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Analysis and Synthesis of Nonuniform Transmission Lines or Stratified Layers[1]

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Nonuniform lossless or lossy transmission lines or layers used as broadband matching or absorbing devices are studied.

When the refraction index, $n(x)$, and the characteristic impedance $Z_0(x)$, are given, the reflection spectrum, $\rho_0(\eta)=\rho_0(4\pi/\lambda)$, for $x=0$ can always be computed by solving numerically a Riccati differential equation (RDE). (Analysis)

Conversely, not only for $n=\text{const}$ [Bolinder, 1950, 1956] but also for $n(x)$ real and $\mu=\mu_0$, a tapered transformer can be synthesized starting from a given $\rho_0(\eta)$ spectrum by using Fourier transform techniques. (Synthesis)

For broadband absorbers, the synthesization procedure can be approximately applied, under certain conditions, to only the part of the spectrum which represents the reflection of the matched (lossy) line.

1. Introduction

The first part of this paper deals with the problem of matching two uniform transmission lines (or layers) by means of a nonuniform line (or layer) in such a way as to "minimize the reflections."

A variational solution of the problem does not seem to exist, and, therefore, instead of trying to minimize the average value of the reflection over a given frequency band, we will solve the problem of keeping the reflection lower than a given value over the largest possible frequency band, for a given length of the matching line.

To find an acceptable solution for this problem, two methods can be followed. The first one (Analysis) consists in varying tentatively the parameters of the line in such a way as to obtain a satisfactory reflection spectrum, i.e., reflection coefficient versus wavelength. According to the second method (Synthesis), we start from a favorable reflection spectrum and find the corresponding law of variation of the parameters of the line.

The second method is made possible by the fact that, in a number of cases, the line parameters can be obtained operating on the inverse Fourier Transform of the reflection spectrum.

In the last paragraphs of this paper (par. 6 and 7) some tentative techniques are described to utilize even in the case of absorption the synthesis method, which is rigorously appliable only in the case of matching.

Let us now recall the basic equations to which recourse will be made.

For monochromatic waves, the two wave equations describing the propagation of em waves along an inhomogenous line or a stratified medium can be condensed into the single Riccati differential equation (RDE)

$$Z'+\gamma Z_0-\gamma\frac{Z^2}{Z_0}=0 \tag{1.1}$$

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where

Z=input impedance at abscissa x,

Z'=its derivative with respect to x,

Z_0=intrinsic impedance of the line or layer at abscissa x,

$\gamma = j\frac{2\pi}{\lambda} n(x)$=propagation constant, and

$n(x) = n_1(x) - jn_2(x)$=relative refraction index.

Equation (1.1) can be written in terms of the reflection coefficient $\rho(x)$, which can be defined in the "Schelkunoff form"

$$\rho(x) = \frac{Z(x) - Z_0(x)}{Z(x) + Z_0(x)}, \tag{1.2}$$

obtaining the following RDE:

$$\rho' - 2\gamma\rho + \frac{Z'_0}{2Z_0}(1 - \rho^2) = 0. \tag{1.3}$$

Let us note here that (1.1) and (1.3) are valid for layered media not only under normal incidence, but also under all conditions of incidence and polarization, provided the normal equivalent wave is considered.

If we multiply both $\mu_r(x)$ and $\epsilon_r(x)$ by a constant C, the first and last terms in (1.3) remain unchanged, while the second becomes $-j\frac{4\pi}{\lambda} Cn(x)\rho(x)$. It follows that we find the same reflection coefficient $\rho(x)$, for the wavelength λ, that we found previously for the wavelength λ/C; that is, on the average, the reflection coefficient is lowered. This property cannot be, in general, easily utilized in the design of inhomogeneous layers, since high values of μ_r are usually accompanied by still higher values of ϵ_r. The above remark shows, however, that an increase, even if gradual, of μ_r (which is generally a complex quantity), increases the effective length of the line, and lowers, consequently, $\rho(x)$.

When μ=const, and only ϵ varies with x, a single function is sufficient to describe the inhomogeneous line: either $\gamma(x)$, or $Z_0(x)$, or $n(x)$. Equation (1.3) can then be written in the form

$$\rho' - j\eta n\rho - \frac{n'}{2n}(1 - \rho^2) = 0, \tag{1.4}$$

where

$$\eta = 4\pi/\lambda.$$

2. Analysis of an Em Line or Layer

The problem of analyzing the behavior of an em line or layer can be summarized as follows: For given $\gamma(x)$ or $n(x)$ and $Z_0(x)$ functions (i.e., for given $\epsilon_r(x) = \epsilon_{r1}(x) - j\epsilon_{r2}(x)$ and $\mu_r(x) = \mu_{r1}(x) - j\mu_{r2}(x)$ functions) and length L of the layer, find the reflection coefficient at abscissa $x=0$, $\rho(0, \eta) = \rho_0(\eta)$, as a solution of (1.3) or (1.1) plus (1.2), with a given limit condition $\rho(\eta, L)$. Obviously, when μ=const it is sufficient to start from only one of the previous functions, e.g., from the refraction index $n(x)$.

In the case of matching, $\rho(\eta, L) = 0$; in some cases of absorption, $\rho(\eta, L) = -1$, i.e., the line is short circuited at the far end. It must be pointed out that (1.1) is more general than (1.3), since it allows an initial discontinuity. If there is such a discontinuity, only if both this one and ρ_0 are very small, the overall reflection coefficient can be approximately calculated by adding the two reflections.

Generally, it is not possible to have analytic solutions of (1.1) and/or (1.3), and it is necessary to carry out computations by means of digital or analog computers.

Equation (1.4) can be linearized as follows:

$$\rho' - j\eta n \rho - \frac{n'}{2n} = 0, \tag{2.1}$$

provided that $|\rho(x)|^2 << 1$ everywhere. This condition is generally fulfilled in a matching device, at least in the η region of low reflection.

In the case of matching and for $\mu_r = 1$, ten different real $n(x)$ functions were tested, and the corresponding $\rho_0(\eta)$ computed from (2.1), by means of a PACE–TR–10 analog computer.

