictly speaking, R_v is defined as the limit for $z=-z_0$, where $z_0\to\infty$, it is of interest to first demonstrate how R_v approaches this limit. An example of such calculations is shown in figures 2a and 2b. Here $C=0.1$, corresponding to highly oblique incidence, and $\lambda_0=15$ km, or $f=20$ kc/s.
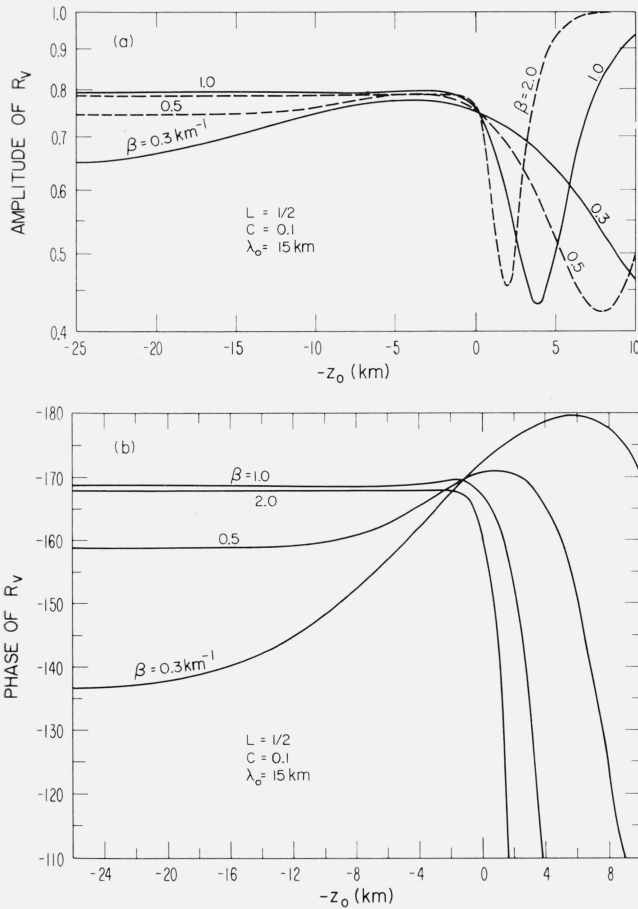
FIGURE 2. *The reflection coefficient as a function of height for an exponential profile and a fixed angle of incidence.*

FIGURE 3. *The amplitude of the reflection coefficients for vertical and horizontal polarization as a function of C which is the cosine of the angle of incidence.*

Various values of β are shown on the curves. It is immediately apparent that for sufficiently negative values of z (i.e., sufficiently far below the layer), both the amplitude and phase of R_v approach a limit.

Actually, the curves in figures 2a and 2b have more than just a mathematical interest. Physically, the R_v, corresponding to $z = -z_0$ where z_0 is finite, is the reflection coefficient for a permittivity profile of the form

$$K(z) = K_0 \left[1 - \frac{i}{L} e^{\beta z} \right] \text{ for } z > -z_0$$

$$= 0 \text{ for } z < -z_0. \quad (25)$$

Therefore, these curves describe reflection from a sharply bounded ionosphere which behaves exponentially above its lower edge. Of course, as z_0 tends to ∞, the discontinuous profile becomes a continuous exponential. It is rather interesting to observe that as the discontinuity is moved from above the reference level ($z=0$) to below, the reflection coefficient passes through a minimum. This is related to a Brewster angle phenomenon
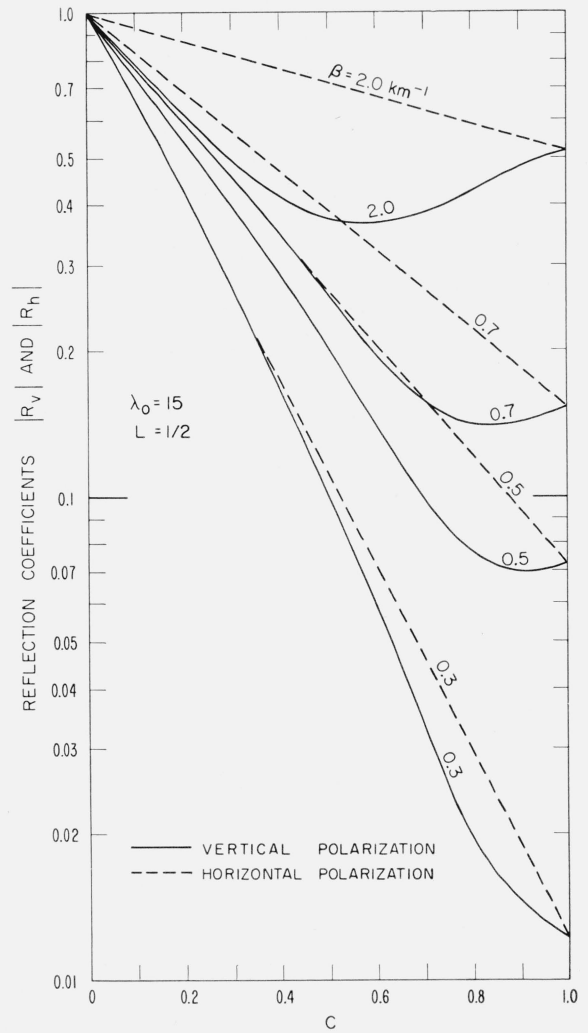
since to the right of this minimum in figure 2a the reflection is metalliclike while, to the left, the reflection is dielectriclike.

For the remainder of the present paper, attention will be confined to the limiting value of R_v far below the reference level $z=0$ (i.e., $z \cong -z_0 \cong -\infty$). On the other hand, it should be emphasized that R_v is always referred to the level $z=0$.

The magnitude of the reflection coefficients R_v and R_h are shown plotted in figure 3 for vertical and horizontal polarization, respectively. The values of $|R_v|$ were obtained from (14) using the multislab model described. The values of $|R_h|$ were calculated directly from (9) and, on this log-linear scale they are merely straight lines. As an important check, the value of $|R_h|$ was also obtained from a multislab model using the same procedure as described for vertical polarization. Within four-figure accuracy, the values of $|R_h|$ obtained by the two methods were the same.

A number of significant features are evident in figure 3. In the first place, $|R_v|$ exhibits a Brewster angle phenomenon, provided β is sufficiently large. Here the reflection process is dielectriclike at grazing angles and is more or less metalliclike at normal incidence. Of course, $|R_h|$ does not exhibit this phenomenon. However, for very small values of β, corresponding to a relatively slowly varying medium, the curves for $|R_h|$ and $|R_v|$ become rather close to one another. This is consistent with the optical behavior of waves in an inhomogeneous medium. In fact, by a direct application of the phase integral method of Eckerskey, the equation for $|R_h|$ is found also to be applicable to $|R_v|$. The use of the phase integral method for such applications is necessarily restricted to slowly varying media [Wait, 1962]. It is apparent, that for VLF radio waves, where β is of the order of 0.5, the phase integral method is inapplicable to vertical polarization.

Another interesting feature of figure 3 is the near linear dependence of all the curves for small values of C. Fortunately, it is just these values which are important in the long-distance propagation of VLF radio waves. The linearity of the phase curves for R_v are indicated in figure 4 for the same conditions. Here, it is evident that they all approach $-180°$ at grazing incidence. The phase curves for R_h exhibit a similar property but they are not shown here since they have only an academic interest at VLF.

The variation of the reflection coefficient R_v as a function of the gradient parameter β is illustrated in figures 5a and 5b at oblique incidence. It is rather remarkable that $|R_v|$ is relatively insensitive to β if it is in the range from 0.7 to 3.0. Furthermore, it appears that $|R_v|$ has a broad maximum for β approximately equal to 1.2.