Assuming $n(0) = 1$, the "transformation rates" $n(L)/n(0) = 2$, 5, and 10, were considered. It was shown [Franceschetti, 1962] that, for $n(0) \neq 1$, the results are still valid, provided that we refer to wavelengths measured in the medium of refraction index $n(0)$, i.e., to the wavelength $\lambda/n(0)$. The scattering matrix (S) of the lossless matching junction can be easily deduced starting from the knowledge of $\rho_0(\eta) = S_{11}$, computed by means of the linearized (2.1). Actually it is possible to express the coefficient S_{22} in terms of $\rho_0(\eta)$, and, by means of the matrix equation $(S)(\tilde{S})^* = (I)$, to calculate (S), obtaining the following expression [Franceschetti, 1962]:

$$(S) = \left\| \begin{matrix} \rho_0(\eta) & \sqrt{1-|\rho_0(\eta)|^2} \cdot \exp\left[-j\frac{\eta}{2}\int_0^L n(x)dx\right] \\ \sqrt{1-|\rho_0(\eta)|^2} \cdot \exp\left[-j\frac{\eta}{2}\int_0^L n(x)dx\right] & -\rho_0^*(\eta) \cdot \exp\left[-j\eta\int_0^L n(x)dx\right] \end{matrix} \right\| \tag{2.2}$$

where $\rho_0^*(\eta)$ is the complex conjugate of $\rho_0(\eta)$.

The results of the $\rho_0(\eta)$ calculations, in the ten different cases considered here, are referred to under table 1, where the ratio $\xi = \lambda/L$ up to which $|\rho_0(\eta)|$ % is less than 10 percent is given for the ten $n(x)$ functions, and the three transformation rates 2, 5, and 10. This means that, for wavelengths $\lambda < L\xi$, the matching junctions under consideration give a power reflection less than 1 percent. It is apparent from this table that, assuming $|\rho_0(\eta)| \leq$ 10 percent, a generally "optimizing" $n(x)$ function cannot be defined, since the behavior of each function depends on the transformation ratio. For example, the best matching function seems to be $n(x) = \exp\,[kx/L]$, for $n(L) = 2$ ($\xi = 3.85$); and $n(x) = \exp\,[k(x/L)^3]$, for $n(L) = 5$ ($\xi = 2.82$), and $n(L) = 10$ ($\xi = 2.18$).

TABLE 1. *Values of* $\xi = \lambda/\mathrm{L}$ *such that for* $\lambda \leq \mathrm{L}\xi, |\rho_0(\mathrm{L}/\lambda)| \leq$ *10 percent (matching case)*

Functions $n(x)$	exp $[kx/L]$	exp $[k(x/L)^2]$	exp $[k(x/L)^3]$	$1+kx/L$	$1+k(x/L)^2$	$1+k(x/L)^3$	$\frac{1}{1-k\,x/L}$	exp $[e^{kx/L-1}]$	cosh $[kx/L]$	$1+k$ $(1-\cos\pi x/L)$
$n(L)=2$	3.85	2.60	2.04	3.78	3.35	2.62	3.57	3.64	3.08	2.94
$n(L)=5$	1.36	2.0	2.82	<1.0	2.50	1.99	1.58	2.41	2.41	1.33
$n(L)=10$	1.44	1.89	2.18	<1.0	1.61	1.46	1.45	1.80	1.77	<1.0

It must be pointed out that values of n up to 8 are easily attainable by means of metal powders suspended in paraffin wax [Kelly et al., 1953].

In the case of absorption, ten different nonmagnetic and magnetic functions $n(x)$ were tested, and their behavior computed, solving (1.1) and (1.2) (i.e., the nonlinearized equation) by means of the IBM 1620 digital computer of the Faculty of Science of the University of Naples. An initial discontinuity was often assumed.

The results are summarized under tables 2 and 3, where the power reflection coefficient $|\rho_0|^2$% is given for several values of the parameter λ/L.

TABLE 2. *Power reflection coefficient* $|\rho_0(L/\lambda)|^2$ *percent, for several values of* λ/L *(absorbing case)*

$\epsilon = A\exp\,[2(m-j\varphi)x/L]$ $\mu = B\exp\,[2(n-j\psi)x/L]$

$A.m.\varphi$ $B.n.\psi$	1.0 1.0 0.7 1.0 0.0 0.0	1.5 1.0 0.7 1.0 0.0 0.0	1.5 1.5 0.6 1.0 0.6 0.6	1.5 1.0 0.7 1.0 1.0 0.6	1.0 1.0 0.7 1.0 0.5 0.5
$\lambda/L=1.0$	0.30	0.75	----------	----------	0.054
2.0	1.20	1.0	2.31	0.61	.29
3.0	2.30	2.70	2.92	.60	.45
4.0	.80	2.90	3.07	.82	.59
5.0	8.2	.50	4.12	.80	.66
6.0	23.8	3.75	4.95	.80	1.06
7.0	----------	14.8	5.60	.72	1.99
8.0	----------	30.16	6.04	.59	3.45

TABLE 3. *Power reflection coefficient* $|\rho_0(L/\lambda)|^2$ *percent, for several values of* λ/L *(absorbing case)*

$\epsilon = A\,[\cos 2\varphi \frac{x}{L} - j\,\frac{\lambda}{\bar{\lambda}} \sin 2\varphi\, x/L]\cdot \exp\,[2m\, x/L]$ $\mu = B\exp\,[2(n-j\psi)\, x/L]$

$A.\ m.\varphi$ $B.\ m.\psi$ $\bar{\lambda}$	1.5 1.0 0.7 1.0 0.0 0.0 5	1.5 1.5 0.3 1.0 0.4 0.4 3	1.5 1.5 0.3 1.0 0.4 0.4 5	1.5 1.5 0.3 1.0 0.6 0.4 3	1.5 1.5 0.3 1.0 0.6 0.4 5
$\lambda/L=1$	1.70	1.83	1.56	1.75	1.58
2	.46	2.13	2.24	1.91	2.01
3	5.90	2.99	3.26	2.92	2.12
4	2.90	3.87	3.76	3.12	3.34
5	.50	5.46	5.51	3.94	3.76
6	4.50	7.47	7.52	5.36	5.20
7	12.4	9.35	8.21	7.03	6.45
8	22.3	11.0	8.26	8.74	7.03

The values of the parameters are listed in the tables under each case.

In table 3 the conditions for physical realizability (i.e., the Kramers-Krönig equations) [Latmiral et al., 1961] have been taken into account for the electric losses (purely conductive electric losses have been assumed).

Cases 1, 2, and 6 are nonmagnetic; besides, for $x \rightarrow L$, they present values of the rate ϵ_2/ϵ_1 so high as to be difficult to obtain practically.

Cases 3, 4, and 5, and particularly the last, approach too closely to Heaviside conditions; they assume values of μ_r practically unattainable in the microwave range.

Cases 7, 8, 9, and 10 are probably those which may be realized without excessive difficulties; the electric losses are purely ohmic, and the rate $\sigma/\omega\epsilon_1$ seems to be everywhere realizable.

3. Problem of the Synthesis

The problem of synthesizing an em line (or layer) can be summarized as follows:

Starting from a given (generally complex) reflection spectrum $\rho_0(\eta)$ at the beginning of the line of length L and from given limit conditions, find the two corresponding $\gamma(x)$ (or $n(x)$) and $Z_0(x)$ functions. If $\mu_r=\text{const}$, only one real function, e.g., $n(x)$, must be found. In the case of matching, obviously, the limit conditions are the values of $n(0)$ and $n(L)$.

In the following sections it will be shown that the synthesizing problem for the matching case, at least for nonmagnetic lines, can be completely solved by means of the Fourier transform techniques.

On the contrary, in the case of absorption (except for some special cases), only the following problem can be solved: for a given $\Sigma_0(\eta)$ function, and length L of the line, find a complex function $n(x)$ such that its reflection coefficient $\rho_0(\eta)$ is more favorable, on the average, than the given $\Sigma_0(\eta)$.

4. Synthesis in the Matching Case

As was pointed out in section 2, in the case of matching the linearized RDE,

$$\rho' - 2\gamma\rho + \frac{Z_0'}{2Z_0} = 0 \tag{4.1}$$

may be considered instead of (1.3).

Solving (4.1) for $\rho(\eta, L)=0$, we have

$$\rho_0(\eta)=\int_0^L \frac{Z_0'}{2Z_0}\exp\left[-j\eta\int_0^x n(x)dx\right]dx. \tag{4.2}$$

When $n=$const (this is the case of TEM guided waves), and, e.g., $n=1$, (4.2) becomes

$$\rho_0(\eta)=\int_0^L \frac{Z_0'}{2Z_0}\exp\left[-j\eta x\right]dx. \tag{4.3}$$

Equation (4.3) shows that $\rho_0(\eta)$ is the Fourier transform [Bolinder, 1950, 1956] of the function (of x) $\frac{Z_0'}{2Z_0}$, which is zero out of the interval $(0, L)$ in which the inhomogeneity is confined, and which in turn can be represented as an inverse Fourier transform as follows:

$$\frac{Z_0'}{2Z_0}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\rho_0(\eta)\exp[j\eta x]\,d\eta. \tag{4.4}$$

The fact that $\rho_0(\eta)$ and $\frac{Z_0'}{2Z_0}$ are a transform pair provides the argument for making the synthesis of a broadband matching line with $n=$const.

For example, if we choose for $\rho_0(\eta)$ the well known "sampling function," we must have $\frac{Z_0'}{2Z_0}$ constant in the interval $(0, L)$ (and zero out of it). Analogously, if $\rho_0(\eta)$ is the square of the sampling function (which is a very favorable "spectrum" of reflection), $\frac{Z_0'}{2Z_0}$ will become a "triangular pulse" in the interval $(0, L)$ [Bolinder, 1950, 1956].

When $\mu=$const, we can get a transform pair even when n is not a constant, but a real function of x.

The constancy of μ allows us to start with (2.1) instead of (4.1), so that (4.2) becomes

$$\rho_0(\eta)=-\int_0^L \frac{n'}{2n}\exp\left[-j\eta\int_0^x n(x)dx\right]dx. \tag{4.5}$$

As $n(x)$ is always positive, the integral function

$$y=y(x)=\int_0^x n(x)dx \tag{4.6}$$

is always increasing and admits, consequently, the inverse function $x=x(y)$.

Let us introduce in (4.5) the new variable of integration y. We obtain

$$\rho_0(\eta)=-\int_0^{y_L}\frac{n'[x(y)]}{2n^2[x(y)]}\exp\left[-j\eta y\right]dy, \tag{4.7}$$

where $y_L=y(L)$, and the derivative is with respect to x.

As (4.7) shows, $\rho_0(\eta)$ and

$$\varphi(y)=-\frac{n'[x(y)]}{2n^2[x(y)]} \tag{4.8}$$

is a Fourier transform pair.

As a matter of fact, even when y_L is finite, the function $\varphi(y)$ is zero out of the $(0, y_L)$ interval and therefore the integration can always be understood as extending along the whole positive y axis.

If the $\rho_0(\eta)$ function is given, the $\varphi(y)$ function result determined by means of the inverse transform and the $n(x)$, which is the datum of practical interest, can be obtained by solving the integral DE

$$-\frac{n'(x)}{2n^2(x)}=\varphi\left(\int_0^x n(x)\,dx\right), \tag{4.9}$$

which becomes a DE of the second order in the unknown $y=\int_0^x n(x)dx$ and, precisely,

$$-\frac{y''}{2y'^2}=\varphi(y). \tag{4.10}$$

Let us start from a family of spectra $\rho_0(\eta, A, y_L)$, whose inverse transforms are zero out of a finite interval; the parameter y_L represents the extent of said interval.

Equation (4.10) becomes

$$-\frac{y''}{2y'^2}=\varphi(y, A, y_L), \tag{4.11}$$

where the derivatives are with respect to x.

This is a DE of the second order containing two parameters (A, y_L).

Integrating the above DE with the two initial conditions $y(0)=0$ and $y'(0)=n(0)$, we get a solution which contains the parameters A, y_L:

$$y=y(x, A, y_L). \tag{4.12}$$

The parameters must be chosen in such a way that

$$y_L=y(L, A, y_L), \tag{4.13}$$

$$n(L)=y'(L, A, y_L). \tag{4.14}$$

Obviously, for $\eta\rightarrow 0$, i.e., for very long wavelengths, the reflection coefficient $\rho_0(\eta)$ must approach the value $[n(0)-n(L)]\cdot[n(0)+n(L)]^{-1}$. However, it must be pointed out that we do not deal here with the complete RDE, but with the linearized one, and that the latter does not require the above condition, which will be satisfied only for low "transformation rates."

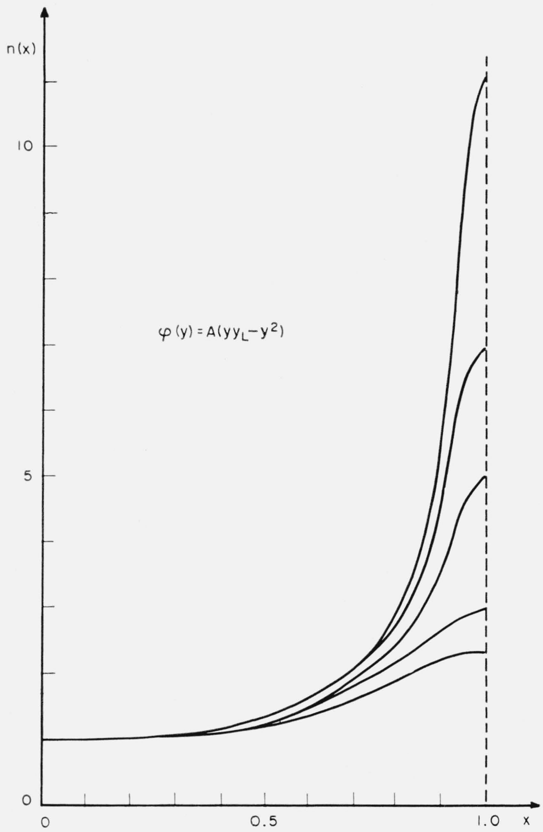

FIGURE 1. n(x) *functions referring to table 5, 1st row.*

The various curves are related to different values of the transformation ratio $n(L)/n(0)=n(L)/1=n(L)$.