The general behavior of the amplitude and phase of the reflection coefficients at highly oblique incidence suggests that, if R_v is written in the form
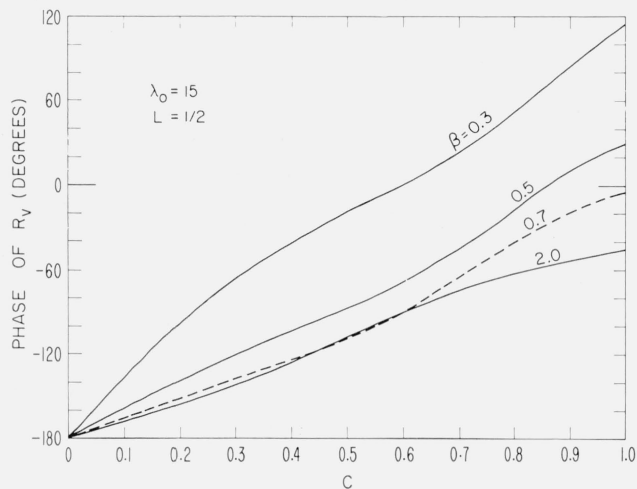
$$R_v = -\exp(\alpha C),$$

the function α should be almost independent of C. Writing

$$\alpha = \alpha_1 + i\alpha_2,$$

where α_1 and α_2 are real, it is a simple matter to compute the complex coefficient α from the numerical data of R_v. The results are shown in figures 6a and 6b where $(-\alpha_1)$ and α_2, respectively, are plotted as a function of C in the range 0.05 to 0.30. It is apparent that over this range of C, the coefficient α can be regarded essentially as a complex constant.

For all the results given in the foregoing figures, it has been assumed that $L=1/2$ and $\lambda_0=15$. Actually, the results for the magnitude of R_v and R_h do not depend on L. In fact, $|R_v|$ and $|R_h|$ are identical for $\beta\lambda_0=$constant. Furthermore, the phase of the reflection coefficients is also simply related although the situation is slightly complicated by the choice of the reference level where L is required to have a special value. In fact, it is convenient to choose the reference level (i.e., $z=0$) so that

$$L = \frac{\omega\nu}{\omega_0^2} = \frac{1}{2} \times \frac{15}{\lambda_0}.$$

In this way, the results can be readily compared as a function of frequency or wavelength. At the reference level $z=0$, the effective conductivity parameter ω_r or ω_0^2/ν has the value 2.51×10^5.
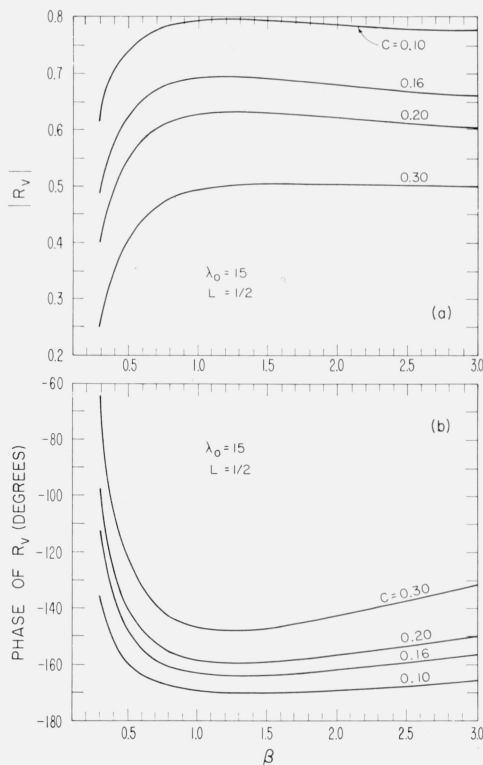

FIGURE 4. *The phase of the reflection coefficient for vertical polarization as a function of* C.