In some special cases, the problem can be solved analytically [Latmiral et al., 1962]. For example, when $\rho_0(\eta)$ is of the type of the "sampling function," the corresponding $n(x)$ function becomes of the type proposed by Jacobs [1958] on the basis of empirical arguments.

Analogously, when $\rho_0(\eta)$ is an exponential integral (which is a very convenient type of reflection spectrum), the corresponding $n(x)$ function becomes the well-known exponential one [Latmiral et al., 1962].

In general, however, the problem admits of only numerical solutions, and it is necessary to solve (4.11) under conditions (4.13)–(4.14) by means of analog or digital computers.

Five different types of $\varphi(y)$ functions (to which favorable spectra of reflection are related) were chosen, and the corresponding $n(x)$ functions computed, for several rates of transformation, by means of the above procedure.

The $n(x)$ functions are traced in figures 1 through 5, under each of which the corresponding $\varphi(y)$ function is referred. The first four cases were computed by means of an IBM 1620 digital computer, and the last one by means of a PACE–TR–10 analog computer.

Besides, in table 4, the ratio $\xi=\lambda/L$, up to which the reflection coefficient $|\rho_0(\eta)|$ is less than 10 percent, is given for the computed $n(x)$ functions and three transformation rates (2, 5, and 10). The complete $\rho_0(\eta)$ function can be easily computed as the Fourier transform of the corresponding $\varphi(y)$ functions, and are referred, together with the numerical values of the parameters, in table 5.

In all computations we have chosen $n(0)=1$, but this assumption does not cause any loss of generality (see sec. 2).

As in the case of analysis (see sec. 2), it is apparent that a generally "optimizing" function $n(x)$ cannot be defined. It is interesting to point out the very good results obtained for high "transformation rates." For example, for $n(L)/n(0)=10$, the $n(x)$ function synthesized starting from the function $\varphi(y)=A\ \sin[\pi y/y_L]$, presents a value of $\xi=3.80$, i.e., wavelengths up to about four times the length L of the matching junction are almost completely transmitted.

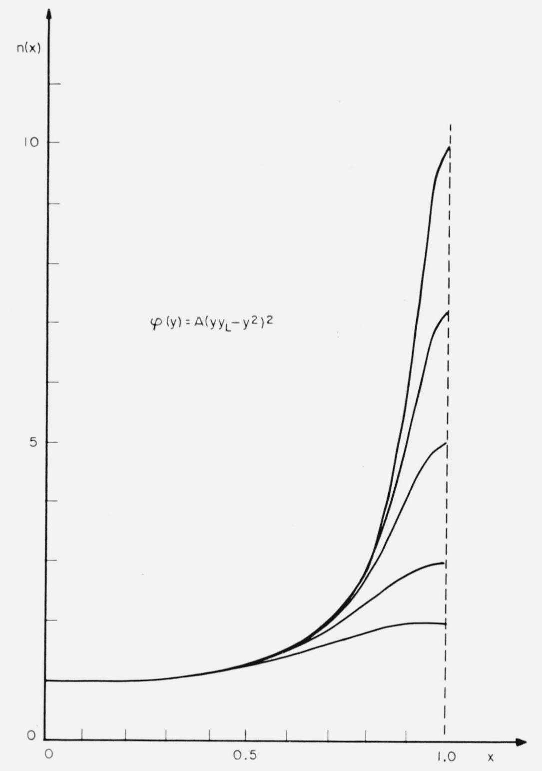

FIGURE 2. n(x) *functions referring to table 5, 2d row.*

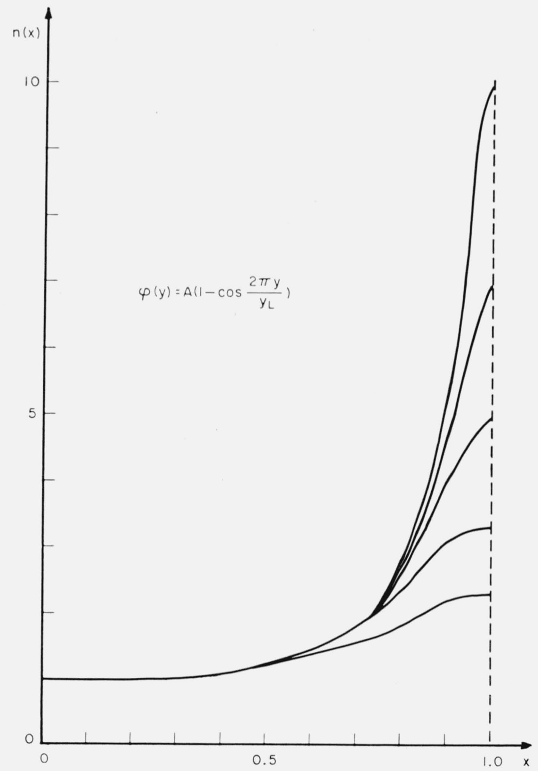

FIGURE 3. n(x) *functions referring to table 5, 3d row.*

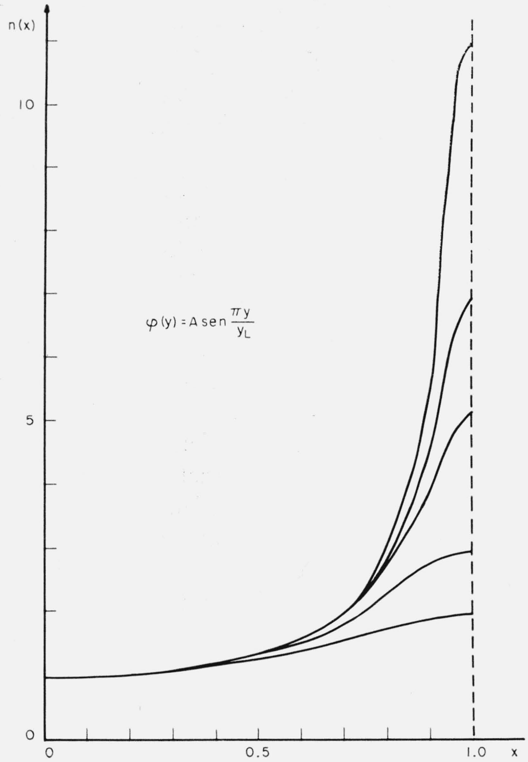

FIGURE 4. n(x) *functions referring to table 5, 4th row.*

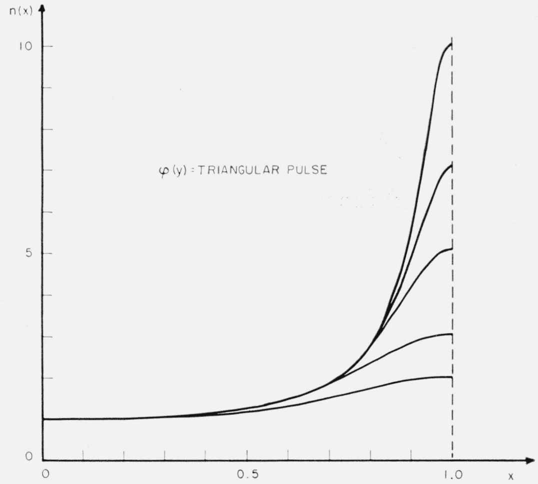

FIGURE 5. n(x) *functions referring to table 5, 5th row.*

TABLE 4. *Values of $\xi=\lambda/L$, such that for $\lambda\leq L\,\xi$, $|\rho_0(L/\lambda)|^2\leq 10$ percent (matching case)*

Functions $\varphi(y)$	$A[yy_L-y^2]$	$A[yy_L-y^2]^2$	$A(1-\cos 2\pi\, y/y_L)$	$A\sin\pi\, y/y_L$	Triangular pulse between $0\text{-}y_L$
$n(L)=2$	2.90	2.08	2.02	2.90	2.27
$n(L)=5$	3.20	2.50	2.50	3.20	2.63
$n(L)=10$	3.70	2.80	2.63	3.80	3.03

TABLE 5. *Matching case: Functions $\varphi(y)$, $\rho_0(\eta)$, and numerical values of the parameters*

Fig.	$\varphi(y)$ $0\leq y\leq y_L$	$\rho_0(\eta)$	$n(L)$ (exact values)	A	y_L
1	$A(yy_L-y^2)$	$\frac{2A}{\eta^2}\left[\frac{2}{\eta}\sin\frac{\eta y_L}{2}-y_L\cos\frac{\eta y_L}{2}\right]\cdot\exp\left[-j\eta\frac{y_L}{2}\right]$	2.05 2.94 5.05 6.99 11.0	−0.64 −.66 −.61 −.55 −.46	1.5 1.7 2.0 2.2 2.5
2	$A(yy_L-y^2)^2$	$\frac{4A}{\eta^3}\left[\frac{12}{\eta^2}\sin\frac{\eta y_L}{2}-\frac{6y_L}{\eta}\cos\frac{\eta y_L}{2}-y_L^2\sin\frac{\eta y_L}{2}\right]\cdot\exp\left[-j\eta\frac{y_L}{2}\right]$	2.01 2.97 4.95 7.20 9.85	−1.95 −1.56 −0.75 −.72 −.53	1.4 1.6 2.0 2.1 2.3
3	$A\left(1-\cos\frac{2\pi y}{y_L}\right)$	$Ay_L\left[\frac{\sin\frac{\eta y_L}{2}}{\frac{\eta y_L}{2}}+\frac{1}{2}\left\{\frac{\sin\left\{\left[\frac{2\pi}{y_L}+\eta\right]\frac{y_L}{2}\right\}}{\left\{\left[\frac{2\pi}{y_L}+\eta\right]\frac{y_L}{2}\right\}}+\frac{\sin\left[\left\{\frac{2\pi}{y_L}-\eta\right\}\frac{y_L}{2}\right]}{\left[\left\{\frac{2\pi}{y_L}-\eta\right\}\frac{y_L}{2}\right]}\right\}\right]\cdot\exp\left[-j\eta\frac{y_L}{2}\right]$	2.01 2.99 4.97 7.01 10.07	−.25 −.32 −.40 −.49 −.52	1.4 1.7 2.0 2.0 2.20
4	$A\sin\frac{\pi y}{y_L}$	$\frac{Ay_L}{2}\left\{\frac{\sin\left[\left\{\frac{\pi}{y_L}+\eta\right\}\frac{y_L}{2}\right]}{\left[\left\{\frac{\pi}{y_L}+\eta\right\}\frac{y_L}{2}\right]}+\frac{\sin\left[\left\{\frac{\pi}{y_L}-\eta\right\}\frac{y_L}{2}\right]}{\left[\left\{\frac{\pi}{y_L}-\eta\right\}\frac{y_L}{2}\right]}\right\}\cdot\exp\left[-j\eta\frac{y_L}{2}\right]$	1.99 3.10 5.20 6.68 10.90	−.36 −.54 −.65 −.68 −.75	1.50 1.68 2.00 2.20 2.50
5	Triangular pulse of height A	$\frac{Ay_L}{8}\left[\frac{\sin\,\eta y_L/4}{\eta y_L/4}\right]^2\cdot\exp\left[-j\eta\frac{y_L}{2}\right]$	2.0 3.0 5.0 7.0 10.0	−.50 −.704 −.864 −.936 −1.02	1.35 1.60 1.86 2.11 2.28

5. Quarter Wavelength Law

Let us now define the interval $\Delta\eta$ according to the following relation (analogous to the one defining the "radius of gyration" in mechanics, or the root-mean-square error in error theory):

$$(\Delta\eta)^2=\frac{1}{I}\int_{-\infty}^{+\infty}\eta^2|\rho_0(\eta)|^2d\eta, \tag{5.1}$$

where

$$I=\int_{-\infty}^{+\infty}|\rho_0(\eta)|^2d\eta. \tag{5.2}$$

Let Δy be the analogous interval for the function $\varphi(y)$.

It is apparent that the two functions $\rho_0(\eta)$ and $\varphi(y)$ will be noticeably different from zero only inside the two intervals $\Delta\eta$ and Δy.

The product of the intervals $\Delta\eta$ and Δy cannot be less than a given positive value K, which, according to the adopted criterion of the square mean, equals 2π [Persico, 1950]:

$$\Delta\eta\cdot\Delta y\geq 2\pi. \tag{5.3}$$

If we refer to the positive half interval, $\Delta\eta=4\pi/\lambda_{\max}$, where $\lambda_{\max}$ is the longest wavelength almost completely transmitted, and if we put $\Delta y\simeq y_L=\bar{n}L$, where $\bar{n}$ is the mean value of $n(x)$ in the interval $(0, L)$, (5.3) becomes

$$2\,\frac{4\pi}{\lambda_{\max}}\,\bar{n}L\geq 2\pi \qquad \lambda_{\max}\leq 4\bar{n}L. \tag{5.4}$$

When $n=\text{const}=1$, the two Fourier related functions are $\rho_0(\eta)$ and $Z_0'/2Z_0$ (see sec. 4), and inequality (5.4) becomes

$$\lambda_{\max}\leq 4L. \tag{5.5}$$

This last inequality shows that, for a given length L (in air), the matching line becomes ineffective for wavelengths beyond $4L$, whatever the function $Z_0'/2Z_0$ may be.