FIGURE 5. *The reflection coefficient as a function of the gradient parameter* β *for an exponential profile.*
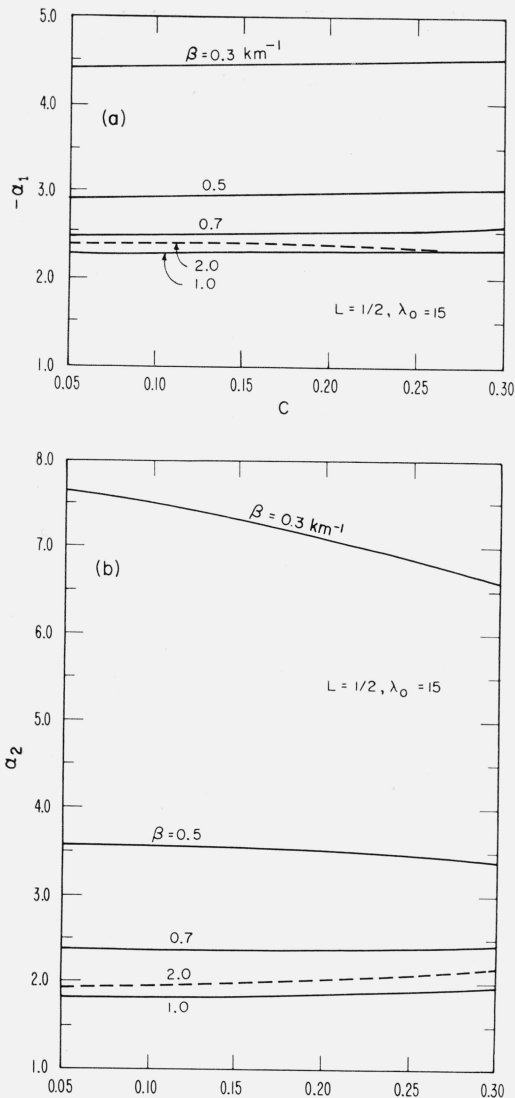
FIGURE 6. *The real and imaginary parts of the function α defined by $R_v = -exp(\alpha C)$ for $\lambda_0 = 15$ and various values of β.*

For wavelengths other than $\lambda_0 = 15$, it is also found that R_v may be approximated by the function— $\exp(\alpha C)$ where α is approximately a complex constant. To illustrate the wavelength dependence, $-\alpha_1$ and α_2 are plotted as a function of λ_0 in figures 7a and 7b, respectively. For these curves, C is chosen to be 0.16; the corresponding curves for other values of C in the range 0.10 to 0.20 are almost indistinguishable. The ordinate in figure 7a is simply related to the magnitude of the reflection coefficient and, thus, small values of $-\alpha_1$ are associated with high reflection coefficients. It is apparent that for small values of β corresponding to a diffuse layer, the reflection coefficient becomes very small for the shorter wavelength. On the other hand, for a

rapidly varying layer, corresponding to large values of β, the reflection coefficient decreases with increasing wavelength. For intermediate values of β, the reflection coefficient has a minimum in the wavelength range between 10 and 30 km. For example, when $\beta = 1.0$, the optimum wavelength is about 17 km or approximately 18 kc/s.

The curves for α_2 in figure 7b also have a particular significance. Noting that

$$\arg R_v = -\pi + \alpha_2 C,$$

it is apparent that $\alpha_2 C$ is the phase shift resulting from the imperfect reflecting properties of the exponential layer. To attach a physical meaning to this term it is often desirable to imagine the reflection taking place at a height Δz_1 *below* the level $z = 0$. In this case, Δz_1 is chosen so that the arg of R_v is always $-\pi$. Clearly,

$$\alpha_2 = 2k\Delta z_1 = 4\pi\Delta z_1/\lambda_0,$$

or

$$\Delta z_1 = \alpha_2\lambda_0/(4\pi).$$

For example, at $\lambda_0 = 15$ km (i.e., 20 kc/s) and $\beta = 1$, the effective height of reflection is depressed by approximately 3 km. For smaller values of β, it is seen from figure 7b that Δz_1 may be much greater.

6. Final Remarks

The results, for the coefficients α_1 and α_2, given here may be introduced into the waveguide mode theory. In this way, attenuation rates and phase velocities of the modes may be obtained. This is a valid subject for another paper.

The influence of nonexponential profiles also appears to be a subject worthy of some attention. Using the iterative method described in this paper, it appears that any reasonably smooth profile may be treated in a straightforward manner. Also, with some modification, the technique may be used to evaluate the effect of discontinuities in gradients and local regions of excess ionization.