On the contrary, when n is a function of x, at least in nonmagnetic cases, an improvement can be expected according to (5.4).

For further improvements due to magnetic properties, see section 1.

6. An Approximate Solution of the RDE in the Case of Absorption

For the synthesis of a broadband termination, an empirical method may consist of adding an arbitrary imaginary part to a purely real $n(x)$ function, found according to the above techniques (see sec. 4), in such a way as to make the phase vary linearly with x; a numerical checking is then obviously necessary. Another way may consist of planning an absorbing device composed by a broadband transformer with a high $n(L)/n(0)$ rate (table 4, $n(L)/n(0)=10$), followed by a thin "Heaviside" absorber, probably realizable without excessive difficulties in these conditions.

We will now find an approximate solution of the RDE in the case of absorption, with the limit condition $\rho(\eta, L)=-1$.

As is well known, a RDE can be transformed into a Bernouilli one, once a particular integral is known. We may consider as such the solution of the RDE for the condition which corresponds to perfect matching at the far end, i.e., to $\rho(\eta, L)=0$.

Calling $\sigma(x)$ such a solution, the substitution $\tau=\rho-\sigma$ transforms (1.3) into a Bernouilli DE in which the further substitution $y=\tau^{-1}$ yields to a linear equation whose general integral is the following:

$$y=\exp\left[\int_x^L\left(2\gamma+\sigma\frac{Z_0'}{Z_0}\right)dx\right]\left\{-\frac{1}{2}\int_L^x\frac{Z_0'}{Z_0}\exp\left[-\int_x^L\left(2\gamma+\sigma\frac{Z_0'}{Z_0}\right)dx\right]\cdot dx+C\right\}\cdot \tag{6.1}$$

As $y=(\rho-\sigma)^{-1}$ must become $(-1-0)^{-1}=-1$ at the end of the shorted line, C must equal -1. Consequently, we can write:

$$\rho(x)=\sigma(x)-\frac{\exp\left[-\int_x^L\left(2\gamma+\sigma\frac{Z_0'}{Z_0}\right)dx\right]}{1-\frac{1}{2}\int_0^L\frac{Z_0'}{Z_0}\exp\left[-\int_x^L\left(2\gamma+\sigma\frac{Z_0'}{Z_0}\right)dx\right]\cdot dx}. \tag{6.2}$$

For $x=0$ we have

$$\rho_0(\eta)=\sigma_0(\eta)-\frac{\exp\left[-\int_0^L\left(2\gamma+\sigma\frac{Z_0'}{Z_0}\right)dx\right]}{1-\frac{1}{2}\int_0^L\frac{Z_0'}{Z_0}\exp\left[-\int_x^L\left(2\gamma+\sigma\frac{Z_0'}{Z_0}\right)dx\right]\cdot dx}. \tag{6.3}$$

As $\sigma(x)$ vanishes at the end of the line, contrary to $\rho(x)$ which equals -1 at the same point, the RDE for $\sigma(x)$ can be linearized, neglecting $|\sigma(x)|^2$ in comparison with unity; consequently, we can consider $\sigma(x)$ as the solution of the following linear DE:

$$\sigma'-2\sigma\gamma=-\frac{1}{2}\frac{Z_0'}{Z_0}. \tag{6.4}$$

Multiplying (6.4) by σ and neglecting σ^2, we have

$$d\sigma^2=-\frac{Z_0'}{Z_0}\,\sigma\,dx. \tag{6.5}$$

By integration we obtain

$$-\int_x^L\frac{Z_0'}{Z_0}\,dx=-\sigma^2(x). \tag{6.6}$$

Consequently

$$\exp\left[-\int_x^L\sigma\frac{Z_0'}{Z_0}\,dx\right]=\exp\left[-\sigma^2(x)\right]\simeq1-\sigma^2(x), \tag{6.7}$$

and, analogously

$$\exp\left[-\int_0^L\sigma\frac{Z_0'}{Z_0}\,dx\right]=\exp\left[-\sigma_0^2(\eta)\right]\simeq1-\sigma_0^2(\eta). \tag{6.8}$$

Substituting (6.7) and (6.8) into (6.3) we obtain

$$\rho_0(\eta)=\sigma_0(\eta)-\frac{[1-\sigma_0^2(\eta)]\exp\left[-2\int_0^L\gamma dx\right]}{1-\frac{1}{2}\int_0^L\frac{Z_0'}{Z_0}\cdot[1-\sigma^2(x)]\cdot\exp\left[-2\int_x^L\gamma dx\right]\cdot dx}. \tag{6.9}$$

Neglecting σ^2 we get the following approximate value for $\rho_0(\eta)$:

$$\rho_0(\eta)=\sigma_0(\eta)-\frac{\exp\left[-2\int_0^L\gamma dx\right]}{1-\frac{1}{2}\int_0^L\frac{Z_0'}{Z_0}\exp\left[-2\int_x^L\gamma dx\right]\cdot dx}=\sigma_0(\eta)-\frac{\exp\left[-2\int_0^L\gamma dx\right]}{1+\bar{\sigma}_L(\eta)}, \tag{6.10}$$

where

$$\sigma_0(\eta)=+\frac{1}{2}\int_0^L\frac{Z_0'}{Z_0}\exp\left[-2\int_0^x\gamma dx\right]\cdot dx, \tag{6.11}$$

and $\bar{\sigma}_L(L)$ is the reflection coefficient in $x=L$ for a wave which proceeds from right to left in the layer matched at $x=L$ [Latmiral et al., 1962].

When $|\bar{\sigma}_L(\eta)|<<1$, (6.10) can be further simplified as follows:

$$\rho_0(\eta)\simeq\sigma_0(\eta)-\exp\left[-2\int_0^L\gamma dx\right], \tag{6.12}$$

even when γ is complex, and μ is not a constant but a complex function of x.

It must be pointed out that (6.12) is the solution of the RDE (1.3) linearized by neglecting $|\rho^2|$ against unity and solved with the condition $\rho(L)=-1$.

This simplification is, however, possible only as a consequence of the considerations which have led to (6.10) by splitting the solution of the RDE into two parts by means of the particular integral $\sigma(x)$, which vanishes for $x=L$.

Otherwise, the two assumptions $|\rho|^2<<1$ and $\rho(L)=-1$ would have been almost unjustifiable.

7. Special Techniques for the Synthesis in the Case of Absorption

The splitting of the solution of the RDE into two parts (6.10), the first of which refers to the condition $\sigma(L)=0$, allows us to extend the transform techniques (under the conditions pointed out in sec. 6) even to the case where the layer is not matched at the far end and particularly to the case where $\rho(L)=-1$. Equation (6.10) shows, as a matter of fact, that to minimize $|\rho_0(\eta)|$ is equivalent to minimizing $|\sigma_0(\eta)|$, provided that $\gamma(x)=j\frac{\eta}{2}n(x)$ is chosen in a class of functions which satisfies the condition

$$\left(\text{Real part of}\int_0^L \gamma dx\right)=\frac{\eta}{2}\int_0^L n_2(x)dx>H, \tag{7.1}$$

where H is a given positive quantity.

According to (6.12), this means that the wave reflected from the metal must not exceed a reasonably low value.

Obviously, in the case $\rho(L)=-1$, $n_1(L)$ and $n_2(L)$ are not given, and only $n_1(0)$, $n_2(0)$ and $\int_0^L n_2\,dx$ (see 7.1) are given.

The above considerations allow us to extend, at least in some special cases, the synthesizing procedure referred to under section 4, to the $\sigma_0(\eta)$ function [Latmiral et al., 1962].

For example, when the refraction index is of the type $n(x)=n_1-jK/\eta$, with n_1 and K real positive constants with respect to both η and x, the Fourier transform pair (see sec. 4) becomes $\rho_0(\eta)$ and $(Z_0' \exp[-Kx])/2Z_0$. This case occurs, e.g., when very low ohmic losses are present ($\sigma<<\omega\epsilon$), and ϵ and σ do not depend on x.

Another case in which the extension is possible is $n=n_1-jn_2=\text{const}$, with n_1 and n_2 both independent of x and η. $\rho_0(\eta)$ are then the values that the Laplace transform of the function $Z_0'/2Z_0$ takes on the straight line of complex equation

$$p=(n_2+jn_1)\eta, \tag{7.2}$$

with η in the interval $(0, \infty)$.

When the refraction index is x-dependent, the synthesizing procedure is still applicable in some special cases.

For example, when the refraction index is of the type $n(x)=n_1(x)-jK/\eta$, with K constant with respect to both η and x, (4.11) becomes, under the hypothesis that $\epsilon_{r\,\max}$ is not too high (for example, $\epsilon_{r\,\max}\leq 16$),

$$-\frac{y''}{2y'^2}e^{-Kx}=\sigma_o(y). \tag{7.3}$$

This DE must be solved according to the method explained in section 4. The above case occurs when very low ohmic losses are present, and the quantity $\sigma(x)/\sqrt{\epsilon_r(x)}$ (proportional to the loss angle) is constant with respect to x. Another case is the following:

$$n(x)=(n_1-jn_2)f(x), \tag{7.4}$$

with n_1 and n_2 real constants with respect to both η and x.

Let

$$y=\int_0^x f(x)dx;\ \ y_L=\int_0^L f(x)dx;\ \ Z_0[x(y)]=\overline{Z}_0(y). \tag{7.5}$$

The values of $\rho_0(\eta)$ are those which the Laplace transform of the function $\overline{Z}_0'/2\overline{Z}_0$ takes on the straight line of complex equation [Latmiral et al., 1962]

$$p=(n_2+jn_1)\eta. \tag{7.6}$$

8. General Techniques for the Synthesis in the Case of Absorption

Let us now consider the general problem of synthesizing a nonmagnetic absorber; we will solve this problem by means of an approximate method.

It must be pointed out that, in the absorption problem, not four but only three conditions, $n_1(0)$, $n_2(0)$, and $\int_0^L n_2\,dx$, are given in a nonmagnetic absorber. The following approximate procedure, based on transform techniques, may be helpful. For μ=const and small values of $\rho(x)$, the linearized (1.3) (see sec. 6) gives:

$$\sigma_0(\eta)=-\frac{1}{2}\int_0^L \frac{n_1'-jn_2'}{n_1-jn_2}\exp\left[-\eta\int_0^x n_2dx\right]\cdot\exp\left[-j\eta\int_0^x n_1dx\right]dx. \tag{8.1}$$

Let us compare the above equation with the following:

$$\sum_0(\eta)=-\frac{1}{2}\int_0^L \frac{n_1'-jn_2'}{n_1-jn_2}\exp\left[-j\eta\int_0^x n_1dx\right]dx. \tag{8.2}$$

If some exceptional cases are excluded, the value of $\Sigma_0(\eta)$ are, on the average, less than the values of $\sigma_0(\eta)$.

Consequently, if $n_1(x)$ and $n_2(x)$ are such as to keep Σ_0 small in a given interval of η, the same functions can give a convenient σ_0.

Introducing the new variable of integration

$$y=\int_0^x n_1(x)dx, \tag{8.3}$$

$$\sum_0(\eta)=-\frac{1}{2}\int_0^{y_L}\frac{d}{dy}[\ln(n_1-jn_2)]\cdot\exp[-j\eta y]dy. \tag{8.4}$$

Thus $\Sigma_0(\eta)$ is a Fourier transform.

Once $\Sigma_0(\eta)$ has been chosen, and indicating its inverse transform by $\varphi(y)=\varphi_1(y)-j\varphi_2(y)$, we have

$$\frac{1}{2}\frac{d}{dy}[\ln(n_1-jn_2)]=\varphi_1(y)-j\varphi_2(y), \tag{8.5}$$

$$\frac{1}{2}(\ln[n_1(y)-jn_2(y)]-\ln[n_1(0)-jn_2(0)])=\int_0^y\varphi_1(y)dy-j\int_0^y\varphi_2(y)dy; \tag{8.6}$$

i.e.,

$$\ln[n_1(y)-jn_2(y)=\left(2\int_0^y\varphi_1(y)dy+\ln\sqrt{n_1^2(0)+n_2^2(0)}\right)$$
$$-j\left(2\int_0^y\varphi_2(y)dy+\operatorname{arctg}\frac{n_2(0)}{n_1(0)}\right)=\Psi_1(y)-j\Psi_2(y). \tag{8.7}$$

As $n_1(0)$ and $n_2(0)$ are given, the values of $n_1(y)$ and $n_2(y)$ can be obtained in terms of y, solving (8.7), i.e.,

$$\sqrt{n_1^2(y)+n_2^2(y)}=\exp\,[\Psi_1(y)]; \ \text{arctg}\,\frac{n_2(y)}{n_1(y)}=\Psi_2(y). \tag{8.8}$$

By substituting $y(x)$ in (8.8), $n_1(x)$ and $n_2(x)$ are obtained. Finally it must be tested that inequality (6.3) is verified. To enable us to comply with (6.3), the quantity y_L may be introduced as a parameter in $\Sigma_0(\eta)$.