General conclusions about the influence of the profile of the lower ionosphere on VLF propagation must await completion of extensive and systematic calculations. It is hoped to report such results in the near future. Detailed comparisons of theory and experiment are also deferred until these more complete computations become available.
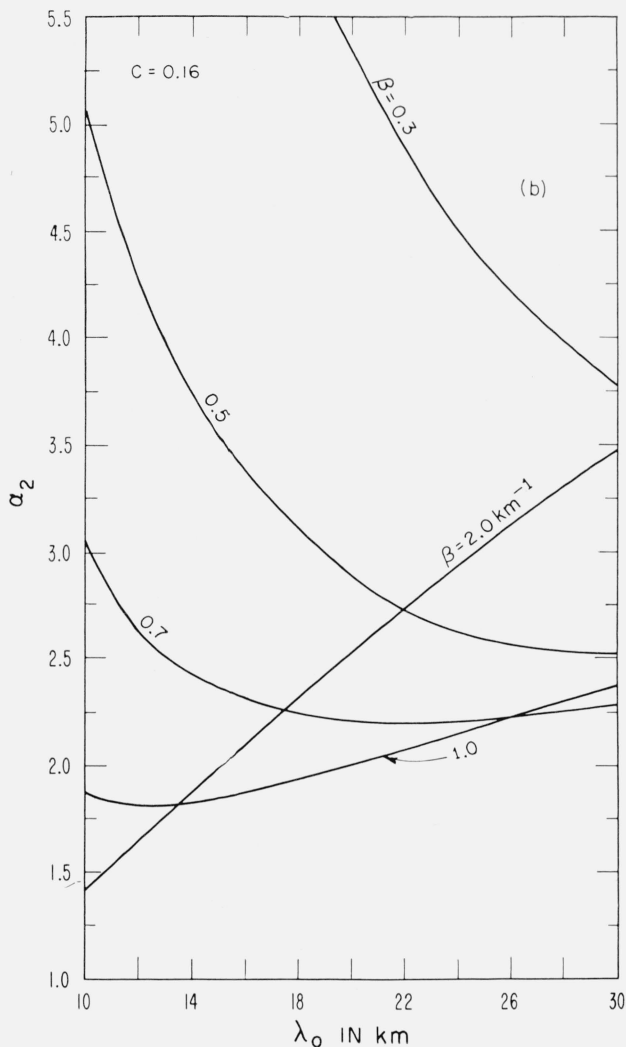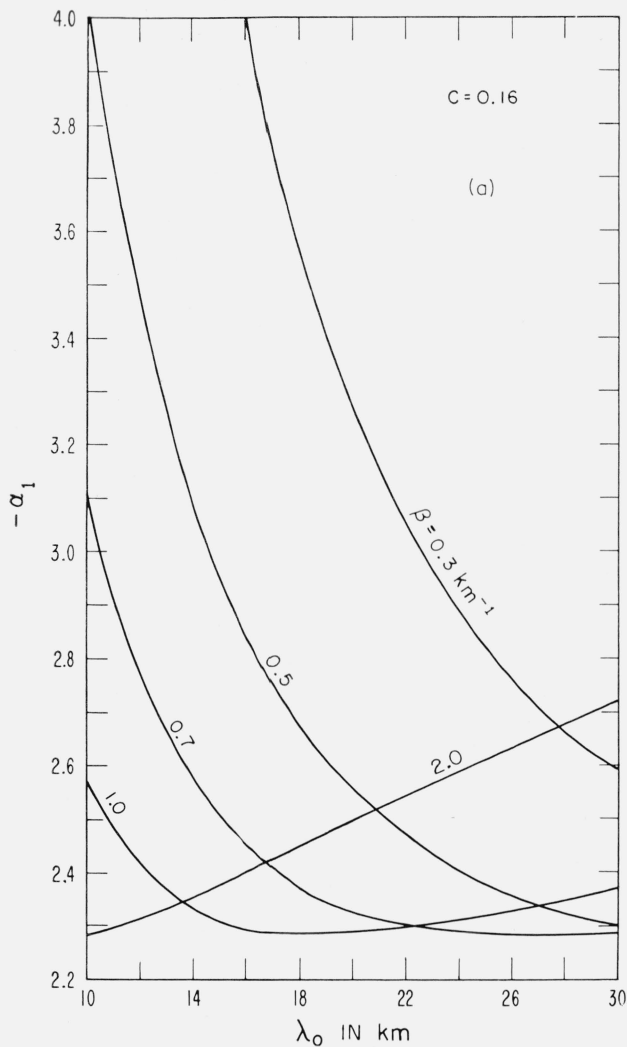
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FIGURE 7. *The real and imaginary parts of the function a plotted as a function of the wavelength λ_0 for C=0.16 and various values of β.*

7. References

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