Obviously, as the above procedure is purely mathematical, it must be verified "a posteriori" that the obtained values of $n_1(x)$ and $n_2(x)$ are practically attainable in a given frequency range. Furthermore, for evaluating $n_1(x)$ and $n_2(x)$, digital computations will be generally necessary. Only if the hypothesis $n_2 << n_1$ is satisfied, the use of an analog computer is practically possible. However, in some special cases, the above procedure can be simplified.

Two different types of $\varphi(y)=\varphi_1(y)-j\varphi_2(y)$ functions (to which a favorable $\Sigma_0(\eta)$ spectrum is related) were chosen, and the corresponding $n(x)=n_1(x)-jn_2(x)$ functions computed by means of the above procedure.

The $n(x)$ functions are traced in figures 6 and 7, under each of which the corresponding $\varphi(y)$ function is listed. The first case was computed by means of an IBM 1620 digital computer (see sec. 2), and the second one by means of a PACE–TR–10 analog computer (see sec. 2).

Table 6 gives the power reflection coefficient $|\rho_0(\eta)|^2$ percent computed by means of the RDE (1.3) and (1.1) plus (1.2), for the two considered cases, and for several values of the rate L/λ. Under table 6 the numerical values of parameters are, as well, referred.

In the second case, an initial discontinuity $(n(0)=1.25)$ was considered.

The results summarized under table 6 seem to be rather poor. But it must be pointed out that no interferential effect (at least for the second case) takes place between the two terms of (6.12). Better results could probably be obtained by lowering the losses.

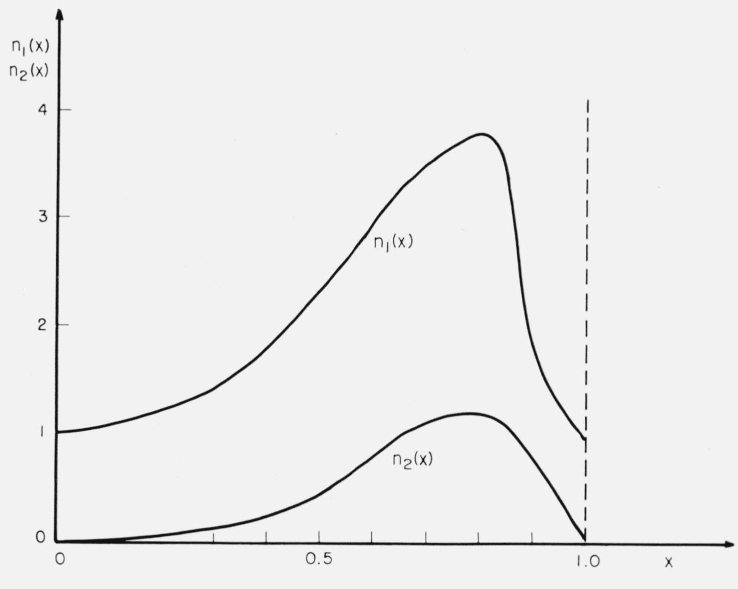

FIGURE 6. *Graphs of the* $n_1(x)$ *and* $n_2(x)$ *functions referring to table 6, 1st column.*

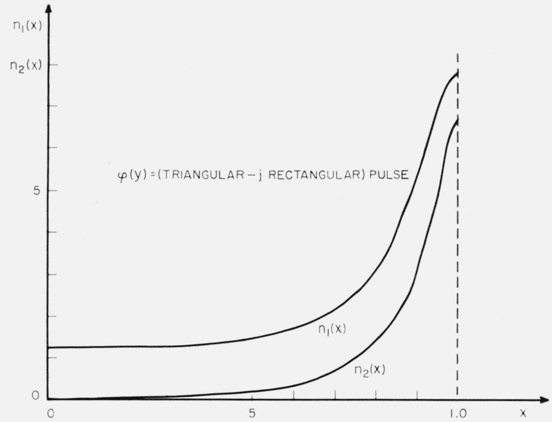

FIGURE 7. *Graphs of the* $n_1(x)$ *and* $n_2(x)$ *functions referring to table 6, 2d column.*

TABLE 6. *Power reflection coefficient* $|\rho_0(L/\lambda)|^2$ *percent for several values of* L/λ *(absorbing case), and numerical values of the parameters*

Functions $\varphi(y)$	$\ln\sqrt{n_1^2+n_2^2}=A\sin\frac{2\pi}{Ty_L}y$ $\text{arctg}\frac{n_2}{n_1}=B\sin\frac{2\pi}{Ty_L}y$	$A=1.4$ $B=0.32$ $y_L=6.28$ $T=3.3\bar{3}$ $0\leq y\leq gy_L$ $6.6\bar{6}$ $gy_L\leq y\leq y_L$ $g=0.83\bar{3}$	Triangular pulse of height A $-j$ rectangular pulse of height B $0\leq y\leq y_L$	$-A=0.936$ $B=0.195$ $y_L=2.11$
$L/\lambda=0.1$	59.9		76.9	
.2	31.2		12.5	
.3	0.55		3.3	
.4	3.82		0.25	
.5	1.14		.02	
.6	0.65			
.7	.68			

9. Conclusion

The possibility of linearizing the basic RDE allows us, in the case of matching, to make easy use of both the analysis and synthesis procedures, the last being based on the Fourier transform techniques. The extension of the method to the $n=n(x)$ case gives it a large field of practical applicability.

In the case of absorption, unfortunately, the transform techniques are, generally, applicable only to a part of the solution of the RDE, and under rather restrictive conditions: μ must be a constant and $n_1(x)$ and $n_2(x)$ must be independent of frequency; owing to the conditions for physical realizability, this can be (approximatively) true only in a limited frequency band. Only the analysis procedure is, therefore, completely reliable.

Furthermore, the construction of a stratified layer closely approaching two given $n_1(x)$ and $n_2(x)$ functions is a hard task, and, if the number of the strata is not sufficiently high, noticeable disagreements with the theoretical values have to be expected. As for the matching, a "band pass" rather than a "high pass" behavior has to be expected [Franceschetti, 1962]. An easy way to design and to construct a broadband absorber may consist of using a single material of known and proper electric and magnetic characteristics $\epsilon(\omega)$ and $\mu(\omega)$, and by tapering it "geometrically," e.g., in the form of dyhedra or pyramids of such dimensions that a sufficient number of them is included in a λ^2 area. Under the above assumptions, the performance of this absorber may be considered approximatively equivalent to that of a stratified absorber, whose $\epsilon(x)$ and $\mu(x)$ parameters are functions of the "geometrical tapering" and, obviously, of the single used material.

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